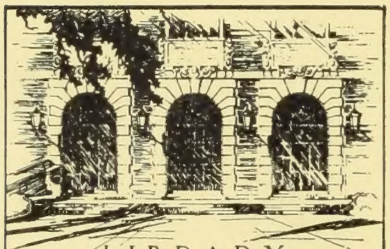

HIGH SCHOOL MATHEMATICS

Unit 1.

THE ARITHMETIC OF THE REAL NUMBERS

UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS

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TABLE OF CONTENTS

Arithmetic by mail. --Stan Brown had a pen pal, Al Moore, who lived in Alaska. Stan and Al corresponded quite frequently. Stan liked to receive letters from Al because he wrote about interesting things like hunting and fishing and prospecting for gold. Al enjoyed hearing about the things Stan did, especially about school, for Al had had very little opportunity to attend school. One day, Al wrote to ask if Stan would mind teaching him some arithmetic. Stan agreed but decided he needed to know how much Al already knew. So, in his next letter to Al he included a simple test, and asked Al to write in the answers and to return the test to him. Al sent the test back immediately; he said it was very easy and asked Stan to send some harder questions next time.

Turn the page to see what Al's test looked like when he returned it.

1.	Take 2 away from 21.	..... 1 .....
2.	What is half of 3?	..... ɔ .....
3.	Add 5 to 7.	..... 57 .....
4.	Does $2 \times 4\frac{1}{2}$ equal 9?	..... no .....
5.	Which is larger, .000065 or .25?	..... .000065 .....
6.	How many times does 3 go into 8?	..... twice .....
7.	How many times does 9 go into 99?	..... twice .....
8.	Which is larger, 3 or 23?	..... 23 .....
9.	What is a number smaller than 4?	..... 4 .....
10.	What is a number larger than 4?	..... 4 .....

Stan was flabbergasted when he looked at Al's answers. Was this a joke? But Al had seemed so serious about wanting to learn arithmetic. Stan decided that Al needed a lot of help. He would start by telling Al about the errors he had made.

Dear Al,

You sure have some funny ideas about numbers. But it won't take long to straighten you out (I hope).

I've enclosed your test with this letter so that you could follow my explanation. Look at the first question. I can see how you thought that the answer was 1 because if you do take 2 away from 21 you are left with 1. But when you take 2 away from 21, you get 19. See? Take the second question. Your answer isn't even a number. What you wrote is half of 3 but half of 3 isn't that, it's $1\frac{1}{2}$. In the third question, you put 5 and 7 together, but adding numbers isn't putting them together. It's, er... well, I'll explain that in

your next lesson. In the fourth question, I can see that you thought that $2 \times 4\frac{1}{2}$ was not the same as 9, but $2 \times 4\frac{1}{2}$ is equal to 9, so your answer is wrong.

Now, in question five, it's true that .000065 is bigger than .25, but really .25 is larger than .000065. You see, .25 is a bigger number than .000065 even though .25 looks smaller than .000065.

I guess I can give you credit for question six, although you really should have said that 3 goes into 8 twice with 2 left over. I guess you just made a careless mistake in dividing in question seven, because 9 goes into 99 eleven times. Question eight you did correctly, so I guess you know how to count.

You made the same kind of mistake in question nine as you did in the very last question. Even though what you wrote for question nine is smaller than 4, I asked for a number smaller than 4 and the one you picked is the same size as 4. Same idea for question ten.

I hope that my explanations of what you did wrong are helpful to you. Let me know when you want your next lesson. Make it soon because we have lots of work to do.

Your pal,

Stan

Al was a little disappointed with Stan's letter. He wrote the following in return.

Dear Stan,

I sure appreciate what you are trying to do for me, but I don't think that <u>you</u> can help me at all. Are you sure <u>you</u> understand this stuff?

Sure I got questions 6 and 8 right. I did them just as I did the others. Anyone can see

that 3 goes into 8 twice, and pretty neatly, too, without any 2 left over, either. You put 3 into 8 the regular way and then you turn another 3 around and put it in on the other side of the 8.

In question 8, you can just see that 23 is larger than 3 because 23 already has a 3 in it and a 2 added on in front. You don't need to know how to count to tell that!

I really laughed at what you said about question 7. If there's one thing I do know it's how to count, and eleven 9's make 99999999999 and not 99. You said that .000065 is bigger than .25. I knew that because I even checked with a ruler. Then you cross yourself up and say that .000065 is really smaller. And in question 4 you don't even need a ruler to tell that $2 \times 4\frac{1}{2}$ is different from 9.

There's no use in trying to learn arithmetic, I guess. I think I'll stick to hunting.

Your friend,

Al

EXERCISES

A. Do you think Stan did a good job explaining Al's errors? What seemed to be Stan's difficulty in doing so? Can you defend what Al did?

B. In certain print shops the printing press is prepared for operation by picking individual type pieces out of boxes and placing the pieces into the bed of the press. Such a print shop has type of various styles and sizes. Here are two examples of remarks you might hear a printer make to his helper:

> Bill, this seven isn't big enough; get me a bigger seven.
>
> I asked you to bring me two threes and you brought me three twos. I don't care what you learned in school, three twos are not the same as two threes.

Make up more examples of confusing statements you might hear in this print shop.

<u>Things and the names of things</u>.--It is easy to see what Al had in mind when he took Stan's test. Al was confusing numbers with the marks that were written on the paper. And even though Stan may have realized this, he certainly didn't get the idea across to Al. It might have helped Al if Stan started by saying something like this:

> Look, Al, if I asked you to tell me something about Alaska, you wouldn't tell me that it started and ended with the same letter, would you? You might tell me how big it was, or how many people lived there, or what its capital city was. So, when I ask you a question about numbers [for example, to take 2 away from 21], I don't expect you to tell me about the marks on the paper. Instead, I expect you to do something with the numbers those marks stand for. You see, those marks aren't numbers. They just stand for numbers. They're really <u>names</u> of numbers, just as the word:
>
> Alaska
>
> is the name of the place in which you live.

Whenever a person writes about numbers, he puts marks on paper; if he talks about numbers, he makes certain sounds. But the marks on paper and the sounds are not the numbers themselves. The marks and the sounds are names of numbers. In order to write <u>about</u> a thing, you must write a <u>name</u> of the thing.

In everyday conversation, one hardly ever confuses a thing with a name of the thing, because in so many cases you can point to the thing and point to its name, and you can see that they are different. But, there are some things which we talk about but which we can't point to. Take justice, for example. You would have a hard time trying to point to something, and saying that that thing is justice. Also, if you were having a serious discussion about justice, you would get pretty annoyed with someone who claimed he could show you what justice is and did it by handing you a piece of paper like this:

Most people would agree that justice isn't something you can touch or see or hear. Numbers are like that, also. Someone can show you 2 apples or 2 toys or a hat and coat, but none of these pairs of things is the number 2 itself. But, just as people can think about justice even though they can't see or touch it, they can also think about the number 2. And, in talking about and working with the ideas of justice and 2, they use marks such as:

justice   2

but they don't [at least they shouldn't] think that the marks themselves are justice and the number 2.

Now, in reading a mathematics textbook or in listening to someone who is trying to teach you mathematics, it is very important that you be able to tell when the book or person is talking about numbers and when the book or person is talking about the marks which are names of numbers. Al had trouble in making sense out of Stan's explanations because Al couldn't tell when

Stan was talking about marks on paper and when he was talking about the numbers which the marks named. And you wouldn't have been able to tell this from Stan's letter either if you didn't already know a lot about arithmetic. Al's trouble with Stan's explanation was that

(1) in writing out the test, Stan had used the mark:

2

and the mark:

21

as names for numbers [so, his first question was about the number two and the number twenty-one], but

(2) Al thought that Stan's first question was about the marks themselves, and

(3) in trying to explain Al's errors to him, Stan began by using the marks as names for themselves ["if you do take 2 away from 21 you are left with 1"], and then used them again as names for numbers ["when you take 2 away from 21 you get 19"].

No wonder Al was confused! In order to write about the marks Stan needed names for them. He might have written:

> Al, let's use:
>
> Tom
>
> as a name for the mark:
>
> 2,
>
> and use:
>
> Dick
>
> and:
>
> Harry
>
> as names for the mark:
>
> 21
>
> and the mark:
>
> 1.
>
> Now, if you do take Tom away from Dick, you are left with Harry. But when you take 2 away from 21, you get 19.

Or, he might have told Al that he would write a name for a mark by drawing a loop around it. When Al saw a loop, he would know that it, together with the mark inside it, was a word, and that this word was a name for the mark. So, Stan might have written:

> I can see how you thought that the answer was 1 because if you do take 2 away from 21, you are left with 1. But, when you take 2 away from 21, you get 19.

Of course, this statement wouldn't teach Al how to subtract, but it would at least tell him that Stan realized that Al was confusing numbers with names of numbers.

Can you think of other devices you could use to write names for marks?

In this book, the device we shall use most frequently to write names for marks is to enclose marks in single quotes ['. . . ']. When you see a mark [such as a word or a sentence, for example] with single quotes around it, you are seeing a name for the mark which is written between the single quotes.

So, the sentence:

'3' is a name for a number

is about the <u>mark</u>:

3

which is <u>named</u> in the sentence. On the other hand, the sentence:

3 is an odd number

is about the <u>number</u> which the same mark <u>names</u>.

To decide whether or not to put single quotes around a mark, ask yourself:

Am I talking about the <u>mark</u>, itself?

or

Am I talking about the <u>thing</u> to which the mark refers?

When you are talking with a person, you can sometimes get along without using names for the things you are talking about. You do this by pointing at a thing and saying 'this'. For instance, Stan could have made his explanation to Al, if they had been together, by pointing, first at the '2' in the '21', then at the '21', and then at the '1', and saying:

If you take this away from this, you are left with this.

Sometimes in writing we shall avoid using a name for a mark by a kind of pointing at the mark. [In fact, we have already done so.]

We can point at a mark by displaying it on a separate line, and putting a colon at the end of the preceding line. Think of the colon as the writer's finger pointing at the mark which follows it.

Sometimes we want to point at a word just to call special attention to it, either because it's especially important or because it's a new word.

(1) 'John' is a <u>word</u>, but John is a <u>boy</u>.

(2) Names of numbers are called <u>numerals</u>.

(3) Notice that in '[8 × (3 + 2)] × 5' we use both parentheses and <u>brackets</u>.

In sentence (1), the words 'word' and 'boy' are underlined merely because we wish to emphasize a contrast. In sentence (2), the word 'numeral' is a new word, and we want to call your attention to it. Actually, the sentence is about this word and, so, should be written:

Names of numbers are called 'numerals'.

But, because the word is underlined in (2), and follows 'are called', our carelessness in omitting the single quotes is not likely to cause confusion. In sentence (3), the word 'brackets' is underlined because it is a word which has not been used before. But [unlike the word 'numeral' in (2)] the sentence isn't about this word, so it would be incorrect to enclose it in single quotes.

Read the following paragraphs carefully and note the use of various signals to tell you when names and marks are being discussed.

> This morning a girl named Ruth received four letters. Four friends had written to Ruth, and each of them wrote 'Ruth' on the envelope. Each letter started with:
>
> Dear Ruth,
>
> Notice that Ruth has four letters and that 'Ruth' has four letters.
>
> Ruth sometimes wished her name were longer so that she could give herself a short nickname. Her friend Margaret was often called Peg. 'Peg' is short even though Peg is tall. Ruth often wondered how you get 'Peg' out of 'Margaret'.

EXERCISES

A. Some of the following sentences make sense and some do not. In the case of each sentence which does not, put single quotes around some of the words in the sentence so that the resulting sentence does make sense.

1. Bill has a dog.
2. Bill found a dog in a book.
3. I have trouble with my pen when I make a 3.
4. Mary is a part of Maryland.
5. He erased the 5 and put a 4 in its place.
6. John has four letters.
7. Ada is taller than Penelope but Ada is shorter than Penelope.
8. Mr. is an abbreviation for Mister.
9. 6 is a number and 6 is a place holder. Similarly, 0 is a number and 0 is a place holder.

B. Use single quotes in punctuating the following paragraph in order to make sense out of it.

> I always put milk on my cereal. Otherwise, it is too dry to eat. But, I have never put milk on my cereal. It would be much easier to put milk on cake frosting than to put milk on corn flakes.

[More exercises are in Part A, Supplementary Exercises.]

Numbers and numerals.--We have said that numbers are things that we can think about, talk about, and write about. In order to talk and write about numbers, we use names for them. Names of numbers are called numerals or, sometimes, numerical expressions. Here are a few of the numerals which are names of the number 4. [There is one numeral among these which doesn't name 4; find it!]

IV

8 - 4

four

2 + 2

7 - 3

4×0

2×2

$\frac{1842 - 1834}{2}$

4

$(1 + 1) + (1 + 1)$

$\frac{6 + 2}{2}$

4

$3 \times 1\frac{1}{3}$

IIII

3 + 1

quatre

$628.424 \div 157.106$

1×4

4×1

4 + 0

$72 \div 18$

the product of 12 and $\frac{1}{3}$.

1 + 3

A number has many, many numerals. Some of a number's numerals are simpler looking than others. For example, '4' is probably the simplest looking of all of 4's numerals. Certainly, it is simpler looking than the numeral '628.424 ÷ 157.106'. Yet,

both of the numerals:

$$4$$

and:

$$628.424 \div 157.106$$

are names of the same number, the number 4.

A short way of saying that each of two numerals names the same number is to write an equality sign between the numerals. Thus, when we write the sentence '5 + 2 = 6 + 1', we are saying that 5 + 2 is the same number as 6 + 1. Our sentence is true because '5 + 2' and '6 + 1' are numerals for the same number. If we write the sentence '9 + 3 = 4 + 7', our sentence is false because '9 + 3' and '4 + 7' are numerals for different numbers. A short way of saying that '9 + 3' and '4 + 7' are numerals for different numbers is to write an '≠' between the numerals. [Pronounce '≠' as you would pronounce 'is not equal to'.] Thus, the sentence '9 + 3 ≠ 4 + 7' is true.

Some of the following true sentences are about numbers, and some are about names of numbers. Be sure that you see that each sentence is true.

(1) 4 + 8 = 9 + 3.

(2) 4 is an even number.

(3) '4' is a numeral for 4.

(4) '4' is not a number.

(5) '8 - 3' and '10 ÷ 2' are numerals for 5.

(6) '2 + 2' is a name for 4.

(7) '2 + 2' is a name for 2 + 2.

(8) '2 + 2' is a name for 3 + 1.

(9) 2 + 2 is the sum of 2 and 2.

(10) 2 + 2 is the sum of 3 and 1.

(11) $4 + 1 \neq 4 \times 1$.

(12) You can put a '4' on paper.

(13) You can't put 4 on paper.

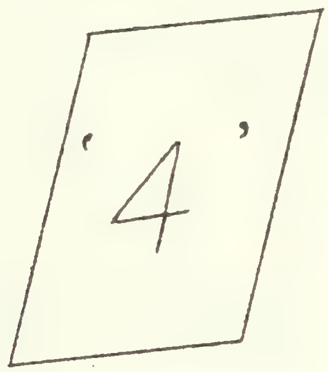
(14) '$\frac{6 + 2}{4}$' and '$\frac{8}{4}$' and '2' are numerals for tne same number, but '2' is the simplest of these numerals.

(15) You can't write 4 but you can write a '4'.

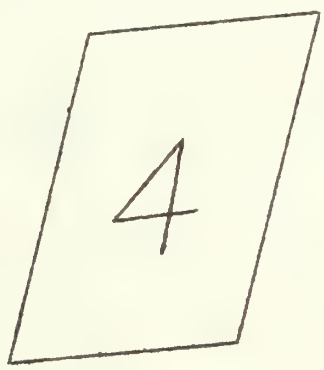
EXERCISES

A. Write 5 numerals for 6 and 5 numerals for 0.

B. A student was asked to write on a sheet of paper a numeral for 4. This is what his paper looked like:



He was wrong. His paper could have looked like this and he would have been correct:



Explain. [Did you make this error in Part A?]

C. The following is a long list of numbers. Although the list contains many numerals, only three numbers are listed. Rearrange the list into three columns such that all of the numerals in a column stand for the same number.

[Notice how the word 'list' is used. If someone asks you for a list of the students in your mathematics class, he doesn't expect you to collect all the students and to march them to him. What he hopes is that you will write down the names of the students, and that you will then hand him a sheet of paper with names written on it. This sheet of paper is a list of the students. And if you wanted to make yourself unpopular, you could write down nicknames for each student as well as their given names. You would still be listing the students, but the person reading the list might think there were many more students in the class. So, it makes sense to say that although the following list contains many numerals, only three numbers are listed.]

9×2 $\quad \dfrac{9-6}{10-5}$ $\quad 3 \times (7-1)$ $\quad \dfrac{4}{3} + \dfrac{50}{3}$ $\quad \dfrac{3-3}{842}$

$12 \times .05$ $\quad \dfrac{7+3}{15-5} - \dfrac{52}{26 \times 2}$ $\quad 29 - 11$ $\quad \dfrac{7}{5} - \dfrac{4}{5}$ $\quad 15 + 3$

$\left(\dfrac{1}{50} + \dfrac{3}{25}\right) + \left(\dfrac{2}{5} + \dfrac{3}{50}\right)$ $\quad \dfrac{60}{5} + \dfrac{30}{5}$ $\quad \dfrac{0}{18}$ $\quad (3+5) + 10$ $\quad 0 + 18$

$\dfrac{6}{10}$ $\quad 8 - 8$ $\quad (74 - 70) - 4$ $\quad 35 - (10 + 7)$ $\quad \dfrac{6+0}{10+0}$

$9.72 + 8.28$ $\quad \dfrac{3 \times 978}{5 \times 978}$ $\quad \dfrac{2}{3} - \dfrac{58}{87}$ $\quad (4 \times 2) + (5 \times 2)$

$\dfrac{(9+3) - (2 \times 4.5)}{(18-3) \div (7-4)}$ $\quad 18 - (3 \times 6)$ $\quad \left(90 \times \dfrac{4}{9}\right) \times (0.3 \times 1.5)$

D. True or false?

1. $7 + 9 = 4 \times 5$
2. $3 + 2 = 4 + 1$
3. $8 + 7 = 8 - 7$
4. $9 \times 5 \neq 40 + 5$
5. $6 \times 2 \neq 10 \times 2$
6. $8 + 5 = 5 + 3$
7. $8 = (2 + 2) + (2 + 2)$
8. $2 + (2 + 2) = 2 \times 2$
9. $52 + 68 = 58 + 62$
10. $52 \times 68 = 58 \times 62$
11. $73 + 92 = 92 + 73$
12. $73 \times 92 \neq 92 \times 73$

E. Each of the following exercises contains a pair of sentences, and each sentence has a blank space in it. For the first of each pair of sentences, write a numeral in the blank space so that the resulting sentence is true. For the second of the pair, write a numeral in the blank space so that the resulting sentence is false.

Sample 1. (a) $9 + \_\_\_ = 7 + 8$
(b) $9 + \_\_\_ = 7 + 8$

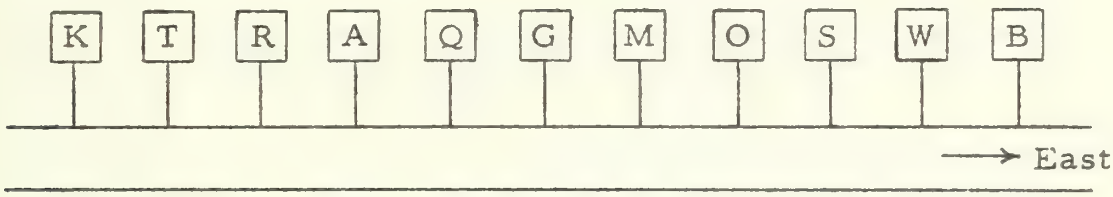
Solution. (a) $9 + \underline{6} = 7 + 8$ *True*
(b) $9 + \underline{79} = 7 + 8$ *False*

Sample 2. (a) $8 \times \_\_\_ \neq 12 \times 2$
(b) $8 \times \_\_\_ \neq 12 \times 2$

Solution. (a) $8 \times \underline{5} \neq 12 \times 2$ *True*
(b) $8 \times \underline{3} \neq 12 \times 2$ *False*

1. (a) $3 + \_\_\_ = 5 + 6$
   (b) $3 + \_\_\_ = 5 + 6$
2. (a) $7 - \_\_\_ = 3 + 2$
   (b) $7 - \_\_\_ = 3 + 2$
3. (a) $11 + \_\_\_ \neq 20 + 7$
   (b) $11 + \_\_\_ \neq 20 + 7$
4. (a) $6 \div \_\_\_ \neq 15 - 12$
   (b) $6 \div \_\_\_ \neq 15 - 12$
5. (a) $\_\_\_ + 984 = 984 + 793$
   (b) $\_\_\_ + 984 = 984 + 793$
6. (a) $\_\_\_ \times 0 = 0$
   (b) $\_\_\_ \times 0 = 0$

1.01 Distance and direction. --Imagine an east-west road which has markers placed one mile apart at the side of the road. The markers are labeled with letters. If you ride a bicycle from A to G, you say



that you have made a trip of 2 miles to the east; if you ride from S to B, you again say that you have made a trip of 2 miles to the east. Describe three other 2-miles-to-the-east trips on this road. Describe three 2-miles-to-the-west trips on this road.

The 2-miles-to-the-east trips and the 2-miles-to-the-west trips are alike in one important way. The length of each trip is 2 miles [or, for each trip, the distance in miles between starting and ending points is 2]. But, the trips are also different in an important way. The trips to the east are made in a direction opposite to that of the trips to the west.

Suppose you and a friend are at the point G on this road. Each of you decides to make a trip that will take you two miles from G. How many miles apart will you be at the end of your trips? It is easy to see that you cannot give a definite answer to this question. It is not enough to know just the distances for the trips; you also have to know something about the directions.

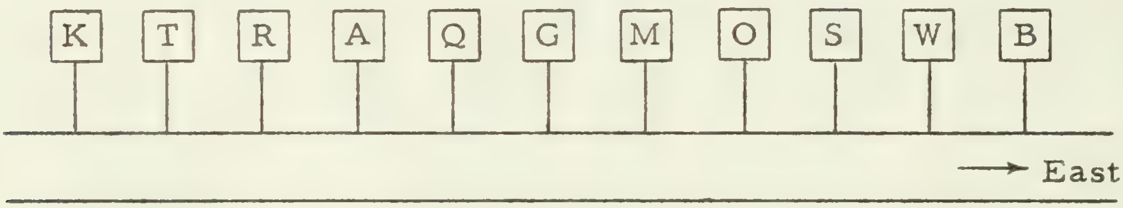
The numbers with which you have been working in school since the first grade can be used in measuring distances for trips along this road. But, using these numbers [let's call them the numbers of arithmetic] doesn't tell whether the trips have been made in the same direction or in opposite directions. If two people start at G and make 2-mile trips, we don't know whether they will be together at the end of their trips or whether they will be four miles apart. In order to measure trips which are made in one of two opposite directions, we need numbers which will take into account both distance

and direction. There are such numbers. They are usually called real numbers.* In this unit, you will learn many things about real numbers--you will learn how to compute with them, you will learn how to use them in solving problems, and you will learn that some of them "act like" our familiar numbers of arithmetic.

NUMERALS FOR REAL NUMBERS

In order to work with real numbers we shall need some system of naming them. We must have numerals for them so that we can talk and write about them. Let us make up a system of numerals which no one we know of has ever used before. After we have seen that it is possible to work with numerals which we have invented, we'll switch to a kind of numerals for real numbers which most other people use. Our ideas of real numbers will not change in switching from one system of numerals to another--only the names will change.

We want to use real numbers to measure trips. Since one important aspect of a trip is the distance between starting and ending points, and since our "old" numbers of arithmetic do serve quite well in measuring distances, we shall use as part of each numeral for a real number a numeral for a number of arithmetic. Then, because we need to include direction in measuring trips, we shall complete the numeral for the real number by including an arrow.



Suppose, again, that you and your friend decide to take 2-mile trips starting at G. If your trip is measured by the real number $\overrightarrow{2}$ and

* Real numbers are not any more (or less) real than other kinds of numbers. In a later course [when you learn about complex numbers] you will see why, historically, the word 'real' came to be used. Sometimes real numbers are called directed numbers or signed numbers.

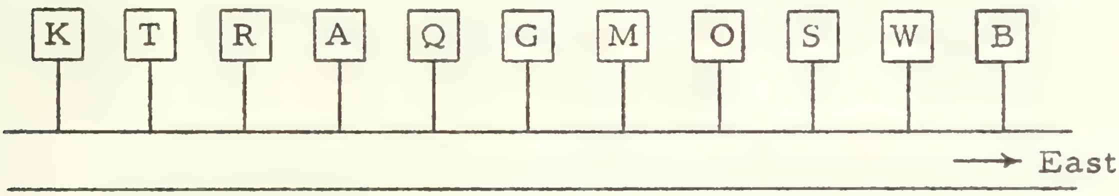
your friend's by the real number $\overleftarrow{2}$, you can be sure that you will be 4 miles apart at the end of your trips. How far apart will you be if each of you makes a trip of $\overrightarrow{2}$ miles? If each makes a trip of $\overleftarrow{2}$ miles? Suppose you take a trip of $\overleftarrow{3}$ miles and your friend takes one of $\overrightarrow{5}$ miles, starting from the same point. What real number measures the trip which your friend will have to take to get from where he is to where you are?

Notice that we have not yet said whether the real number $\overrightarrow{2}$ measures a trip-to-the-east or a trip-to-the-west. This is something which should be decided for each problem in which you want to measure trips made in one of two opposite directions.

In working with numbers like $\overrightarrow{3}$ and $\overleftarrow{9}$, you will want to talk about them as well as write about them. So, you need to decide upon the pronunciation of their numerals. Let's agree to pronounce '$\overrightarrow{3}$' as you would pronounce 'right three' and '$\overleftarrow{9}$' as 'left nine'.

EXERCISES

A. Let us agree that the number $\overrightarrow{2}$ measures a 2-miles-to-the-east trip. Then $\overleftarrow{2}$ measures a 2-mile trip in the opposite direction.



1. Give the real numbers which measure the trips listed.

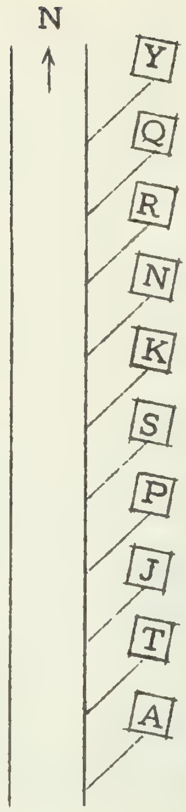
   (a) K to R   (b) R to G   (c) W to O

   (d) B to Q   (e) Q to B   (f) G to S

2. List three trips which are measured by each of the given real numbers.

   (a) $\overrightarrow{8}$ (b) $\overleftarrow{8}$ (c) $\overrightarrow{6}$

   (d) $\overleftarrow{5}$ (e) $\overrightarrow{1}$ (f) $\overrightarrow{2.5}$



B. Consider taking trips along a north-south road. Let $\overrightarrow{2}$ measure a 2-miles-to-the-south trip. Then $\overleftarrow{2}$ measures a 2-mile trip in the opposite direction.

1. Give the real numbers which measure these trips.

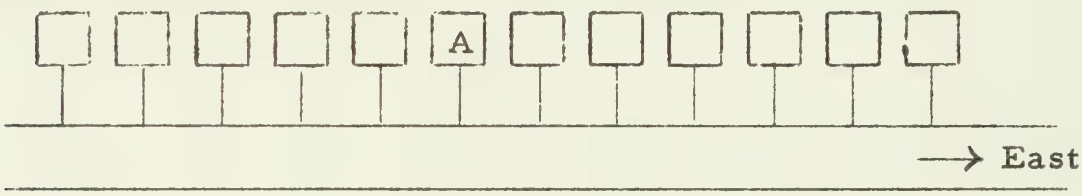
   (a) T to K (b) Y to R

   (c) P to A (d) A to P

2. List three trips which are measured by each of these real numbers.

   (a) $\overrightarrow{3}$ (b) $\overleftarrow{3}$ (c) $\overleftarrow{3.5}$

C. Imagine trips taken along an east-west road, and trips to the east measured by right real numbers. Each of the following statements gives the starting and ending points of a trip and the real number which measures the trip. Use them to fill in the markers on the map.



(1) a trip from A to G is measured by $\overleftarrow{4}$.

(2) a trip from J to G is measured by $\overleftarrow{2}$.

(3) J to B, $\overleftarrow{3}$ (4) I to E, $\overleftarrow{2}$

(5) F to C, $\overleftarrow{2}$ (6) G to D, $\overrightarrow{5}$

(7) L to H, $\overleftarrow{2}$ (8) B to A, $\overrightarrow{5}$

(9) A to F, $\overrightarrow{5}$ (10) E to C, $\overrightarrow{6}$

(11) L to F, $\overrightarrow{1}$ (12) K to E, $\overleftarrow{9}$

* * *

One of the important things you learned in Part C was that if you knew the direction of trips measured by right real numbers, you could then tell the location of a second point with respect to a first point just by knowing the real number which measures the trip from the first point to the second point.

* * *

<u>D</u>. Each of the following exercises is a list of real numbers. These are measures of successive trips along an east-west road. [Right real numbers are used to measure trips-to-the-east.] The first real number in each list measures a trip starting at a point A. The second measures a trip whose starting point is the ending point of the trip from A. The third measures a trip whose starting point is the ending point of the second trip. Etc.

Your job in each exercise is to tell the location of the ending point of the last trip with respect to A by giving the real number which measures the trip from A to that ending point.

<u>Sample 1</u>. $\overrightarrow{3}$, $\overrightarrow{5}$.

<u>Solution</u>. One way to solve this problem is to make a sketch of an east-west road and mark a point A.

Then mark the ending point of the trip measured by $\overrightarrow{3}$. This is a point 3 units to the right of A.

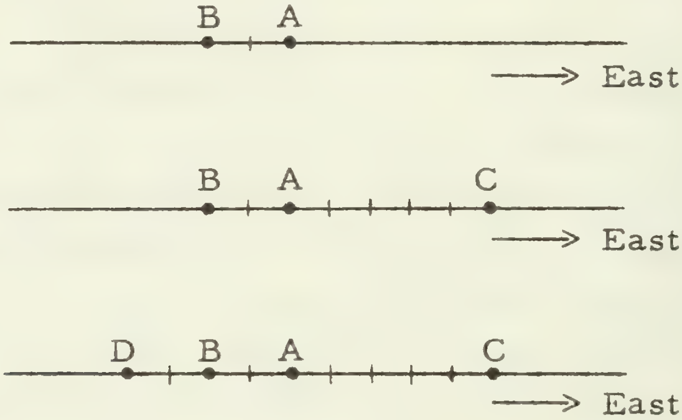
Now mark the ending point of a $\overrightarrow{5}$-unit trip whose starting point is B. This is a point 5 units to the right of B.

So, C is 8 units to the right of A. This means that the ending point of the last trip is 8 units east of A.

The real number which measures the trip from A to the ending point of the last trip in the succession is $\overrightarrow{8}$. <u>Answer</u>: $\overrightarrow{8}$

<u>Sample 2</u>. $\overleftarrow{2}$, $\overrightarrow{7}$, $\overleftarrow{9}$.

<u>Solution</u>.



So, the real number which measures the trip from A to ending point of the last trip in the succession is $\overleftarrow{4}$. <u>Answer</u>: $\overleftarrow{4}$

[<u>Note</u>: Use drawings for doing these exercises as long as you need to. You may be able to find short cuts so that you can give the answer immediately.]

1. $\overrightarrow{2}$, $\overrightarrow{5}$	2. $\overleftarrow{3}$, $\overleftarrow{6}$	3. $\overrightarrow{4}$, $\overleftarrow{3}$
4. $\overrightarrow{6}$, $\overleftarrow{8}$	5. $\overleftarrow{12}$, $\overrightarrow{3}$	6. $\overleftarrow{5}$, $\overrightarrow{8}$
7. $\overrightarrow{3}$, $\overrightarrow{7}$, $\overrightarrow{5}$	8. $\overrightarrow{2}$, $\overrightarrow{9}$, $\overleftarrow{1}$	9. $\overleftarrow{5}$, $\overleftarrow{3}$, $\overleftarrow{10}$
10. $\overrightarrow{5}$, $\overleftarrow{7}$, $\overrightarrow{8}$	11. $\overrightarrow{2}$, $\overleftarrow{9}$, $\overrightarrow{2}$	12. $\overleftarrow{7}$, $\overrightarrow{4}$, $\overrightarrow{8}$
13. $\overrightarrow{2}$, $\overleftarrow{10}$, $\overrightarrow{3}$, $\overleftarrow{5}$	14. $\overleftarrow{6}$, $\overleftarrow{5}$, $\overrightarrow{15}$, $\overleftarrow{20}$	15. $\overrightarrow{3}$, $\overleftarrow{5}$, $\overrightarrow{7}$, $\overleftarrow{21}$
16. $\overrightarrow{78}$, $\overleftarrow{83}$	17. $\overleftarrow{95}$, $\overrightarrow{107}$	18. $\overrightarrow{107}$, $\overleftarrow{95}$
19. $\overleftarrow{50}$, $\overrightarrow{98}$, $\overleftarrow{97}$	20. $\overleftarrow{74}$, $\overrightarrow{10}$, $\overrightarrow{75}$	21. $\overleftarrow{41.5}$, $\overrightarrow{57}$, $\overrightarrow{44.5}$

POSITIVE AND NEGATIVE REAL NUMBERS

In the preceding exercises you worked with real numbers, using them to measure trips. We invented numerals for these numbers. We could continue to use these numerals throughout all of our work with real numbers, but it is important that you become familiar with the more standard ones.

The real numbers which we have called 'right real numbers' and 'left real numbers' are commonly called *positive numbers* and *negative numbers*, respectively. Positive and negative numbers come in pairs. Each pair contains a positive number and a negative number, and both numbers correspond with the same number of arithmetic. The numbers in each pair can be used to measure trips over the *same distance* but in *opposite directions*. The direction of trips measured by positive numbers is called *the positive direction*, and the direction opposite to the positive direction is called *the negative direction*.

In naming positive numbers we shall use a '$^{+}$' instead of the '$\rightarrow$'. For example, we shall write

$$\text{'}\,{}^{+}3\,\text{' instead of '}\,\overrightarrow{3}\,\text{'.}$$

In naming negative numbers, we shall use a '$^{-}$' instead of the '$\leftarrow$'. So, we shall write

$$\text{'}\,{}^{-}3\,\text{' instead of '}\,\overleftarrow{3}\,\text{'.}$$

We form a numeral for a real number by prefixing a '$^{+}$' or a '$^{-}$' to a numeral for the corresponding number of arithmetic. [The numerals '$^{+}3$' and '$^{-}3$' are pronounced as 'positive three' and 'negative three', respectively.]

EXPLORATION EXERCISES

Each of the following exercises gives the measures of successive trips. Find a real number which measures the direct trip from the starting point of the first trip to the ending point of the second trip.

1. $^{+}7$, $^{+}2$
2. $^{-}3$, $^{-}9$
3. $^{+}6$, $^{-}9$
4. $^{-}10$, $^{+}7$
5. $^{-}11$, $^{+}20$
6. $^{+}20$, $^{-}11$

1.02 Addition of real numbers.--In the Exploration Exercises you practiced finding the measure of a trip made from the starting point of the first of two successive trips to the ending point of the second. For example, consider a $^{+}3$-mile trip followed by a $^{+}5$-mile trip. The direct trip which takes you from the starting point of the $^{+}3$-mile trip to the ending point of the $^{+}5$-mile trip is a trip of $^{+}8$ miles. And, similarly, a direct trip which takes you from the starting point of a $^{-}9$-mile trip to the ending point of the $^{-}10$-mile trip which follows it is a trip of $^{-}19$ miles.

The idea of a pair of trips one of which is tacked onto the other suggests the notion of "adding". And, this suggests that what you did with the measures of two such trips to find the measure of the direct trip should be called addition of real numbers. The examples above, then, show that

$$^{+}3 + {}^{+}5 = {}^{+}8$$

and that

$$^{-}9 + {}^{-}10 = {}^{-}19.$$

Take still another example.

$$^{+}9 + {}^{-}5 = \ ?$$

We have decided that the sum of a first real number and a second real number is the measure of a direct trip from the starting point of one trip to the ending point of a following trip, the trips being measured by the first and second real numbers, respectively. So, $^{+}9 + {}^{-}5$ is the measure of a direct trip which takes us from a starting point to that point which would be reached by a $^{+}9$-mile trip followed by a $^{-}5$-mile trip. Do you see that such a direct trip is a $^{+}4$-mile trip? We say that

$$^{+}9 + {}^{-}5 = {}^{+}4.$$

Can you solve the problem:

$$^{-}7 + {}^{+}2 = \ ?$$

EXERCISES

A. Simplify each of the following numerals.

Sample. $^{+}3 + {}^{+}4$

Solution. $^{+}7$

1. $^{+}6 + {}^{-}2$	2. $^{+}3 + {}^{+}8$	3. $^{-}2 + {}^{-}3$
4. $^{+}3 + {}^{-}4$	5. $^{+}3 + {}^{-}5$	6. $^{+}8 + {}^{-}2$
7. $^{+}6 + {}^{+}1$	8. $^{-}2 + {}^{-}8$	9. $^{+}9 + {}^{-}8$
10. $^{+}3 + {}^{+}7$	11. $^{+}11 + {}^{-}7$	12. $^{+}3 + {}^{+}5$
13. $^{+}18 + {}^{-}17$	14. $^{+}18 + {}^{+}17$	15. $^{-}18 + {}^{+}17$
16. $^{-}18 + {}^{-}17$	17. $^{+}4.3 + {}^{+}5.9$	18. $^{-}2.7 + {}^{+}8.3$
19. $^{+}12.4 + {}^{-}19.3$	20. $^{+}81 + {}^{-}102$	21. $^{-}765 + {}^{+}346$

22. $({}^{+}6 + {}^{+}7) + {}^{-}4$ 23. $({}^{-}3 + {}^{+}6) + {}^{+}2$

24. $^{+}4 + {}^{-}3$	25. $^{+}7 + {}^{-}8$	26. $^{+}9 + {}^{-}3$
27. $^{+}5 + {}^{+}10$	28. $^{+}3 + {}^{+}12$	29. $^{+}12 + {}^{+}15$
30. $^{+}2 + {}^{+}5$	31. $^{-}4 + {}^{-}4$	32. $^{-}3 + {}^{+}7$
33. $^{-}7 + {}^{+}3$	34. $^{+}2.5 + {}^{+}3.5$	35. $^{-}1 + {}^{+}2$
36. $\frac{^{+}9}{3} + \frac{^{+}6}{3}$	37. $\frac{^{+}3}{5} + \frac{^{-}2}{5}$	38. $\frac{^{+}1}{3} + \frac{^{+}1}{2}$
39. $^{-}3 + {}^{-}5$	40. $^{+}1 + {}^{+}3$	41. $^{+}10 + {}^{-}12$
42. $^{+}2 + {}^{-}1$	43. $^{-}2 + {}^{+}1$	44. $^{+}7 + {}^{-}3$
45. $^{+}3 + {}^{-}7$	46. $^{-}6 + {}^{+}7$	47. $^{+}8 + {}^{-}9$
48. $^{+}21 + {}^{-}15$	49. $^{-}12 + {}^{+}13$	50. $^{-}32 + {}^{-}42$
51. $^{+}17 + {}^{+}19$	52. $^{-}181 + {}^{+}75$	53. $^{+}181 + {}^{-}75$
54. $\frac{^{+}1}{5} + \frac{^{-}3}{5}$	55. $\frac{^{-}2}{7} + \frac{^{-}3}{7}$	56. $\frac{^{-}1}{4} + \frac{^{+}1}{2}$

(continued on next page)

57. $^{-}1\frac{1}{2} + {}^{-}3\frac{1}{4}$ 58. $^{+}4.6 + {}^{-}3.2$ 59. $^{-}9\frac{1}{6} + {}^{+}3\frac{1}{3}$

60. $^{+}3875 + {}^{-}2431$ 61. $^{-}2431 + {}^{+}3875$ 62. $({}^{+}4 + {}^{-}3) + {}^{+}5$

63. $({}^{-}15 + {}^{+}3) + {}^{+}7$ 64. $({}^{-}15 + {}^{+}28) + {}^{+}2$

65. $({}^{-}29 + {}^{-}98) + {}^{-}2$ 66. $({}^{+}87 + {}^{+}64) + {}^{+}36$

67. $({}^{-}997 + {}^{-}482) + {}^{-}3$ 68. $({}^{+}3 + {}^{-}7) + ({}^{+}9 + {}^{-}2)$

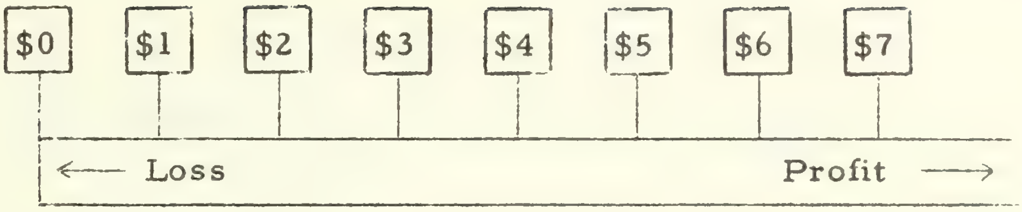
[More exercises are in Part B, Supplementary Exercises.]

B. Bill's father gave him \$3 to start and operate a flower business for one week. His father told him to use the \$3 to buy flowers the first day and to sell as much as he could each day. He also told him to use all of the money he collected on one day to buy flowers the next morning. Although spending all of his money each morning might not be the best business procedure, his father wanted to see how far up he could "run" the \$3.

Here is a record of his week's business.

	Expenses	Sales	Outcome
Monday	3.00	4.00	
Tuesday	4.00	5.20	
Wednesday	5.20	4.80	
Thursday	4.80	4.90	
Friday	4.90	4.70	
Saturday	4.70	6.80	

On Monday, Bill's assets changed by \$1, and this change was an increase in his assets. That is, the outcome of business on Monday was a profit of \$1. The outcome of business on Friday was a loss of \$0.20.



The outcome of a day's business for Bill can be regarded as a trip, for it involves "distance" [difference in assets between the opening and closing of the business day] and "direction" [increase or decrease in assets]. So, the outcome of a day's business can be measured by a real number. If we decide that a profit is to be measured by a positive number then the outcome of Monday's business is $\$^{+}1.00$, and the outcome of Friday's business is $\$^{-}0.20$. In dollars, the measures of the two outcomes are $^{+}1.00$ and $^{-}0.20$.

1. Complete the table given on page 1-10 by listing the [measure of the] outcome of each day's business.

2. Bill started with \$3.00 on Monday morning and ran it up to \$6.80 by Saturday evening. <u>Predict</u> the sum of the outcomes for the six days of business, and check your prediction by actually adding the real numbers listed in the outcome column.

<u>C</u>. Solve the following problems. Use real numbers wherever they apply.

1. Phil made 3 dollars profit the first day of business, lost 6 dollars the second day, and made 5 dollars profit the third day. What was the outcome of the total business for the three days?

2. Ed made \$6.80 profit the first day of business, made \$2.55 profit the second day, lost \$5.42 the third day, and made \$1.53 the fourth day. What was the outcome of the four days of business?

(continued on next page)

3. Zabranchburg High's football team gained 3 yards the first down, lost 4 yards the second down, gained 5 yards the third down, and gained 7 yards the fourth down. Did they make a first down?

4. John and Fred are playing a game. John wins 3 points in the first round, loses 4 points in the second round, and wins 5 points in the third round. What is his score at the end of the third round?

☆5. A department store has 6 floors above the ground floor and 2 floors below the ground floor. The first floor above the ground floor is called 'mezzanine', the next floor above the mezzanine is called 'first floor', the floor next above the first floor is called 'second floor', etc. The first floor below the ground floor is called 'first basement' and the floor below that one is called 'second basement'. An operator makes the following trip: ground floor to mezzanine to second floor to first floor to third floor to fourth floor to mezzanine to first basement to third floor to second basement to first floor.

(a) If you use real numbers to measure the separate trips, what is the sum of these real numbers? Could you have predicted your answer without adding?

(b) If the floors are 17 feet apart, how many feet did the operator travel during the entire trip?

(c) Which traveled the greater distance, the operator's head or his feet?

☆6. Two cyclists start from the same home at the same time. John travels 4 miles east, then 2 miles west, then 3 miles east. He then travels west until he meets Walt. Walt starts by traveling 3 miles west, then 1 mile east, then 3 miles west, then east until he meets John.

(a) If both cyclists travel at the same speed, how far from home do they meet?

(b) In which direction must they travel if they head directly for home together?

(c) How many miles has each cyclist traveled by the time they reach home?

[More exercises are in Part C, Supplementary Exercises.]

TRIPS OF DISTANCE 0

The problems in adding real numbers up to now have not included such problems as finding the sum of $^{+}3$ and $^{-}3$, or finding the sum of $^{+}7$ and $^{-}7$. Just as it is possible to find the sum of any pair of numbers of arithmetic, we would like it to be the case that there is a sum for every pair of real numbers. Our problem, then, is to give an interpretation of, say, '$^{+}3 + ^{-}3$' so that '$^{+}3 + ^{-}3$' is a numeral for a real number. Let us try our trip interpretation. Using the picture on page 1-1, '$^{+}3 + ^{-}3$' should be a numeral for a real number that measures the trip whose starting point is that of the trip, say, from A to M and which has the same ending point as the trip from M to A. In other words, $^{+}3 + ^{-}3$ should measure the trip from A to A. This is an unusual trip! For even though it involves the distance aspect, it does not involve a direction. In fact, we wonder if we can even consider this as a trip at all. If we don't consider it as a trip, and if we wish to continue thinking of real numbers as numbers which measure trips, then we must admit that collections of marks such as

'$^{+}7 + ^{-}7$' and '$^{+}3 + ^{-}3$' and '$^{-}4 + ^{+}4$'

are not numerals for real numbers. In other words, we would have to admit that there are pairs of real numbers which do not have sums. Rather than do this, we prefer to stretch our imaginations a bit and regard a "trip from A to A" as a trip which can be measured by a real number. Naturally, the real number to be used in measuring such a trip is neither a positive number nor a negative number. So, it must be one which we have not yet discussed.

A simple numeral which we shall use for this real number is a numeral which is also used for a number of arithmetic. It is the numeral '0'. Thus, each of the numerals '$^{+}7 + {}^{-}7$', '$^{+}3 + {}^{-}3$', and '$^{-}4 + {}^{+}4$' can be simplified to '0'. Each of these numerals names the <u>real</u> <u>number</u> 0.

Now, just as we do not confuse the positive and negative real numbers with numbers of arithmetic [we don't think, for example, that the real number $^{-}3$ is the same as the number 3 of arithmetic], so we must not confuse the real number 0 with the number 0 of arithmetic. It may seem difficult to avoid this confusion since both numbers have the same name, and you may wonder why we did not invent a new numeral for this real number. We could have used '$\ominus$' or 'ϕ' or '$\otimes$' or even 'ξ', but everyone else uses '0', so we shall also. Actually, it will not be too hard to keep the idea of the number 0 of arithmetic separate from the idea of the real number 0, since any problem in which either number is to be used will tell you which meaning to give to the numeral '0'. For example, the real number 0 measures the outcome of a day's business in which expenses were the same as sales, whereas the number 0 of arithmetic measures the content of the money-box when the box is empty.

EXERCISES

<u>A.</u> Simplify.

1. $^{+}9 + {}^{-}9$
2. $^{-}9 + {}^{+}9$
3. $^{+}18 + {}^{-}18$
4. $^{+}3 + 0$
5. $0 + {}^{+}3$
6. $0 + {}^{-}3$
7. $^{-}3 + 0$
8. $^{+}20 + {}^{+}30$
9. $^{+}30 + {}^{-}50$
10. $(^{+}9 + {}^{-}7) + {}^{+}7$
11. $(^{+}19 + {}^{+}12) + {}^{-}12$
12. $(^{-}843 + {}^{+}726) + {}^{-}726$
13. $(^{+}487 + {}^{-}851) + {}^{-}487$

[More exercises are in <u>Part</u> <u>D</u>, <u>Supplementary</u> <u>Exercises</u>.]

B. Fill in the blanks to make true sentences.

1. $^+8 + \_\_\_\_\_ = 0$
2. $^-3 + \_\_\_\_\_ = ^-3$
3. $^-3 + \_\_\_\_\_ = ^+6$
4. $^-3 + \_\_\_\_\_ = ^-6$
5. $0 + \_\_\_\_\_ = 0$
6. $\_\_\_\_\_ + ^+2 = ^+17$
7. $^-3 + \_\_\_\_\_ = ^+3$
8. $\_\_\_\_\_ + ^+71 = ^-100$
9. $\_\_\_\_\_ + ^+1 = ^-1$
10. $\_\_\_\_\_ + ^+3 = ^+13$
11. $\_\_\_\_\_ + ^+52 = ^+52 + ^-31$
12. $^+87 + \_\_\_\_\_ = \_\_\_\_\_ + ^+87$
13. $^+7 + ^+5 = ^+(7 + \_\_)$
14. $^-2 + ^-11 = ^-(\_\_ + 11)$
15. $^+8 + \_\_\_\_\_ = ^+(8 - 3)$
16. $^-(15 - 2) = ^-15 + ^+\_\_\_\_\_$
17. $^+4 + ^-7 = ^-(7 - \_\_)$
18. $^-(6 - 3) = ^+3 + \_\_\_\_\_$
19. $^-(8 - 4) = \_\_\_\_\_ + ^-4$
20. $^-[(5 + 2) - 3] = (^-5 + ^-2) + \_\_\_\_\_$
21. $(^+351 + \_\_) + ^-284 = ^+351 + (^-937 + ^-284)$

C. You have learned about three kinds of real numbers--positive real numbers, negative real numbers, and 0. Each real number is either positive, negative, or 0. And no real number is of two of these kinds. So, the real numbers can be classified into three subsets:

the set consisting of the positive real numbers,
the set consisting of the negative real numbers, and
the set consisting of the real number 0.

1. Another way of classifying the real numbers is to note that each real number is either a negative real number or a nonnegative real number. Describe the set of nonnegative real numbers.

2. Describe the set of nonpositive real numbers.

3. Does the set of nonnegative real numbers have any numbers in common with the set of nonpositive real numbers?

EXPLORATION EXERCISES

A. Suppose that a pump fills a tank with water at a rate of 3 gallons per minute. What will be the increase (gallons) in the volume of water in the tank

1. 1 minute from now?
2. 4 minutes from now?
3. $10\frac{1}{2}$ minutes from now?
4. 0 minutes from now?

B. Suppose that the tank is full and the pump empties the tank at a rate of 4 gallons per minute. What will be the decrease in the volume of water in the tank

1. 1 minute from now?
2. 4 minutes from now?
3. $10\frac{1}{2}$ minutes from now?
4. 0 minutes from now?

C. Suppose the pump fills the tank at a rate of 5 gallons per minute. How many fewer gallons of water were there in the tank

1. 1 minute ago?
2. 4 minutes ago?
3. $10\frac{1}{2}$ minutes ago?
4. 0 minutes ago?

D. Suppose a full tank is emptied by a pump at a rate of 3 gallons per minute. How many more gallons of water were there in the tank

1. 1 minute ago?
2. 4 minutes ago?
3. $10\frac{1}{2}$ minutes ago?
4. 0 minutes ago?

1.03 <u>Multiplication of real numbers.</u> --In telling what we mean by addition of real numbers, we gave, in terms of trips, interpretations of numerals such as

'$^{+}4 + ^{-}7$', '$^{-}8 + ^{+}9$', '$^{+}3 + ^{+}10$', and '$^{-}5 + ^{-}11$'.

In order to explain multiplication of real numbers, we shall give an interpretation of numerals such as

'$^{+}4 \times ^{+}3$', '$^{-}8 \times ^{+}7$', '$^{-}2 \times ^{-}5$', and '$^{+}4 \times ^{-}2$'.

This interpretation should help us find the product of each pair of real numbers.

We know that real numbers are numbers which can be used to measure trips. What are the characteristics of trips which make this possible? A trip involves

(1) a change in position by a certain amount,
and (2) a change in position in one of two opposite directions.

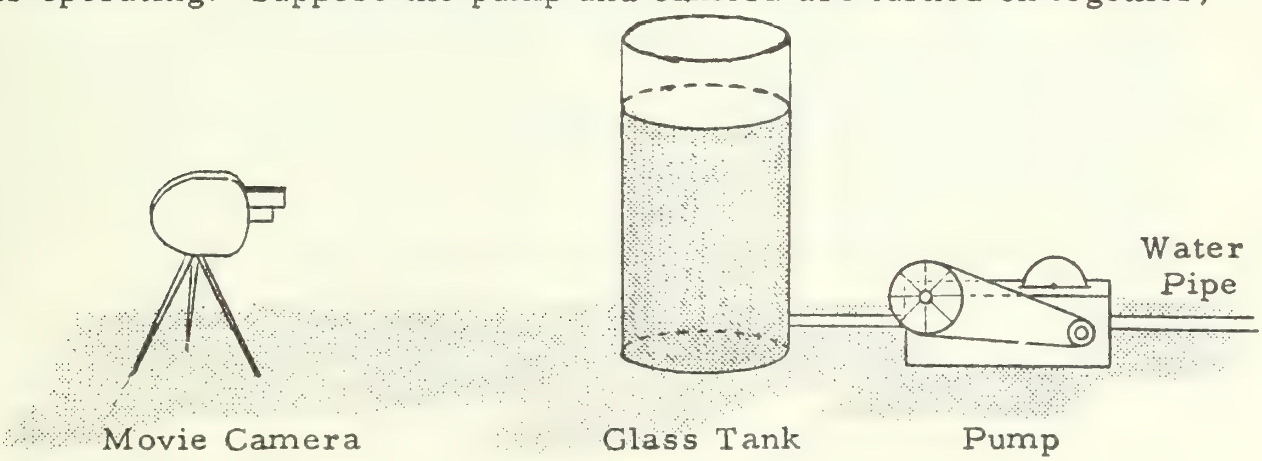
In general, anything which involves an amount and one of two opposite directions can be measured by a real number. So, in looking for an interpretation of a numeral such as

'$^{-}4 \times ^{+}3$',

we look for something which involves both an amount and a direction so that it can be measured by a real number.

A PUMP, A TANK, AND A MOVIE

Think of a pump which can pump water either into or out of a tank, and a camera which takes a movie of the tank while the pump is operating. Suppose the pump and camera are turned on together,



and that 4 gallons of water flow through the pipe each minute. After the pump and camera have run for 3 minutes, they are stopped. The film is then developed and projected on a screen. What change in the water-volume do you observe on the screen? It is easy to predict that the change <u>observed on the screen</u> will be a change of 12 gallons. But, will it be an increase? In order to answer this question, you need to know two more things.

One of the things you need to know is whether the flow of water was into the tank or out of the tank. Suppose that the flow was into the tank. Will the picture you see on the screen show an increase in water-volume? If your answer is 'yes' then you are probably assuming that the film is being run <u>forward</u> through the projector. But suppose the film were run <u>backward</u> through the projector. [Have you ever watched a comedy film in which a man seems to dive up out of the water and land on a diving board, or a film of a race horse running backward on a muddy track, picking up its footprints as it goes?] If the film <u>were</u> run backward, what change in water-volume would you see on the screen?

Now, suppose the water was being pumped out of the tank while the picture was being taken. If the film were run forward through the projector, what change in water-volume would you observe on the screen? If the film were run backward, what change would you see on the screen?

So, in order to predict what change you will observe on the screen, you need to know

(a) the amount of water per minute being pumped into or out of the tank, and

(b) the number of minutes the film is being run forward or backward.

Each of these things involves an amount and a direction, and therefore can be measured by real numbers. We can use real numbers to measure the rate at which the water is being pumped,

deciding to use

positive numbers when water flows into the tank,

and

negative numbers when water flows out of the tank.

Thus, if 4 gallons of water are being pumped into the tank each minute, we say that the rate is $^{+}4$ gallons per minute. Explain what is meant by saying that the rate is $^{-}4$ gallons per minute.

Also, we can use real numbers to measure how long the film is being projected, deciding to use

positive numbers when the film is run forward,

and

negative numbers when the film is run backward.

So, if the film is running for $^{+}3$ minutes, we know it is being projected forward (normally) for 3 minutes. Explain what is meant by saying that the film is running for $^{-}3$ minutes.

Now, how does all of this help us in interpreting a numeral such as

'$^{-}4 \times {}^{+}3$'?

We can think of $^{-}4$ as measuring the rate at which water is being pumped. We can think of $^{+}3$ as telling us how long the film is being projected. And, finally, we can think of the product

$$^{-}4 \times {}^{+}3$$

as measuring the change in water-volume which we see on the screen. [Let's agree to use positive numbers to measure observed increases, and negative numbers to measure observed decreases.] In this case, since water is being pumped out of the tank at 4 gallons per minute and the film is being run forward for 3 minutes, the change in water-volume observed on the screen is a decrease of 12 gallons. So, the change is $^{-}12$ gallons. Since we agreed to think of the product as a measure of the change, we can say that

$$^{-}4 \times {}^{+}3 = {}^{-}12.$$

Let's take another case:

$$^{+}9 \times {}^{-}8 = ?$$

This number, $^{+}9 \times {}^{-}8$, measures the change in water-volume which you would observe on a screen. The first number, $^{+}9$, measures the rate of pumping [is it filling or is it emptying?], and the second number, $^{-}8$, tells how long the film is being projected [is it being run forward or is it being run backward?]. Does $^{+}9 \times {}^{-}8$ measure an observed increase in water-volume or an observed decrease? Since a backward projection of the movie of a tank being filled shows a decrease in water-volume, we can say that

$$^{+}9 \times {}^{-}8 = {}^{-}72.$$

EXERCISES

A. The table below contains problems dealing with the pump-tank-movie interpretation. From each problem you can learn how to multiply a pair of real numbers. We have solved the first problem for you as a sample.

In this problem you are told that a pump is filling the tank at the rate of 4 gallons per minute. Therefore, a '$^{+}4$' is written in the column headed 'Pump'. You learn from the second column that the movie has been run backward for 2 minutes. Therefore, you write a '$^{-}2$' in this column. Now, we ask about the change in water-volume that would be observed on the screen. Since the pump is filling the tank (as indicated by the '$^{+}4$'), and since the film is run backward (as indicated by the '$^{-}2$'), the volume of water appears to be decreasing. So, we observe on the screen a decrease in volume of 8 gallons. The number $^{-}8$ measures this observed change. Finally, we write the corresponding multiplication statement in the last row.

Complete the table.

	Pump	Movie	Observed Change in Volume
1.	Filling 4 gal. per minute	Running backward 2 minutes	decrease of 8 gallons
	$^{+}4$	$^{-}2$	$^{-}8$
	Corresponding multiplication statement:		$^{+}4 \times {}^{-}2 = {}^{-}8$
2.	Emptying 4 gal. per minute	Running forward 2 minutes	
	Corresponding multiplication statement:		
3.	Filling 4 gal. per minute	Running forward 2 minutes	
	Corresponding multiplication statement:		
4.	Emptying 4 gal. per minute	Running backward 2 minutes	
	Corresponding multiplication statement:		
5.	Filling 8 gal. per minute	Running forward 3 minutes	
	Corresponding multiplication statement:		
6.	Emptying 8 gal. per minute	Running forward 3 minutes	
	Corresponding multiplication statement:		
7.	Emptying 8 gal. per minute	Running backward 3 minutes	
	Corresponding multiplication statement:		

[Note: In the rest of the problems you are given real numbers and you should fill in the corresponding blanks.]

	Pump	Movie	Observed Change in Volume
8.	$^-5$	$^-6$	
	Corresponding multiplication statement: $^-5 \times {}^-6 =$		
9.	$^+7$	$^-3$	
	Corresponding multiplication statement: $^+7 \times {}^-3 =$		
10.	$^-8$	0	
	Corresponding multiplication statement: $^-8 \times 0 =$		
11.	$^-6\frac{1}{2}$	$^-4$	
	Corresponding multiplication statement: $^-6\frac{1}{2} \times {}^-4 =$		

B. Simplify. Use the pump-tank-film interpretation as long as you need to, but try to find a short cut.

1. $^+5 \times {}^+2$	2. $^+6 \times {}^+3$	3. $^+8\frac{1}{2} \times {}^+8$
4. $\frac{^+28}{3} \times {}^+6$	5. $^+6 \times {}^-2$	6. $^-2 \times {}^+6$
7. $^-5 \times {}^+7$	8. $^+8 \times {}^-8$	9. $^-9 \times {}^+10$
10. $^+12 \times {}^-10$	11. $^-7 \times {}^-8$	12. $^-15 \times {}^-3$
13. $^-1 \times {}^-1$	14. $^-8 \times {}^-12$	15. $^+7 \times 0$

16. $0 \times {}^{-}6$

17. 0×0

18. ${}^{-}12 \times 0$

19. ${}^{-}16 \times \frac{{}^{-}1}{4}$

20. ${}^{-}100 \times \frac{{}^{+}5}{2}$

21. ${}^{+}5 \times \frac{{}^{-}3}{10}$

22. ${}^{+}3 \times {}^{-}15$

23. ${}^{-}7 \times {}^{+}12$

24. ${}^{-}3 \times {}^{-}8$

25. ${}^{+}6 \times {}^{-}4$

26. ${}^{-}3 \times {}^{-}2$

27. ${}^{-}17 \times {}^{+}2$

28. ${}^{+}47 \times {}^{-}58$

29. ${}^{+}27 \times {}^{-}65$

30. ${}^{+}705 \times {}^{+}15$

31. ${}^{-}86 \times {}^{-}75$

32. ${}^{-}1.83 \times {}^{-}1.81$

33. ${}^{+}9.65 \times {}^{-}7.48$

34. $({}^{+}2 \times {}^{-}3) \times {}^{-}4$

35. $({}^{-}2 \times {}^{+}7) \times {}^{-}3$

36. $({}^{+}5 \times {}^{-}3) \times {}^{+}4$

37. $({}^{+}6 \times {}^{-}2) \times {}^{-}3$

38. ${}^{+}4 \times ({}^{-}3 \times {}^{-}7)$

39. ${}^{-}6 \times ({}^{+}2 \times {}^{-}5)$

40. $({}^{+}73 \times {}^{-}81) \times 0$

41. $({}^{+}5 \times {}^{-}17) \times ({}^{-}3 \times 0)$

[More exercises are in Part E, Supplementary Exercises.]

C. Simplify. [Be careful not to confuse addition signs with multiplication signs.]

1. $({}^{+}5 + {}^{-}3) \times {}^{-}7$

2. $({}^{+}3 \times {}^{-}4) + {}^{-}6$

3. $({}^{+}8 \times {}^{-}3) + {}^{-}5$

4. $({}^{+}12 + {}^{-}11) \times {}^{-}3$

5. $({}^{+}1 + {}^{+}1) + {}^{+}1$

6. $({}^{+}1 \times {}^{+}1) + {}^{+}1$

7. $({}^{-}1 + {}^{-}1) + {}^{-}1$

8. $({}^{-}1 \times {}^{-}1) \times {}^{-}1$

9. $({}^{-}4 \times {}^{-}2) + ({}^{-}5 \times {}^{+}6)$

10. $({}^{-}3 \times {}^{+}7) + ({}^{+}8 \times {}^{-}4)$

11. $({}^{+}71 + {}^{-}11) \times ({}^{+}6 + {}^{-}4)$

12. $({}^{-}8 + 0) \times ({}^{+}4 \times {}^{+}2)$

D. Fill in the blanks to make true sentences.

1. ${}^{+}5 \times$ _____ $= {}^{-}20$

2. ${}^{-}3 \times$ _____ $= {}^{+}18$

3. ${}^{+}7 \times$ _____ $= {}^{+}21$

4. ${}^{-}7 \times$ _____ $= {}^{-}21$

5. _____ $+ {}^{-}3 = {}^{+}9$

6. _____ $\times {}^{-}3 = {}^{+}9$

7. ${}^{+}8 +$ _____ $= {}^{-}8$

8. ${}^{+}8 \times$ _____ $= {}^{-}8$

9. ${}^{+}3 \times$ _____ $= {}^{+}1$

10. ${}^{+}3 +$ _____ $= 0$

11. ${}^{+}3 \times$ _____ $= 0$

12. ${}^{+}3 +$ _____ $= {}^{+}1$

13. ${}^{+}8 \times {}^{-}2 = {}^{-}(8 \times$ ____ $)$

14. ${}^{-}5 \times$ _____ $= {}^{+}(5 \times 2)$

EXPLORATION EXERCISES

A. Consider the table of pairs of numbers at the right. One of the interesting features of this table is that you can carry out some computations with the numbers listed in one column by doing computations with the corresponding numbers listed in the other column. For example, suppose you want to find the sum of, say, 267 and 445, two numbers listed in the lefthand column. To simplify

'267 + 445'

merely simplify

'3 + 5'.

3 and 5 correspond with 267 and 445, respectively. '3 + 5' simplifies to '8', and 8 corresponds with 712. And we find that

267 + 445 = 712.

Here is another example.

$$\begin{array}{r} 534 \\ +\ 801 \\ \hline ? \end{array} \quad \cdots\cdots \quad \begin{array}{r} 6 \\ +\ 9 \\ \hline 15 \end{array}$$

89 . . .	1
178 . . .	2
267 . . .	3
356 . . .	4
445 . . .	5
534 . . .	6
623 . . .	7
712 . . .	8
801 . . .	9
890 . . .	10
979 . . .	11
1068 . . .	12
1157 . . .	13
1246 . . .	14
1335 . . .	15
1424 . . .	16
1513 . . .	17
1602 . . .	18
1691 . . .	19
1780 . . .	20

15 corresponds with 1335, and it turns out that 534 + 801 = 1335.

Use the table and the illustrated procedure to simplify each of the following. Check your results by carrying out the simplification directly.

1. 356 + 534 2. 712 + 979 3. 1068 + 267

4. 445 + 1157 5. 1424 - 1068 6. 1157 - 178

*

7. Is it possible to multiply pairs of numbers listed in the lefthand column by multiplying the corresponding numbers listed in the righthand column? Try simplifying '178 × 534' that way.

B. Here is another table in which you can add some pairs of numbers listed in one column by adding the corresponding numbers of the other column. For example:

$$\begin{array}{r} .027432 \\ +\ .041148 \\ \hline ? \end{array} \quad \ldots\ldots \quad \begin{array}{r} 32\frac{2}{3} \\ +\ 49 \\ \hline 81\frac{2}{3} \end{array}$$

$81\frac{2}{3}$ corresponds with .068580. So, .068580 is the sum of .027432 and .041148. [Check this by adding.]

.030861	$36\frac{3}{4}$
.013716	$16\frac{1}{3}$
.041148	49
.020574	$24\frac{1}{2}$
.006858	$8\frac{1}{6}$
.024003	$28\frac{7}{12}$
.037719	$44\frac{11}{12}$
.027432	$32\frac{2}{3}$
.034290	$40\frac{5}{6}$
.068580	$81\frac{2}{3}$

Use the table to simplify each of the following. Check your results by direct simplification. [Note that Exercise 2 requires that you work with numbers listed in the lefthand column.]

1. .013716 + .020574
2. $8\frac{1}{6} + 16\frac{1}{3}$
3. $16\frac{1}{3} + 24\frac{1}{2}$
4. .020574 + .020574
5. $40\frac{5}{6} + 40\frac{5}{6}$
6. .030861 + .037719

C. Here is a table which can be used to find products of some pairs of numbers listed in one column by computing products of the corresponding numbers of the other column. For example:

$$\begin{array}{rcr} & .5 \;\ldots\ldots & 2 \\ \times & .25 \;\ldots\ldots & \times\; 4 \\ \hline & ? & 8 \end{array}$$

8 corresponds with .125, and .125 is the product of .5 and .25.

.5	....	2
.25	....	4
.2	....	5
.125	....	8
.1	....	10
.0625	....	16
.05	....	20
.04	....	25
.03125	....	32
.025	....	40

Use the table to simplify each of the following, and check by direct simplification.

1. $.5 \times .2$ 2. 8×5 3. $.0625 \times .5$

4. 8×4 5. $.125 \times .25$ 6. $.2 \times .2$

*

7. Can you use this table for addition?

D. Here is another table which can be used for multiplication.

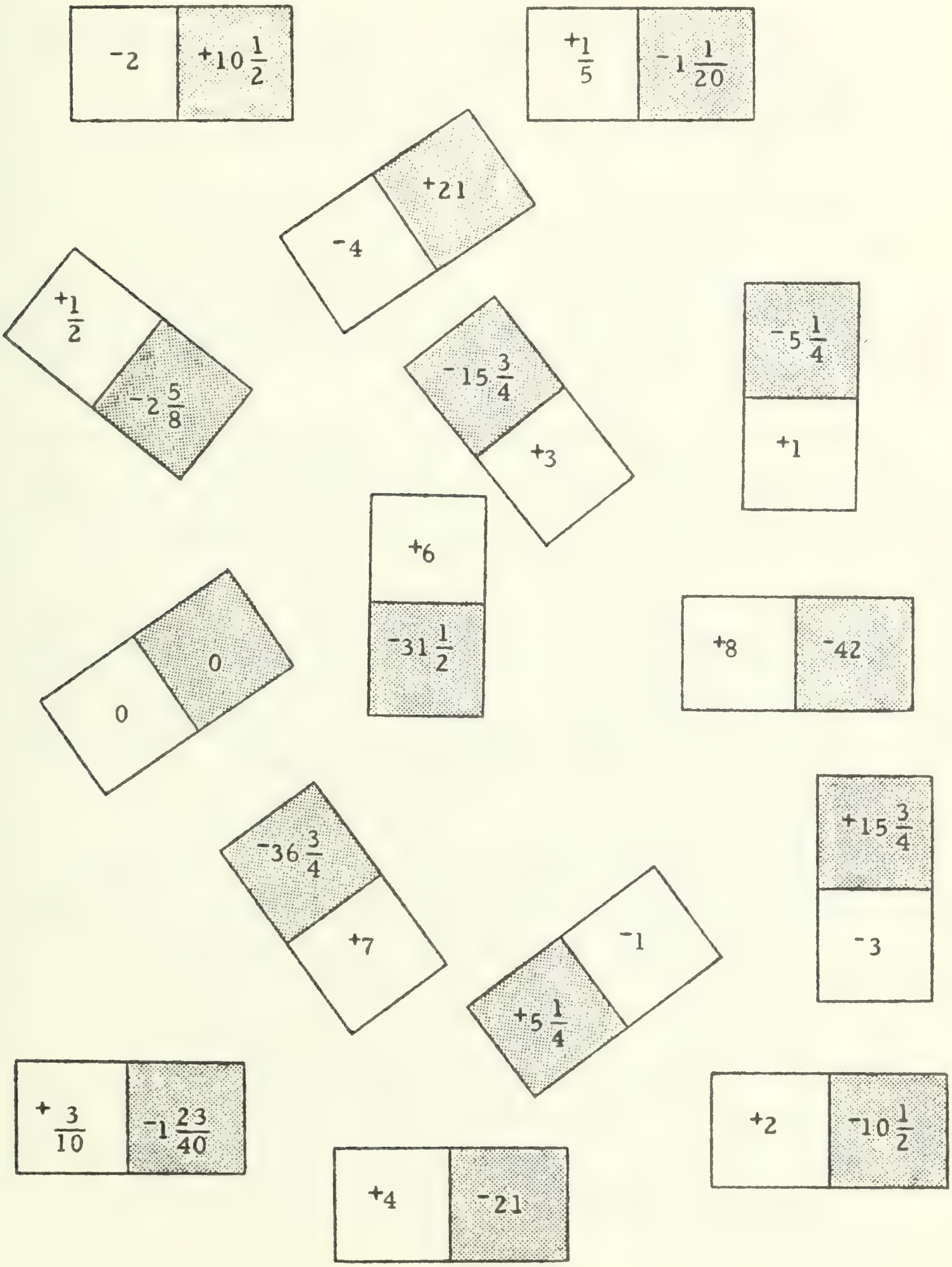
$^+2$	$^+5$	$^+\frac{1}{3}$	$^+\frac{1}{5}$	$^+\frac{1}{9}$	$^+\frac{2}{3}$	$^+1$	$^+\frac{5}{9}$	$^+\frac{10}{27}$	$^+\frac{20}{81}$
$\frac{1}{2}$	$\frac{1}{5}$	3	5	9	$1\frac{1}{2}$	1	$1\frac{4}{5}$	2.7	4.05

Use the table to find the products, and check by direct simplification.

1. $3 \times \frac{1}{2}$ 2. $^+\frac{1}{3} \times {^+2}$ 3. $9 \times \frac{1}{5}$

4. $^+\frac{1}{9} \times {^+5}$ 5. $1\frac{1}{2} \times 2.7$ 6. $^+\frac{2}{3} \times {^+\frac{10}{27}}$

☆<u>E</u>. Here is a table which can be used just as you used the other tables. Figure out how to use it, and see if it works for multiplication or for addition.



F. For each table, see if it can be used for finding sums or products of some pairs of numbers listed in one row by computing sums or products of the corresponding numbers listed in the other row.

1.

4	9	16	25	36	64	100	144	225	324	400	576
$^+2$	$^+3$	$^+4$	$^+5$	$^+6$	$^+8$	$^+10$	$^+12$	$^+15$	$^+18$	$^+20$	$^+24$

2.

$^-3$	$^-2$	$^-1$	0	$^+\frac{1}{4}$	$^+\frac{1}{2}$	$^+\frac{3}{4}$	$^+1$	$^+2$	$^+3$	$^+4$	$^+6$	$^+8$
$^+9$	$^+6$	$^+3$	0	$^-\frac{3}{4}$	$^-1\frac{1}{2}$	$^-2\frac{1}{4}$	$^-3$	$^-6$	$^-9$	$^-12$	$^-18$	$^-24$

3.

0	1	2	3	4	5	6	7	8	9	12	18	35	54
0	1	2	3	4	5	6	7	8	9	12	18	35	54

4.

1	2	4	8	16	32	64	128	256	1024	2048	4096
0	1	2	3	4	5	6	7	8	10	11	12

5.

0	$^-1$	$^-2$	$^-3$	$^-4$	$^-5$	$^-6$	$^-7$	$^-8$	$^-10$	$^-12$	$^-13$	$^-20$	$^-42$
0	1	2	3	4	5	6	7	8	10	12	13	20	42

6.

0	$^+1$	$^+2$	$^+3$	$^+4$	$^+5$	$^+6$	$^+7$	$^+8$	$^+10$	$^+12$	$^+13$	$^+20$	$^+42$
0	1	2	3	4	5	6	7	8	10	12	13	20	42

1.04 <u>Numbers of arithmetic and real numbers</u>.--Think carefully about what you do when you compute the sum of a pair of nonnegative or a pair of nonpositive real numbers. For example, what do you think about when you do the following problems?

$$^{+}7 + {}^{+}12 = ? \qquad \text{and} \qquad {}^{-}7 + {}^{-}12 = ?$$

Most likely, you first simplify:

$$7 + 12$$

and get:

$$19,$$

and then you say that

$$^{+}7 + {}^{+}12 = {}^{+}19 \qquad \text{and} \qquad {}^{-}7 + {}^{-}12 = {}^{-}19.$$

When you do this, you no doubt imagine a table which links up each nonpositive number and each nonnegative number with a number of arithmetic.

0	0	0
$^{-}\frac{1}{2}$	$\frac{1}{2}$	$^{+}\frac{1}{2}$
$^{-}1$	1	$^{+}1$
$^{-}1\frac{1}{2}$	$1\frac{1}{2}$	$^{+}1\frac{1}{2}$
$^{-}2$	2	$^{+}2$
$^{-}3$	3	$^{+}3$
$^{-}4$	4	$^{+}4$
$^{-}5$	5	$^{+}5$
$^{-}6$	6	$^{+}6$
.	.	.
.	.	.
.	.	.

In this table, each real number corresponds with that number of arithmetic which gives the distance part of a trip measured by the real number. Thus, in order to find sums of pairs of nonpositive or nonnegative real numbers, you should begin by finding the arithmetic numbers which correspond with the real numbers, and then add the arithmetic numbers. We can summarize these remarks by saying that

> the nonpositive real numbers and the nonnegative real numbers <u>act like</u> the numbers of arithmetic with respect to addition.

You probably discovered this idea when you were looking for short cuts in computing sums.

What about multiplication? Do the nonpositive real numbers act like the corresponding numbers of arithmetic with respect to multiplication? A quick check of the table above shows that they do not.

$$\begin{array}{rcr} ^-2 & \ldots\ldots\ldots & 2 \\ \times\ ^-3 & \ldots\ldots\ldots & \times\ 3 \\ \hline ? & & 6 \end{array}$$

6 corresponds with $^-6$ but $^-2 \times {}^-3 \neq {}^-6$.

Do the nonnegative real numbers act like the corresponding numbers of arithmetic with respect to multiplication? The answer is 'yes'. So,

> the nonnegative real numbers <u>act like</u> the numbers of arithmetic with respect to <u>both</u> addition and multiplication.

SHORTER NAMES FOR POSITIVE NUMBERS

In view of the fact that the nonnegative real numbers act like the numbers of arithmetic with respect to both addition and multiplication, it will cause no trouble if we use the names of numbers of arithmetic as names of the nonnegative real numbers. When we want to state a fact about real numbers, for example, that

(1) $^{+}9 + {}^{-}3 = {}^{+}6$,

we can just write:

(2) $9 + {}^{-}3 = 6$.

Anyone who looks at sentence (2) and believes that it should make sense must conclude that '9' and '6' are numerals for positive numbers rather than numerals for numbers of arithmetic. [He would conclude this because it wouldn't make sense to add a negative real number to a number of arithmetic.] So, when you look at (2) you "see" it as (1). And writing (2) instead of (1) saves you the trouble of writing the little plus signs.

Consider another example.

(3) $7 \times 4 = 28$.

Is this a statement about numbers of arithmetic or is it a statement about real numbers? Unless you know the problem which led to someone's writing (3), you are free to interpret it either way.

Numerals such as those in sentence (3) which name more than one number are said to be ambiguous. We have already seen an ambiguous numeral in the case of '0'. '0' names the number 0 of arithmetic and it names the real number 0. We are now preparing to deal with many more cases of ambiguous numerals.

Ambiguous words or names may cause confusion. For example, suppose there are two students in your class each having the name 'Ann Brown'. If this message is sent from the principal's office:

> Ann Brown is to report to the principal's office at 3:30 for a conference

there is likely to be confusion since 'Ann Brown' refers to either of these students, and the teacher would not know which student should get the message. On the other hand, suppose the following message is sent:

> Ann Brown is to report to the principal's office at 3:30 for a conference with the other freshmen representatives to the Student Council.

It is likely that the ambiguity of 'Ann Brown' in this message would cause no trouble since the rest of the message makes clear which Ann Brown is intended.

Similarly, although the '9' in each of the sentences:

$$(4) \qquad 9 \times 4 = 36$$

and:

$$(5) \qquad 9 \times {}^{-}7 = {}^{-}63$$

is ambiguous, we know from the rest of sentence (5) that it refers to the real number $^{+}9$ and not to the number 9 of arithmetic.

EXERCISES

A. Each of the following sentences contains at least one ambiguous numeral. In which sentences are you unable to tell whether real numbers or numbers of arithmetic are intended?

1. $2 \times {}^{-}3 = {}^{-}6$
2. $8 + 4 = 3 + 9$
3. $5 + {}^{-}5 = 0$
4. $4 \times 7 = 14 \times {}^{+}2$
5. $8 \times 3 = 6 \times 4$
6. $10 \times 5 = {}^{-}25 \times {}^{-}2$

B. Simplify.

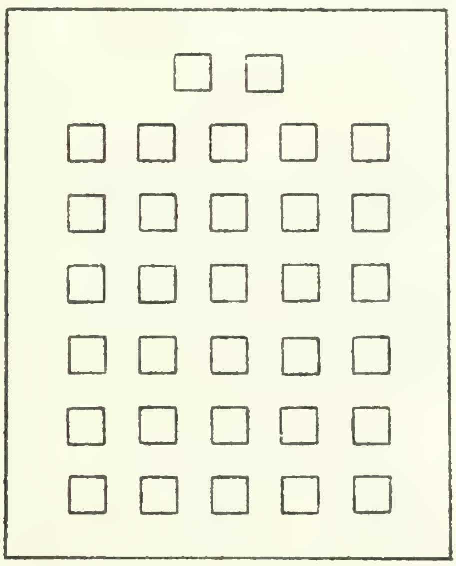
1. $3 \times {}^{-}4$
2. ${}^{-}7 \times 5$
3. ${}^{-}8 \times (6 + 4)$
4. ${}^{-}2 \times {}^{-}5$
5. $15 \times {}^{-}3$
6. ${}^{-}2 \times {}^{-}7$
7. $3 + {}^{-}4$
8. $7 + {}^{-}5$
9. ${}^{-}15 + 12$
10. $(17 + {}^{-}8) + 3$
11. $({}^{-}21 + 5) + {}^{-}5$
12. $({}^{-}53 \times 6) \times \frac{1}{6}$
13. $({}^{-}751 \times 7) \times {}^{-}\frac{1}{7}$

C. Each of the following sentences contains ambiguous words or phrases. Be prepared to give two interpretations for each sentence.

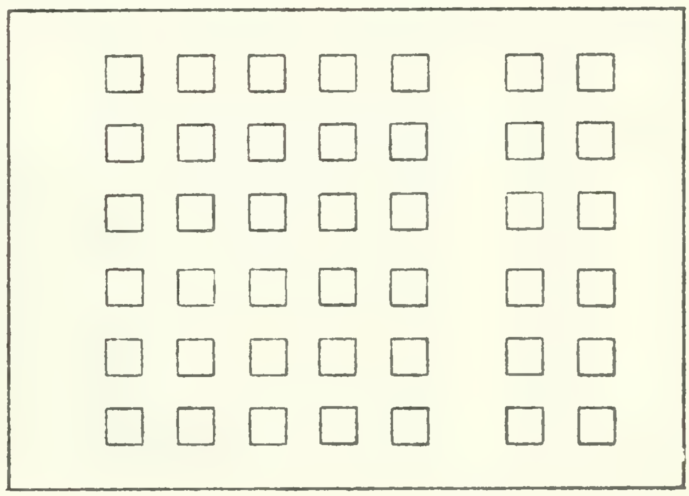
1. There will be little change in ladies' pocketbooks next year.
2. Charles ran after Henry.
3. Please leave the table.
4. One of the horses was scratched.
5. This is a fine day for the race.
6. Look at the scales.
7. Mr. Blattingham gave his address.
8. What does that ring mean?
9. 

10. 

1.05 Punctuating numerical expressions.--The pictures below show the seat arrangements of two classrooms in a school building. Under each picture someone has tried to indicate the number of seats in the room and, at the same time, show his method of computing that number.



$6 \times 5 + 2$



$6 \times 5 + 2$

It is easy to see that the room shown on the left has 32 seats and that the room shown on the right has 42 seats. If the collection of marks:

$$6 \times 5 + 2$$

is a numeral, it would seem to be the case that this numeral stands for 32 and for 42! Without the pictures, the numeral '$6 \times 5 + 2$' is of little help. This is a case of ambiguity which could cause trouble. What can be done to remove the ambiguity?

Here is a case of an English sentence which is ambiguous:

(1) John will play and Bill will sing or Mary will sing.

This sentence could mean that

(2) Either John will play and Bill will sing or Mary will sing.

Or, it could mean that

(3) John will play and either Bill will sing or Mary will sing.

To make an English sentence unambiguous you usually have to rewrite it and use grouping words like 'either...or...'.

In mathematics we can remove ambiguity from expressions by punctuating them with parentheses. We could use parentheses to punctuate sentence (1) to show the first meaning. We would write:

(2′) (John will play and Bill will sing) or Mary will sing.

To show the other meaning we would write:

(3′) John will play and (Bill will sing or Mary will sing).

Similarly, we can punctuate '$6 \times 5 + 2$' this way:

$$(6 \times 5) + 2$$

when we mean 32, and like this:

$$6 \times (5 + 2)$$

when we mean 42. [To read '$(6 \times 5) + 2$' aloud, say 'the sum of 6×5 and 2' or 'the quantity 6×5, plus 2' or 'parenthesis 6×5 close parentheses, plus 2'. How do you read aloud '$6 \times (5 + 2)$'?]

Here is another expression which is ambiguous:

$$8 \times 3 + 2 \times 5.$$

Give some numbers which it might name. Here are some of the ways in which this expression can be punctuated to make it unambiguous.

If we want to name 34, we can write:

$$(8 \times 3) + (2 \times 5).$$

To name 130, we can write:

$$[(8 \times 3) + 2] \times 5.$$

If we mean 200 then we should write:

$$[8 \times (3 + 2)] \times 5.$$

Is there another possibility?

Notice that when we want to show a grouping inside of another grouping, we use both parentheses and brackets. [Why don't we just use two pairs of parentheses?] Here is a punctuated expression in which parentheses, brackets, and braces are used:

$$\{[(7 \times 2) + 6] \times 5\} + 7.$$

This expression is a name for a number. Often it is helpful to have a simpler looking name for such a number. [For example, if someone asked you how much your new cap cost, and you replied that it cost $\{[(7 \times 2) + 6] \times 5\} + 7$ cents, he would have left before you had finished speaking.] To find a simpler looking expression equivalent to:

$$\{[(7 \times 2) + 6] \times 5\} + 7$$

[that is, to find a simpler expression which names the same number], we must find a simpler expression equivalent to:

$$\{[(7 \times 2) + 6] \times 5\};$$

this will be easier to do if we find a simpler one equivalent to:

$$[(7 \times 2) + 6],$$

and this is done by first finding a simpler name for the number named by:

$$(7 \times 2).$$

Hence, to simplify our original expression we would think through the following steps to obtain equivalent expressions.

$$\begin{array}{l}
\{[(7 \times 2) + 6] \times 5\} + 7 \\
\{[\quad 14 \quad + 6] \times 5\} + 7 \\
\{\qquad\quad 20 \ \times 5\} + 7 \\
\qquad\qquad\quad 100 \ + 7 \\
\qquad\qquad\qquad\quad 107
\end{array}$$

Most people would agree that '107' is a simpler looking name for 107 than is '$\{[(7 \times 2) + 6] \times 5\} + 7$'.

EXERCISES

A. Simplify. [Be careful not to confuse '+'s with '×'s.]

1. $(3 + 4) \times 2$
2. $3 + (4 \times 2)$
3. $(7 \times 5) + 4$
4. $(8 + {}^{-}3) \times {}^{-}2$
5. ${}^{-}1 \times ({}^{-}6 + 4)$
6. $({}^{-}1 \times {}^{-}6) + 4$
7. $(9 \div 3) + 3$
8. $9 \div (3 + 3)$
9. $(8 \div 2) \times 4$
10. $8 \div (2 \times 4)$
11. $(6 \times 5) \div 5$
12. $6 \times (5 \div 5)$
13. $(8 + 2) + 5$
14. $8 + (2 + 5)$
15. $(2 \times 5) \times 3$
16. $[(8 + 2) \times 3] + 5$
17. $[({}^{-}3 + {}^{-}2) \times 6] + {}^{-}7$
18. $5 + [{}^{-}1 \times (8 + {}^{-}2)]$
19. $(5 + {}^{-}1) \times (8 + {}^{-}2)$
20. $(8 \times 7) + (2 \times 7)$
21. $(8 + 2) \times 7$
22. $(3 \times {}^{-}6) + (7 \times {}^{-}6)$
23. $(3 + 7) \times {}^{-}6$
24. $({}^{-}2 \times {}^{-}12) + (2 \times {}^{-}12)$
25. $({}^{-}2 + 2) \times {}^{-}12$
26. $2 \times \{[(3 + 2) \times 4] + [(5 + 9) \times 4]\}$
27. ${}^{-}3 \times \{[{}^{-}2 \times (8 + 12)] + [(9 + 31) \times {}^{-}4]\}$
28. $\{[5 \times (8 + {}^{-}1)] + 3\} \times \{{}^{-}2 + [4 \times (3 + {}^{-}3)]\}$
29. $\dfrac{5 + 21}{2 \times 13}$ [Note: When a bar is used to indicate division, it also acts as a grouping symbol. So, '$\dfrac{5 + 21}{2 \times 13}$' means $(5 + 21) \div (2 \times 13)$.]
30. $\dfrac{28 - 3}{100 - 50}$
31. $\dfrac{15 + 4}{(7 \times 5) + 3}$
32. $\dfrac{17 + (6 \times 2)}{17 - (6 \times 2)}$
33. $\dfrac{[(9 \times 3) + (5 \times 2)] + 3}{[8 + (5 \times 5)] - 13}$
34. $\dfrac{[(5 \times 6) - (4 \times 2)] + [(3 \times 4) - 3]}{2 + \{5 + [3 + 2 \times (27 - 1)]\}}$

B. 1. Joey and Jane went to the candy store; Joey bought 5 candy bars at 6 cents each and a 10-cent package of bubble gum. Jane bought 6 suckers that were 5 cents each and 10 sacks of peanuts that were also 5 cents each. What single expression could you write which (if no grouping symbols were used) might name the cost of either Joey's or Jane's purchases?

2. Mrs. Gardner and Mrs. Plantin were making vegetable gardens. Mrs. Gardner said she had planted 3 rows of cabbages with 7 plants in a row, and 5 rows of tomatoes with 3 plants in each row. Could the expression '$3 \times 7 + 5$' represent the number of cabbage and tomato plants in Mrs. Gardner's garden?

   Her friend, Mrs. Plantin, explained that she had set out a row containing 5 green pepper plants; also, she had planted potatoes so that she would have 3 potato plants in each of 7 rows. Could the expression above which indicates the number of plants set out by Mrs. Gardner also be used to indicate the number of green pepper and potato plants in Mrs. Plantin's garden?

CONVENTIONS FOR OMITTING GROUPING SYMBOLS

You have seen how parentheses and other grouping symbols can be used to remove ambiguity from numerical expressions. In fact, there would never be an ambiguous expression if everyone followed the rule that each operation sign [$+$, $\times$, $-$, $\div$] required a pair of grouping symbols. You would have expressions like these:

$$(5 + 4), \quad [(6 - 2) + 9], \quad [3 \times (6 \times 5)],$$

$$\{[(4 + 1) + 3] + (6 - 5)\}, \quad \{[(3 \times 4) + (6 - 2)] \div (12 \div 3)\}.$$

In the expression '$(5 + 4)$', the parentheses go with the '$+$'.

In the expression '$[3 \times (6 \times 5)]$', the parentheses go with the second '$\times$', and the brackets go with the first '$\times$'.

In the expression '$\{[(4 + 1) + 3] + (6 - 5)\}$', the braces go with the third '$+$', the brackets go with the second '$+$', the first pair of parentheses goes with the first '$+$', and the second pair goes with the '$-$'.

Tell which pairs of grouping symbols go with which operation signs in the other two expressions written above. Make up three expressions each of which contains at least two operation signs and a pair of grouping symbols which go with each sign.

If an expression contains many operation signs, this rule requires that the expression contain just as many pairs of grouping symbols.

Such expressions are frequently hard to read and look quite complicated. So, people follow certain agreements [or conventions] which permit them to omit some grouping symbols and still avoid ambiguity. Under such conventions, an expression like:

$$9 + 2 \times 4 - 4 \div 2$$

is unambiguous. Our job now is to learn what these conventions are so that when we come upon an expression such as the one above, there will be no doubt what number is intended by the person who followed our conventions in writing it.

One convention [which we have been using throughout this book] is to omit the outermost grouping symbols. For example, we would write

'$5 + 4$' as an abbreviation for '$(5 + 4)$',

and '$(6 - 2) + 9$' as an abbreviation for '$[(6 - 2) + 9]$'.

This convention was followed in all of the expressions in Part A on page 1-36. Turn to that page now and, for the first ten expressions, tell which operation sign goes with the omitted grouping symbols. When we use such abbreviated expressions to form larger expressions, we often have to unabbreviate them by replacing the omitted grouping symbols. For example, we name the product of $3 + 2$ and $4 + 5$ by '$(3 + 2) \times (4 + 5)$'. Although '$3 + 2$' and '$4 + 5$' are not ambiguous, '$3 + 2 \times 4 + 5$' is ambiguous.

Another convention which we shall adopt is illustrated in these examples.

'$3 + 5 + 6$' is an abbreviation for '$(3 + 5) + 6$'.

'$5 \times 2 \times 7$' is an abbreviation for '$(5 \times 2) \times 7$'.

'$4 + 3 + 2 + 9$' is an abbreviation for '$[(4 + 3) + 2] + 9$'

'$12 - 2 - 3$' is an abbreviation for '$(12 - 2) - 3$'.

'$18 \div 3 \div 2$' is an abbreviation for '$(18 \div 3) \div 2$'.

'$2 + (3 + 5) + 6$' is an abbreviation for '$[2 + (3 + 5)] + 6$'.

'$2 \times 3 \times (5 \times 6)$' is an abbreviation for '$(2 \times 3) \times (5 \times 6)$'.

Notice that each of the abbreviated expressions contains only one kind of operation sign. The expressions which contain no grouping symbols are unabbreviated by introducing a pair of grouping symbols for each operation

sign, proceeding from left to right. So, we unabbreviate:

$$4 + 3 + 2 + 9$$

by first writing:

$$(4 + 3) + 2 + 9,$$

and then writing:

$$[(4 + 3) + 2] + 9.$$

[As before, we don't include the outermost grouping symbols.] As in the step from '(4 + 3) + 2 + 9' to '[(4 + 3) + 2] + 9', when some grouping symbols are already present in an expression to be unabbreviated, we bring in grouping symbols for those operation signs which do not already have them, again proceeding from left to right.

Examples:

(1)
$$\begin{array}{c} 2 + (3 + 5) + 6 \\ \downarrow \\ [2 + (3 + 5)] + 6 \end{array}$$

(2)
$$\begin{array}{c} 2 \times 3 \times (5 \times 6) \\ \downarrow \\ (2 \times 3) \times (5 \times 6) \end{array}$$

This same convention is used for unabbreviating expressions in which the operation signs which lack grouping symbols are either all addition and subtraction signs or all multiplication and division signs.

(3)
$$\begin{array}{c} 4 + 3 - 2 + 7 \\ \downarrow \\ (4 + 3) - 2 + 7 \\ \downarrow \\ [(4 + 3) - 2] + 7 \end{array}$$

(4)
$$\begin{array}{c} 8 \times 2 \div 4 \times 3 \\ \downarrow \\ (8 \times 2) \div 4 \times 3 \\ \downarrow \\ [(8 \times 2) \div 4] \times 3 \end{array}$$

(5)
$$\begin{array}{c} 9 - (3 \times 2) + 4 - 5 \\ \downarrow \\ [9 - (3 \times 2)] + 4 - 5 \\ \downarrow \\ \{[9 - (3 \times 2)] + 4\} - 5 \end{array}$$

(6)
$$\begin{array}{c} 10 \div 5 \div (2 + 2) \times 2 \\ \downarrow \\ (10 \div 5) \div (2 + 2) \times 2 \\ \downarrow \\ [(10 \div 5) \div (2 + 2)] \times 2 \end{array}$$

Consider this expression:

$$12 + [3 \times (5 + 6) \div 2] - 5.$$

To unabbreviate it, we first attack the expression in the brackets:

$$12 + [\{3 \times (5 + 6)\} \div 2] - 5.$$

And, now we unabbreviate this expression, getting:

$$\{12 + [\{3 \times (5 + 6)\} \div 2]\} - 5.$$

Here are some more examples of this.

(7) $4 \times (3 \times 2 \times 5 \times 3)$	(8) $38 + [3 \times 4 \div 6 \times (2 + 4)]$
$\downarrow$	$\downarrow$
$4 \times ([3 \times 2] \times 5 \times 3)$	$38 + [(3 \times 4) \div 6 \times (2 + 4)]$
$\downarrow$	$\downarrow$
$4 \times (\{[3 \times 2] \times 5\} \times 3)$	$38 + [\{(3 \times 4) \div 6\} \times (2 + 4)]$

EXERCISES

A. On a separate sheet of paper, rewrite each of these expressions in unabbreviated form. [You need not put in outermost grouping symbols.]

Sample. $6 + 4 + 3 + 9$

Solution. $[(6 + 4) + 3] + 9$

1. $2 + 8 + 3$
2. $7 \times 5 \times 3$
3. $9 - 5 - 3$
4. $6 \div 2 \div 3$
5. $8 + 3 + 5 + 4$
6. $9 \times 2 \times 3 \times 5$
7. ${}^{+}5 + {}^{+}2 + {}^{+}3 + {}^{+}7$
8. ${}^{+}3 \times {}^{+}2 \times {}^{+}6 \times {}^{+}7$
9. $3 + 8 - 2 + 5$
10. $15 - 7 + 5 - 9$
11. $5 \times 6 \div 3 \times 4$
12. $3 \times 4 \div 2 \div 3$
13. $15 - 8 - 3 - 7$
14. $24 \div 3 \div 2 \div 4$
15. $5 + 9 + (6 + 8)$
16. $7 + (9 + 5) + 8$
17. $2 \times (5 \times 3) \times 4$
18. $6 \times 8 \times (3 \times 7) \times 4$
19. $5 + (3 \times 2 \times 8)$
20. $6 \times (4 + 9 + 3 - 8)$
21. $8 + [3 \times (7 + 2) \times 5] - 6 + 7 - 5$
22. $(5 \times 3) + (4 \times 7) - (8 \times 2) + (8 - 2)$
23. $6 + [5 \times (3 + 7)] + [4 \times (8 - 5)]$
24. $4 \times [5 - 3 - 1] \times [7 + (2 \times 4)]$

B. According to the conventions we have discussed, each of the expressions in Part A is unambiguous. You should be able to simplify each expression without rewriting it in unabbreviated form. Do so.

* * *

In simplifying abbreviated expressions which contain multiplication or division signs <u>and</u> addition or subtraction signs such as:

$$9 + 2 \times 4 - 4 \div 2,$$

you first do the multiplications and divisions, working from left to right, and then the additions and subtractions in the same order. In particular, the expression:

$$9 + 2 \times 4 - 4 \div 2$$

is unabbreviated in two steps:

$$9 + (2 \times 4) - (4 \div 2),$$

and then:

$$[9 + (2 \times 4)] - (4 \div 2).$$

So, the given expression is, in view of our conventions, completely unambiguous. It stands for 15.

<u>Example 9</u>. Simplify by first unabbreviating:

$$5 \times 9 + 3 - 8 \div 2 \times 3.$$

<u>Solution</u>. Unabbreviate the expression.

$$\begin{aligned} & 5 \times 9 + 3 - 8 \div 2 \times 3 \\ = \; & (5 \times 9) + 3 - [(8 \div 2) \times 3] \\ = \; & [(5 \times 9) + 3] - [(8 \div 2) \times 3]. \end{aligned}$$

Now simplify it.

$$\begin{aligned} & [(5 \times 9) + 3] - [(8 \div 2) \times 3] \\ = \; & [\ 45 \ + 3] - [\ 4 \ \times 3] \\ = \; & \quad 48 \quad - \quad 12 \\ = \; & 36. \end{aligned}$$

<u>Example 10</u>. Simplify by first unabbreviating:

$$3 + (6 + 2 \times 5) - (17 - 4 \times 3).$$

<u>Solution</u>. Unabbreviate first.

$$\begin{aligned} & 3 + (6 + 2 \times 5) - (17 - 4 \times 3) \\ = \; & 3 + (6 + [2 \times 5]) - (17 - [4 \times 3]) \\ = \; & \{3 + (6 + [2 \times 5])\} - (17 - [4 \times 3]) \end{aligned}$$

(continued on next page)

Simplify next.

$$\begin{aligned} &\{3 + (6 + [2 \times 5])\} - (17 - [4 \times 3]) \\ &= \{3 + (6 + 10)\} - (17 - 12) \\ &= \{3 + 16\} - 5 \\ &= 19 - 5 \\ &= 14. \end{aligned}$$

* * *

C. Rewrite each of the following expressions in unabbreviated form, and then simplify it.

1. $3 + 5 \times 10$
2. $9 \times 2 + 4$
3. $8 \div 2 + 5$
4. $7 \div 2 \times 5$
5. $4 \times 3 - 1$
6. $2 + 5 \times 6$
7. $4 \times 5 + 3 \times 7 + 2 \times 3$
8. $6 \div 2 + 15 \div 3 + 20 \div 10$
9. $3 \times 7 - 2 \times 5 + 4 \div 8$
10. $10 \times 5 \div 2 \times 4 \div 100$
11. ${}^{+}5 \times {}^{-}4 + {}^{-}3 \times {}^{-}2$
12. $4 + {}^{-}3 \times 7 + {}^{-}2 \times {}^{-}4$
13. $8 + {}^{-}2 + {}^{-}3 \times 6$
14. $8 + {}^{-}2 \times {}^{-}5 + 10 \times 3$
15. $7 \times (8 + 3) \times (6 \div 2)$
16. $12 \times (4 + {}^{-}2) + (20 + {}^{-}3)$
17. $3 + (16 \div 2 \div 4) + (5 \times 2 - 3 \times 3)$
18. $5 \times 3 - 6 \div 2 + [(4 + 5) \times 3 + 2 + 4 \times 2]$
19. $(5 + 3) \times (7 + 1) - 8 \times 2 \times (3 - 1) + 7 \times 5 + 1$
20. $6 + 3 + 8 \times (6 - 3 \times 2 + 5 + 4 \div 2) + (4 - 3)$

D. Simplify.

1. $3 \times 6 - 4 \times 2$
2. $5 \times 4 - 3 \times 2$
3. $7 \times 4 \div 2 + 3 \times 7$
4. $6 \times 5 \div 2 + 4 \times 9 \div 2$
5. $18 - 7 + 2 - 8 \div 2$
6. $12 - (5 + 3) \div 2 + 6$
7. $18 - (7 + 2) - 8 \div 2$
8. $12 - (5 + 3) \div (2 + 6)$
9. $84 - \{5 \times [5 \times 2 - 3 + (6 \times 4 - 5) - 7] - (6 \div 2 + 8)\}$
10. $16 + 3 \times [(3 + 5) \times 2 - (2 + 1) \times 4] + 6 \times (8 - 3)$

[More exercises are in Part F, Supplementary Exercises.]

EXPLORATION EXERCISES

A. For each number listed below, give a whole number (other than itself and 1) which divides it. ['divides it' means the same thing as 'divides it exactly'. For example, 2, 3, 4, and 6 each divides 12, but 5 does not divide 12.]

Sample. $3 \times 5 + 7 \times 5$

Solution. This simplifies to '50'. So, 5 is a number which divides $3 \times 5 + 7 \times 5$. [Other numbers are 2, 10, and 25.]

1. $3 \times 7 + 5 \times 7$
2. $8 \times 9 + 2 \times 9$
3. $2 \times 13 + 18 \times 13$
4. $5 \times 31 + 2 \times 31$
5. $93 \times 6 + 7 \times 6$
6. $3 \times 593 + 2 \times 593$

*

For each number listed below, give two numbers which divide it.

7. $3 \times 19 + 12 \times 19$
8. $6 \times 7 + 7 \times 7$
9. $67 \times 41 + 71 \times 41$
10. $51 \times 87 + 56 \times 87$
11. $1319 \times 547 + 1409 \times 547$
12. $3163 \times 3833 + 3163 \times 3847$

B. Simplify mentally.

1. $(387 + 9) + 1$
2. $(7452 + 75) + 25$
3. $583 + 92 + 8$
4. $927 + 152 + {}^{-}52$
5. $(37 \times 5) \times 2$
6. $(98 \times 25) \times 4$
7. $487 \times 25 \times 40$
8. $6871 \times 20 \times 5$
9. $(82 + 47) + 53$
10. $13 + (987 + 5426)$
11. $894 + 751 + {}^{-}94$
12. ${}^{-}6341 + {}^{-}275 + {}^{-}659$
13. $(12 \times 15) \times 2$
14. $(5 \times 84) \times 20$
15. $55 \times 7 \times 2$
16. $40 \times 812 \times 50$
17. $98 + 76 + {}^{-}76$
18. ${}^{-}583 + {}^{-}9624 + 583$
19. $634 \times 5 \times \frac{1}{5}$
20. $97 \times \frac{1}{17} \times 17$
21. $\frac{^{-}1}{48} \times {}^{-}79 \times {}^{-}48$
22. ${}^{-}384 \times 5627 \times \frac{^{-}1}{384}$

1.06 Principles for the numbers of arithmetic. --The numbers of arithmetic have certain properties which you make use of time and again as you do problems with these numbers.

(I) Here is a start on a multiplication table. Your job is to fill in the empty spaces.

×	$\frac{2}{3}$	1200	$\frac{3}{4}$	87	21
21	14		$\frac{63}{4}$	1827	441
87	58	104, 400		7569	
$\frac{2}{3}$	$\frac{4}{9}$		$\frac{1}{2}$		
$\frac{3}{4}$		900	$\frac{9}{16}$	$\frac{261}{4}$	
1200	800	1, 440, 000			25, 200

(II) Sort the expressions below into pairs of numerals for the same number.

21×17	657×891	$1984 \times \frac{3}{4}$	$\frac{2}{3} \times 96$
$\frac{3}{4} \times 1984$	17×21	$\frac{396}{59} \times 243$	891×657
$96 \times \frac{2}{3}$	27×31	$243 \times \frac{396}{59}$	93×9

How many products did you need to compute to make your list of pairs? How many products did you need to compute to fill the table in the first problem?

If you had to do more than two computations for problem (II) and any for problem (I), you failed to recognize places to make use of an important property of the numbers of arithmetic. This is that if you multiply a pair of numbers, you get the same answer no matter what order you use in multiplying. This is <u>the commutative principle for multiplication</u>. Instances of this principle are:

$$5 \times 7 = 7 \times 5,$$

$$21 \times \frac{3}{4} = \frac{3}{4} \times 21.$$

Notice that you don't have to simplify the numerals connected by the equality signs in these two sentences in order to know that the sentences are true. If you believe that the commutative principle for multiplication is true then you believe that each instance of it is true, also.

Is there a corresponding property of the numbers of arithmetic for addition? If Mr. Brown has 12 Black Angus cows and 8 Holstein cows on his farm, how many cows of each kind must he buy to have twice as many cows, and the same number of Black Angus as Holstein?

To solve this problem quickly you need to recognize that

$$12 + 8 = 8 + 12,$$

and this is an instance of <u>the commutative principle for addition</u>. Other instances are:

$$986 + 724 = 724 + 986,$$

$$16.357 + 5.009 = 5.009 + 16.357.$$

Notice, again, that you don't feel any urge to compute to see if the last two sentences are true. As long as you believe that the commutative principle for addition is true, you also believe that each instance of it is true.

There are still other properties of the numbers of arithmetic which are useful in solving problems, especially in finding short cuts.

For example, suppose you were trying to find the total number of raffle tickets sold on three consecutive days.

First day. . . . 47, Second day. . . . 75, Third day. . . . 25.

One way of doing this is to find the total for the first and second days, and then add to this total the number sold on the third day. So,

$$(47 + 75) + 25 = 122 + 25 = 147.$$

But a much easier way, which most likely has already occurred to you, is to do the problem this way:

$$47 + (75 + 25) = 47 + 100 = 147.$$

There was probably no doubt in your mind that you would get the same sum in doing the problem the second way as you would in doing it the first way. You feel sure about this because you feel sure about another property of the numbers of arithmetic, a property which is expressed by *the associative principle for addition*. Other instances of this principle are:

$$(8 + 5) + 19 = 8 + (5 + 19)$$

$$23 + 91 + 9 = 23 + (91 + 9),$$

$$15 + (85 + 38) = 15 + 85 + 38.$$

And, as you have probably guessed by now, there is also *the associative principle for multiplication*. Notice how it gives you short cuts.

$$(27 \times 5) \times 2 = ?$$

$(27 \times 5) \times 2 = 27 \times (5 \times 2)$. So, $(27 \times 5) \times 2 = 27 \times 10 = 270$.

$$(897 \times 4) \times 25 = ?$$

$(897 \times 4) \times 25 = 897 \times (4 \times 25)$. So, $(897 \times 4) \times 25 = 89700$.

$$50 \times (2 \times 68) = ?$$

$50 \times (2 \times 68) = (50 \times 2) \times 68$. So, $50 \times (2 \times 68) = 6800$.

Sometimes you use short cuts which depend upon more than just one of these principles. For example, suppose you want to find the total number of points you made in a test which had three parts:

Part I . . . . 27 Part II . . . . 39 Part III 23

The straight-forward procedure would be:

$$(27 + 39) + 23 = 66 + 23 = 89.$$

But a short cut might involve thinking through these steps:

$$(27 + 39) + 23 = 27 + (39 + 23),$$

because of the associative principle for addition, and

$$27 + (39 + 23) = 27 + (23 + 39),$$

because the commutative principle for addition tells us that

$$39 + 23 = 23 + 39.$$

Finally,

$$27 + (23 + 39) = (27 + 23) + 39,$$

because of the associative principle for addition. So, we know that $(27 + 39) + 23 = (27 + 23) + 39$. Since $27 + 23$ is 50, and $50 + 39$ is 89, we know that

$$(27 + 39) + 23 = 89.$$

[Did you, without realizing it, use the associative and commutative principles for addition in finding that $27 + 23$ is 50?]

Suppose you were asked to do long "column addition" as in these examples. Do them.

5	2	5	14
3	9	7	25
5	1	2	32
7	8	3	51
6	9	3	98

Did you skip around to find the easy combinations? Do you believe that you can get a correct total this way? That you do get a correct total even though you skip around is a consequence of the associative and commutative principles for addition.

EXERCISES

A. Each of the following sentences is an instance of one of the four principles you have just learned. Tell which principle.

1. $9 + 7 = 7 + 9$
2. $3 \times 5 = 5 \times 3$
3. $61 + 17 = 17 + 61$
4. $97 \times 816 = 816 \times 97$
5. $81 + (9 + 13) = (81 + 9) + 13$
6. $(93 \times 5) \times 100 = 93 \times (5 \times 100)$
7. $71 \times (51 + 47) = (51 + 47) \times 71$
8. $523 + 43 + 79 = 523 + (43 + 79)$
9. $657 \times 982 \times 539 = 657 \times (982 \times 539)$
10. $(841 + 56) + (75 + 37) = (75 + 37) + (841 + 56)$
11. $[(72 + 45) + 63] + 85 = (72 + 45) + (63 + 85)$
12. $72 + [(45 + 63) + 85] = [72 + (45 + 63)] + 85$
13. $72 + 45 + (63 + 85) = 72 + [45 + (63 + 85)]$
14. $72 + (45 + 63) + 85 = 85 + [72 + (45 + 63)]$
15. $(81 + 37) + (92 + 54) = (92 + 54) + (81 + 37)$
16. $7 + 3\frac{1}{4} = (7 + 3) + \frac{1}{4}$ [‘$3\frac{1}{4}$’ is an abbreviation for ‘$(3 + \frac{1}{4})$’.]
17. $85\frac{1}{5} \times (25 + 48) = (25 + 48) \times 85\frac{1}{5}$
18. $72 + (45 + 63) + 85 = 72 + [45 + 63 + 85]$
19. $(72 + 45 + 63) + 85 = (72 + 45) + (63 + 85)$
20. $(3 \times 7 \times 2 \times 5) \times (8 + 4 + 3 + 2) = (8 + 4 + 3 + 2) \times (3 \times 7 \times 2 \times 5)$

B. None of the following sentences is an instance of any of the four principles you have just learned. However, some of the sentences are consequences of the principles. Tell which sentences are consequences of which principles.

Sample 1. $5 + (7 \times 2) = 5 + (2 \times 7)$.

Solution. Since ‘$7 \times 2 = 2 \times 7$’ is an instance of the commutative principle for multiplication, the given sentence is a consequence of that principle.

<u>Sample 2</u>. $72 + (45 + 63) + 85 = [72 + (45 + 63)] + 85$

<u>Solution</u>. The expression on the left of the equality sign is merely an abbreviation for the expression on the right. So, none of the four principles is required in showing that the given sentence is true.

1. $6 \times 5 \times 3 = 5 \times 6 \times 3$
2. $4 + 7 + 3 = 7 + 4 + 3$
3. $4 + (7 + 3) = 4 + (3 + 7)$
4. $(9 + 3) \times 5 = (3 + 9) \times 5$
5. $7 + 4 \times 3 = 7 + (4 \times 3)$
6. $8 + 2 + 9 = (8 + 2) + 9$
7. $9\frac{1}{5} + 6\frac{4}{5} = 9\frac{1}{5} + (\frac{4}{5} + 6)$
8. $14 \times 2\frac{1}{7} = 14 \times (2 + \frac{1}{7})$
9. $(6 + 5) \times 3 = 3 \times (5 + 6)$
10. $8 + 5 + 3 = 3 + 5 + 8$
11. $(4 + 7) + (3 + 8) = (8 + 3) + (7 + 4)$
12. $(10 \times 3) + (15 \times 3) = (3 \times 10) + (3 \times 15)$

<u>C</u>. Fill the blanks to make true sentences.

1. $19 + \underline{\qquad} = 72 + 19$
2. $31 \times \underline{\qquad} = 59 \times 31$
3. $6 \times \underline{\qquad} \times 9 = 6 \times (\underline{\qquad} \times 9)$
4. $85 + 97 \times \underline{\qquad} = 85 + \underline{\qquad} \times 97$
5. $10 + 7 + 3 = 3 + \underline{\qquad} + 7$
6. $(8 + 5) \times 2 = 8 + \underline{\qquad} \times 2$
7. $9 + \underline{\qquad} + 4 = 9 + (\underline{\qquad} + 4)$
8. $(\underline{\qquad} + 5) + 7 = 7 + 5 + \underline{\qquad}$
9. $\frac{3 \times 7}{7 \times 5} = \frac{3 \times 7}{5 \times \underline{\qquad}}$
10. $\frac{1}{3} \times (7 \times \frac{1}{5}) = \frac{1}{15} \times \underline{\qquad}$
11. $3 \times 5 \times 3\frac{1}{3} = \underline{\qquad} \times 5$
12. $61 + (\underline{\qquad} + 39) = 184$
13. $6 \div 3 = \underline{\qquad} \div 6$
14. $9 - 5 = \underline{\qquad} - 9$
15. $24 \div 4 \div 2 = 24 \div (4 \div \underline{\qquad})$
16. $15 - 6 - 1 = 15 - (\underline{\qquad} - 1)$
17. $107 \times \underline{\qquad} + 372 \times 76 = 76 \times 372 + 859 \times 107$
18. $(8 + 7) \times (9 + 16) = (16 + \underline{\qquad}) \times (7 + 8)$

<u>D</u>. 1. Are subtraction and division commutative operations? Give examples to justify your answers.

2. Are subtraction and division associative? Give examples.

ANOTHER PRINCIPLE

Perhaps you have found short cuts for some problems which involve both multiplication and addition.

$$7 \times 11 + 3 \times 11 = ?$$

Do you see a short way of solving this problem? If you don't, you may see it after you have filled in the blanks in the following sentences.

$$4 \times 15 + 6 \times 15 = \underline{\ 60\ } + \underline{\ 90\ } = \underline{\ 150\ } = \underline{\ 10\ } \times 15$$

$$8 \times 29 + 2 \times 29 = \underline{\qquad} + \underline{\qquad} = \underline{\qquad} = \underline{\qquad} \times 29$$

$$13 \times 21 + 17 \times 21 = \underline{\qquad} + \underline{\qquad} = \underline{\qquad} = \underline{\qquad} \times 21$$

$$5 \times 9 + 6 \times 9 = \underline{\qquad} + \underline{\qquad} = \underline{\qquad} = \underline{\qquad} \times 9$$

$$21 \times 8 + 19 \times 8 = \underline{\qquad} + \underline{\qquad} = \underline{\qquad} = \underline{\qquad} \times 8$$

$$\frac{2}{5} \times 7 + \frac{3}{5} \times 7 = \underline{\qquad} + \underline{\qquad} = \underline{\qquad} = \underline{\qquad} \times 7$$

Can you do these problems by a short cut?

$$8 \times 7 + 3 \times 7 = ?$$

$$6 \times 582 + 4 \times 582 = ?$$

The same short cut can be used in the following problem.

> Suppose you have two vacation jobs, one of which pays 85 cents an hour, and the other \$1.15 an hour. How much have you earned if you worked 35 hours on each job? Some people may solve this problem the hard way by first multiplying to tell how much was earned on each job, and then adding the results.
>
> $$.85 \times 35 + 1.15 \times 35 = 29.75 + 40.25 = 70.00.$$
>
> Do you see an easy way? First, add the rates of pay, and then multiply.
>
> $$(.85 + 1.15) \times 35 = 2.00 \times 35 = 70.00$$

The fact that you are sure that this easy way will give you the total earned is a consequence of your belief in a principle called <u>the distributive principle for multiplication over addition</u>. The instance of it which we just used is:

$$.85 \times 35 + 1.15 \times 35 = (.85 + 1.15) \times 35.$$

Here are other examples of how this principle makes computations easier.

<u>Example 1</u>. $8\frac{1}{6} \times 5 = ?$

<u>Solution</u>. $$\begin{aligned}(8 + \tfrac{1}{6}) \times 5 &= 8 \times 5 + \tfrac{1}{6} \times 5 \\ &= 40 + \tfrac{5}{6} \\ &= 40\tfrac{5}{6}.\end{aligned}$$

<u>Example 2</u>. $91 \times 61 + 9 \times 61 = ?$

<u>Solution</u>. $$\begin{aligned}91 \times 61 + 9 \times 61 &= (91 + 9) \times 61 \\ &= 100 \times 61 \\ &= 6100.\end{aligned}$$

EXERCISES

<u>A</u>. Fill in the blanks to make true sentences, and then tell which are instances of the distributive principle for multiplication over addition.

1. $5 \times 7 + 3 \times 7 = (5 + \_\_\_) \times 7$
2. $8 \times 5 + 2 \times 5 = (\_\_\_ + 2) \times 5$
3. $7 \times 9 + 3 \times 9 = (7 + 3) \times \_\_\_$
4. $(2 + 8) \times 5 = 2 \times 5 + \_\_\_ \times 5$
5. $(6 + 3) + 2 = 6 + (\_\_\_ + 2)$
6. $2 \times 7 + \_\_\_ \times 7 = (2 + 3) \times 7$
7. $10 + 3 \times 5 = \_\_\_ \times 5 + 3 \times 5$
8. $(6 + 5) \times \_\_\_ = 8 \times (6 + 5)$
9. $7 \times (3 + 2) = 7 \times 3 + 7 \times \_\_\_$
10. $5 \times (4 + 8) = \_\_\_ \times 4 + \_\_\_ \times 8$
11. $9 \times 7 + 9 \times 3 = \_\_\_ \times (7 + 3)$

* * *

If you wrote a '9' in the blank in Exercise 3 of Part A, you obtained the true sentence:

$$7 \times 9 + 3 \times 9 = (7 + 3) \times 9,$$

and this sentence is an instance of the distributive principle for multiplication over addition. An instance of this principle tells you a way of computing the sum of two products which have the <u>same</u> multiplier.

If you wrote a '9' in the blank in Exercise 11 of Part A, you obtained the true sentence:

$$9 \times 7 + 9 \times 3 = 9 \times (7 + 3).$$

This sentence tells you a way of computing the sum of two products which have <u>different</u> multipliers. So, this sentence is <u>not</u> an instance of the distributive principle for multiplication over addition. However, it is an instance of another principle which is called <u>the left distributive principle for multiplication over addition</u>. Here are some other instances of this new principle.

$$12 \times 3 + 12 \times 97 = 12 \times (3 + 97)$$
$$20 \times (4 + \tfrac{1}{2}) = 20 \times 4 + 20 \times \tfrac{1}{2}.$$

Write three instances of the left distributive principle for multiplication over addition, and three instances of the distributive principle for multiplication over addition.

* * *

<u>B</u>. Fill in the blanks to make true sentences, and then tell what principles they are instances of.

1. $4 \times 9 + 7 \times 9 = (4 + \_\_\_) \times 9$
2. $9 \times 4 + 9 \times 7 = 9 \times (\_\_\_ + 7)$
3. $6 \times 8\tfrac{1}{3} = 6 \times 8 + \_\_\_ \times \tfrac{1}{3}$
4. $4 \times 7 \times 3 = 4 \times (7 \times \_\_\_)$
5. $(8 + 2) \times (9 + 3) = (9 + 3) \times (8 + \_\_\_)$
6. $(6 + 5) \times 9 + (7 + 3) \times 9 = [(6 + 5) + (\_\_\_ + 3)] \times 9$
7. $(8 + 5) \times (6 + 2) = (8 + 5) \times \_\_\_ + (8 + 5) \times 2$
8. $(8 + 5) \times (6 + 2) = 8 \times (6 + \_\_\_) + 5 \times (6 + 2)$

MORE PRINCIPLES

We have mentioned and illustrated six principles which express certain properties of operations with the numbers of arithmetic. There are also certain numbers of arithmetic which have interesting properties. For example, there is a number of arithmetic which when added to any number of arithmetic gives you back that number. In each of the blanks below, write a numeral which makes the sentence true.

$$4 + \_\_\_ = 4$$

$$5\frac{1}{2} + \_\_\_ = 5\frac{1}{2}$$

$$\frac{23}{18} + \_\_\_ = \frac{23}{18}$$

Is there a number of arithmetic which does a similar thing for multiplication? What must you write in each blank below to make the sentence true?

$$4 \times \_\_\_ = 4$$

$$5\frac{1}{2} \times \_\_\_ = 5\frac{1}{2}$$

$$\frac{23}{18} \times \_\_\_ = \frac{23}{18}$$

These properties of 0 and 1 are expressed by <u>the principle for adding 0</u> and <u>the principle for multiplying by 1</u>. [Can you guess what <u>the principle for multiplying by 0</u> is? Give some instances of it.]

The principle for multiplying by 1 is used in adding fractional numbers. For example, suppose you want to add $\frac{2}{7}$ and $\frac{3}{5}$. In order to simplify the expression:

$$\frac{2}{7} + \frac{3}{5},$$

we need to find names for $\frac{2}{7}$ and $\frac{3}{5}$ which have the same denominator. The procedure usually followed is to multiply $\frac{2}{7}$ by $\frac{5}{5}$ to get $\frac{10}{35}$, and to multiply $\frac{3}{5}$ by $\frac{7}{7}$ to get $\frac{21}{35}$. Then we write:

$$\frac{10}{35} + \frac{21}{35} = \frac{31}{35}.$$

So,

$$\frac{2}{7} + \frac{3}{5} = \frac{31}{35}.$$

Although we know that $\frac{31}{35}$ is the sum of $\frac{10}{35}$ and $\frac{21}{35}$, how do we know it is the sum of $\frac{2}{7}$ and $\frac{3}{5}$? The principle for multiplying by 1 assures us that since $\frac{5}{5}$ is 1, the product of $\frac{2}{7}$ and $\frac{5}{5}$ is $\frac{2}{7}$. So, $\frac{10}{35}$ is $\frac{2}{7}$. Similarly, the principle for multiplying by 1 assures us that $\frac{21}{35}$ is $\frac{3}{5}$. So, the sum of $\frac{10}{35}$ and $\frac{21}{35}$ is the sum of $\frac{2}{7}$ and $\frac{3}{5}$.

Here is a summary of the principles we have discussed for the numbers of arithmetic.

The Commutative Principle for Addition [cpa]

$$7 + 29 = 29 + 7$$

$$6\frac{1}{2} = \frac{1}{2} + 6$$

$$9.34 + 5 = 5 + 9.34$$

The Associative Principle for Addition [apa]

$$(9 + 5) + 15 = 9 + (5 + 15)$$

$$6\frac{3}{5} + (7\frac{2}{5} + 8\frac{1}{3}) = (6\frac{3}{5} + 7\frac{2}{5}) + 8\frac{1}{3}$$

$$87 + 9 + 91 = 87 + (9 + 91)$$

The Commutative Principle for Multiplication [cpm]

$$590 \times 2 = 2 \times 590$$

$$81.3 \times 17.7 = 17.7 \times 81.3$$

$$4\frac{1}{2} \times 6\frac{1}{4} = 6\frac{1}{4} \times 4\frac{1}{2}$$

The Associative Principle for Multiplication [apm]

$$(84 \times 5) \times 20 = 84 \times (5 \times 20)$$

$$16\frac{2}{3} \times (6 \times 59) = (16\frac{2}{3} \times 6) \times 59$$

$$97 \times 25 \times 4 = 97 \times (25 \times 4)$$

The Distributive Principle for Multiplication over Addition [dpma]

$$5 \times 98 + 95 \times 98 = (5 + 95) \times 98$$

$$(\frac{1}{3} + \frac{1}{2}) \times 12 = \frac{1}{3} \times 12 + \frac{1}{2} \times 12$$

$$\frac{5}{7} \times 59 + \frac{2}{7} \times 59 = (\frac{5}{7} + \frac{2}{7}) \times 59$$

The Left Distributive Principle for Multiplication over Addition [ℓdpma]

$$5 \times 8 + 5 \times 12 = 5 \times (8 + 12)$$

$$7 \times 3\frac{1}{7} = 7 \times 3 + 7 \times \frac{1}{7}$$

The Principle for Adding 0 [pa 0]

$$6 + 0 = 6$$

$$87\frac{1}{3} + 0 = 87\frac{1}{3}$$

The Principle for Multiplying by 1 [pm 1]

$$19 \times 1 = 19$$

$$86.73 = 86.73 \times 1$$

The Principle for Multiplying by 0 [pm 0]

$$7 \times 0 = 0$$

$$0 = 318 \times 0$$

EXERCISES

A. Each of the following sentences is an instance of one of the principles for the numbers of arithmetic. Tell which principle.

1. $7 + 0 = 7$
2. $3 \times 1 = 3$
3. $4 + 1 = 1 + 4$
4. $5 \times 8 = 8 \times 5$
5. $0 = 9 \times 0$
6. $17 = 17 \times 1$
7. $4 \times (5 + 8) = (5 + 8) \times 4$
8. $(5 + 8) \times 4 = (5 \times 4) + (8 \times 4)$
9. $3.59 \times 8.61 \times 7.32 = 3.59 \times (8.61 \times 7.32)$
10. $758 \times (321 + 684) = 758 \times 321 + 758 \times 684$
11. $67 \times 531 + 33 \times 531 = (67 + 33) \times 531$
12. $(85 + 3) \times (17 + 12) = (85 + 3) \times 17 + (85 + 3) \times 12$
13. $8 \times (17 \times 9) + 12 \times (17 \times 9) = (8 + 12) \times (17 \times 9)$
14. $(97 + 35) \times (9 + 2) = (9 + 2) \times (97 + 35)$
15. $\frac{94}{35} \times 1 = 1 \times \frac{94}{35}$
16. $(\frac{2}{3} + \frac{1}{2}) \times 6 = \frac{2}{3} \times 6 + \frac{1}{2} \times 6$

[More exercises are in Part G, Supplementary Exercises.]

* * *

As we have said earlier, the principles for the numbers of arithmetic are very useful in computing because they give you different ways of reaching the same result. This means that you have different ways of carrying out a given computational task. With practice you will learn how to pick the easiest way. Here are some examples.

<u>Example 1</u>. $493 + 39 + 7 = ?$

The uninspired way of simplifying '$493 + 39 + 7$' consists of two steps. First, add 39 to 493; second, add 7 to this sum.

<u>Hard way</u>

$$\begin{aligned} 493 + 39 + 7 &= 532 + 7 \\ &= 539. \end{aligned}$$

A more sensible approach is to observe that it is easier to add 7 to 493 than it is to add 39 to 493, and that once you've done this, it is easy to add 39 to the sum.

<u>Easy way</u>

$$\begin{aligned} 493 + 7 + 39 &= 500 + 39 \\ &= 539. \end{aligned}$$

But, wait a minute! How were you able to <u>predict</u> that you would get the same answer the easy way? You could have based your prediction on the principles. For example:

$$\begin{aligned} 493 + 39 + 7 &= 493 + (39 + 7) \quad [\text{apa}] \\ 493 + (39 + 7) &= 493 + (7 + 39) \quad [\text{cpa}] \\ 493 + (7 + 39) &= 493 + 7 + 39 \quad [\text{apa}]. \end{aligned}$$

Or, you could have used the principles like this:

$$\begin{aligned} 493 + 39 + 7 &= 7 + (493 + 39) \quad [\text{cpa}] \\ 7 + (493 + 39) &= 7 + 493 + 39 \quad [\text{apa}] \\ 7 + 493 + 39 &= 493 + 7 + 39 \quad [\text{cpa}], \end{aligned}$$

or even like this:

$$493 + 39 + 7 = 39 + 493 + 7 \quad [\text{cpa}]$$

$$39 + 493 + 7 = 39 + (493 + 7) \quad [\text{apa}]$$

$$39 + (493 + 7) = 493 + 7 + 39 \quad [\text{cpa}].$$

In any case the principles assure you that

$$493 + 39 + 7 = 493 + 7 + 39.$$

<u>Example 2</u>. $987 \times 593 + 593 \times 13 = ?$

<u>Hard way</u>

$$\begin{aligned} 987 \times 593 + 593 \times 13 &= 585291 + 7709 \\ &= 593000. \end{aligned}$$

<u>Easy way</u>

$$\begin{aligned} 593 \times (987 + 13) &= 593 \times 1000 \\ &= 593000. \end{aligned}$$

The principles justify the easy way by enabling us to show that

$$987 \times 593 + 593 \times 13 = 593 \times (987 + 13).$$

Here is one way of showing this:

$$987 \times 593 + 593 \times 13 = 987 \times 593 + 13 \times 593 \quad [\text{cpm}]$$

$$987 \times 593 + 13 \times 593 = (987 + 13) \times 593 \quad [\text{dpma}]$$

$$(987 + 13) \times 593 = 593 \times (987 + 13) \quad [\text{cpm}].$$

And, here is another way:

$$987 \times 593 + 593 \times 13 = 593 \times 987 + 593 \times 13 \quad [\text{cpm}]$$

$$593 \times 987 + 593 \times 13 = 593 \times (987 + 13) \quad [\ell\text{dpma}].$$

These examples and the exercises which follow help you to learn how to use the principles to check short cuts. With practice, the checking procedure becomes almost automatic, and even suggests short cuts.

* * *

B. Each sentence below suggests a short cut in carrying out a computation. Your job is to justify the short cut by showing that the sentence is a consequence of the principles. In connection with Example 1 above, we have given three samples of how to show that the sentence:

$$493 + 39 + 7 = 493 + 7 + 39$$

is a consequence of the principles. In Example 2, we gave two samples of how to derive the sentence:

$$987 \times 593 + 593 \times 13 = 593 \times (987 + 13)$$

from the principles. For each of the following sentences, it is enough if you give just one derivation of the sentence from the principles.

1. $5 \times (9 \times \frac{3}{5}) = \frac{3}{5} \times 5 \times 9$
2. $43 \times 31 + 31 \times 57 = 31 \times (43 + 57)$
3. $8\frac{2}{7} + \frac{3}{7} = 8 + (\frac{2}{7} + \frac{3}{7})$
4. $799 + (58 + 1) = 799 + 1 + 58$
5. $\frac{1}{2} \times (85 + 85) = \frac{1}{2} \times (1 + 1) \times 85$
6. $9 \times 75 + 75 = 75 \times (9 + 1)$
7. $5\frac{1}{5} + 3\frac{2}{5} = (5 + 3) + (\frac{1}{5} + \frac{2}{5})$
8. $9 \times 38 + 70 \times 38 + 38 \times 21 = 38 \times (21 + 9 + 70)$
9. $27 + 13 = (2 + 1) \times 10 + (7 + 3)$
   [Recall that '13' is an abbreviation for '$(1 \times 10 + 3)$'.]
10. $25 + 3 \times (7 + 25) = 25 \times (1 + 3) + 3 \times 7$

☆11. $27 \times 13 = (2 \times 1) \times (10 \times 10) + (2 \times 3 + 7 \times 1) \times 10 + 7 \times 3$

C. Simplify, using as many short cuts and doing as little writing as possible. [Be prepared to justify your short cuts on the basis of the principles.]

1.	$7 \times (8 \times \frac{4}{7})$	2.	$9 \times (\frac{2}{9} \times 8)$
3.	$16 + (27 + 4)$	4.	$75 + (18 + 25)$
5.	$88 \times 21 + 21 \times 12$	6.	$19\frac{1}{4} \times 12 + 19\frac{1}{4} \times 88$
7.	$15 \times (10 + 1) + 85$	8.	$65 + 7 \times (10 + 5)$
9.	$30 \times 31 + 70$	10.	$29 \times 25 + 25$
11.	$68 \times (\frac{1}{2} \times 1000)$	12.	84×500
13.	$67 \times 25 + 33 \times 25$	14.	$(18 \times 75) \times (93 \times 0)$
15.	102% of 30	16.	$0.77 \times 0.01 + 1.23 \times \frac{1}{100}$
17.	$\frac{3 \times 9 + 8 \times 9}{11}$	18.	$\frac{7}{7 \times 15 + 12 \times 7}$
19.	$\frac{5 \times 7 + 4 \times 7}{9 \times 3 + 4 \times 9}$	20.	$\frac{6 \times 5 + 7 \times 5}{5 \times 8 + 25}$

[More exercises are in Part H, Supplementary Exercises.]

D. Suppose you want to solve the problem:

$$39 \times 83 + 39 \times 17 = ?$$

You know a short cut which is based on the left distributive principle for multiplication over addition. Remember Stan and Al? Suppose Stan hasn't told Al about the left distributive principle. Could Al justify the short cut on the basis of the other principles? In other words, could Al derive the sentence:

$$39 \times 83 + 39 \times 17 = 39 \times (83 + 17)$$

from just the principles mentioned on page 1-54? How would he do it?

1.07 Principles for the real numbers. --The principles you have learned about in the last section refer to the system of numbers of arithmetic. It is natural to ask if there are similar principles for the system of real numbers. If we asked this question about just the nonnegative real numbers, the answer would be 'yes' [Why?]. But, the system of real numbers includes the negative numbers as well; so, for example, the question whether multiplying by a real number distributes over adding requires some investigation.

(I) For each sentence below, simplify both sides and label the sentence 'True' or 'False'.

(a) $^{+}3 + ^{-}2 = ^{-}2 + ^{+}3$

(b) $(^{+}6 + ^{-}4) + ^{+}7 = ^{+}6 + (^{-}4 + ^{+}7)$

(c) $(^{+}7 + ^{-}12) \times ^{-}3 = ^{+}7 \times ^{-}3 + ^{-}12 \times ^{-}3$

(d) $(^{+}6 \times ^{-}4) \times ^{+}7 = ^{+}6 \times (^{-}4 \times ^{+}7)$

(e) $^{+}3 \times ^{-}2 = ^{-}2 \times ^{+}3$

(f) $(^{-}3 \times ^{-}5) + ^{+}4 = (^{-}3 + ^{+}4) \times (^{-}5 + ^{+}4)$

(g) $^{-}5 + ^{-}6 = ^{-}6 + ^{-}5$

(h) $^{-}3 + (^{-}12 + ^{-}5) = (^{-}3 + ^{-}12) + ^{-}5$

(i) $^{-}6 \times ^{+}4 + ^{-}8 \times ^{+}4 = (^{-}6 + ^{-}8) \times ^{+}4$

(j) $(^{+}9 + ^{-}3) \times ^{-}5 = ^{+}9 + (^{-}3 \times ^{-}5)$

(k) $^{-}3 \times ^{-}12 \times ^{-}5 = ^{-}3 \times (^{-}12 \times ^{-}5)$

(l) $^{-}5 \times ^{-}6 = ^{-}6 \times ^{-}5$

(m) $^{-}2 + ^{+}4 + ^{-}6 = ^{-}2 + (^{+}4 + ^{-}6)$

(n) $^{-}2 \times (^{+}4 \times ^{-}6) = (^{-}2 \times ^{+}4) \times ^{-}6$

(II) Classify as many as possible of the true sentences in (I) as instances of these five principles <u>for real numbers</u>.

Commutative principle for addition

Commutative principle for multiplication

Associative principle for addition

Associative principle for multiplication

Distributive principle for multiplication over addition

(III) Make up one more instance of each principle mentioned in (II) for real numbers. How many of your five new sentences are true?

Correct lists for Exercise II, page 1-60, might have appeared as follows.

The Commutative Principles for Addition and Multiplication

$${}^{+}3 + {}^{-}2 = {}^{-}2 + {}^{+}3 \qquad {}^{+}3 \times {}^{-}2 = {}^{-}2 \times {}^{+}3$$

$${}^{-}5 + {}^{-}6 = {}^{-}6 + {}^{-}5 \qquad {}^{-}5 \times {}^{-}6 = {}^{-}6 \times {}^{-}5$$

The Associative Principles for Addition and Multiplication

$$({}^{+}6 + {}^{-}4) + {}^{+}7 = {}^{+}6 + ({}^{-}4 + {}^{+}7) \qquad ({}^{+}6 \times {}^{-}4) \times {}^{+}7 = {}^{+}6 \times ({}^{-}4 \times {}^{+}7)$$

$${}^{-}3 + ({}^{-}12 + {}^{-}5) = ({}^{-}3 + {}^{-}12) + {}^{-}5 \qquad {}^{-}3 \times {}^{-}12 \times {}^{-}5 = {}^{-}3 \times ({}^{-}12 \times {}^{-}5)$$

$${}^{-}2 + {}^{+}4 + {}^{-}6 = {}^{-}2 + ({}^{+}4 + {}^{-}6) \qquad {}^{-}2 \times ({}^{+}4 \times {}^{-}6) = ({}^{-}2 \times {}^{+}4) \times {}^{-}6$$

The Distributive Principle for Multiplication over Addition

$$({}^{+}7 + {}^{-}12) \times {}^{-}3 = {}^{+}7 \times {}^{-}3 + {}^{-}12 \times {}^{-}3$$

$${}^{-}6 \times {}^{+}4 + {}^{-}8 \times {}^{+}4 = ({}^{-}6 + {}^{-}8) \times {}^{+}4$$

EXERCISES

A. 1. Make up two more instances of each of the five principles and check each of your ten sentences.

2. The fact that the real numbers have these same five properties as the numbers of arithmetic is important because they give us short cuts in working with the real numbers. The other properties of the numbers of arithmetic--those expressed by the principle for adding 0 and the principles for multiplying by 1 and by 0--also hold for the real numbers. That is, there is a *principle for adding the real number 0*, a *principle for multiplying by the real number ${}^{+}1$*, and a *principle for multiplying by the real number 0*. Make up three instances of each of these three real number principles and check each of the nine sentences.

3. Is there a left distributive principle for multiplication over addition for the system of real numbers? What must be your answer if you accept the five principles mentioned at the top of this page?

B. For each of the following sentences fill in the blank to make it true, and tell what principle the true sentence illustrates.

1. $^{+}5 \times \_\_\_\_ = {}^{-}12 \times {}^{+}5$ 2. $^{-}3 \times {}^{-}7 = \_\_\_\_ \times {}^{-}3$

3. $({}^{-}3 + {}^{+}7) \times {}^{+}5 = {}^{-}3 \times {}^{+}5 + \_\_\_\_ \times {}^{+}5$

4. $^{-}6 \times ({}^{+}8 \times \_\_\_) = ({}^{-}6 \times {}^{+}8) \times {}^{+}15$

5. $^{+}9 + {}^{-}3 + {}^{+}5 = {}^{+}9 + (\_\_\_ + {}^{+}5)$

6. $^{-}2 \times {}^{-}7 + {}^{-}2 \times \_\_\_ = {}^{-}2 \times ({}^{-}7 + {}^{+}17)$

7. $^{+}9 \times \_\_\_ = 0$ 8. $^{-}3 \times \_\_\_ = {}^{-}3$

9. $^{-}5 + \_\_\_ = {}^{-}5$ 10. $^{+}5 \times {}^{-}7 \times \_\_\_\_ = 0$

11. $0 = \_\_\_\_ \times 0$ 12. $\_\_\_\_ + 0 = 0$

13. $4 \times (7 + \_\_\_) = 4 \times 7 + 4 \times \_\_\_\_$

14. $^{-}25 = \_\_\_\_ + 0$ 15. $^{-}25 = \_\_\_\_ \times 0$

C. Use the principles for the real numbers to help you simplify the following numerical expressions. Do as little writing as you can.

1. $({}^{-}18\frac{1}{2} + 85) + 3\frac{1}{2}$ 2. $(212 + {}^{-}473) + 473$

3. $7 \times 8 + 8 \times {}^{-}9$ 4. $^{-}7 \times 8 + 2 \times {}^{-}7$

5. $({}^{-}185 \times \frac{1}{3}) \times 3$ 6. $(\frac{^{-}1}{5} \times 5.792) \times {}^{-}5$

7. $({}^{-}793 \times \frac{3}{5}) \times \frac{5}{3}$ 8. $^{-}12 + 876 + 512$

9. $^{-}18 \times 57 + 57 \times 68$ 10. $^{-}3 \times 15789.6 + 15789.6 \times 13$

11. $^{-}16\frac{1}{2} + 86 + {}^{-}3\frac{1}{2}$ 12. $^{-}972.75 \times {}^{-}37 + {}^{-}27.25 \times {}^{-}37$

13. $(892 \times \frac{1}{2}) \times ({}^{-}37 \times 0) \times 18$ 14. $27 \times {}^{-}117 + 17 \times 27$

15. $^{-}8 + ({}^{+}3 + {}^{+}8)$ 16. $({}^{-}3 + {}^{-}7) + ({}^{+}3 + {}^{+}7)$

17. $^{-}453 + ({}^{+}624 + {}^{+}453)$ 18. $({}^{-}587 + {}^{-}426) + ({}^{+}587 + {}^{+}426)$

19. $\frac{1}{16} \times 45.678 \times 16 + {}^{-}45.678$

20. $891.23 \times 386.9 + (\frac{1}{7} \times {}^{-}891.23) \times (7 \times {}^{+}386.9)$

EXPLORATION EXERCISES

A. Guess the number.

1. Multiply it by 8, and you get 24.
2. Double it, and you get 18.
3. Add 10 to it, and you get 17.
4. Add 23 to it, subtract 3 from the sum, and you get 60.
5. Add 12 to it, subtract 12 from the sum, and you get 8.
6. Multiply it by 5, multiply the product by 2, and you get 40.
7. Multiply it by 7, multiply the product by $\frac{1}{7}$, and you get 10.
8. Multiply it by 9, multiply the product by $\frac{1}{9}$, and you get 4.
9. Multiply it by $\frac{1}{5}$, multiply the product by 5, and you get 20.
10. Add 98 to it, subtract 98 from the sum, and you get 45.
11. Add 974 to it, subtract 974 from the sum, and you get 283.
12. Add $81\frac{2}{3}$ to it, subtract $81\frac{2}{3}$ from the sum, and you get $57\frac{1}{8}$.
13. Multiply it by $\frac{1}{78}$, multiply the product by 78, and you get 253.
14. Add 2.7 to it, subtract 2.7 from the sum, and you get 185.9.

B. Fill in the blanks to make true sentences.

1. (___ + 4) - 4 = 7	2. (___ + 9) - 9 = 20
3. (___ + 8) - 8 = 12	4. (___ + 5) - 5 = 900
5. (___ + 51) - 51 = 9	6. (___ + 79) - 79 = 8423
7. (7 + 5) - 5 = ____	8. (19 + 3) - 3 = ____
9. (29 + 2) - 2 = ____	10. (876 + 15) - 15 = ____
11. (82 + 769) - 769 = ____	12. (2431 + 1893) - 1893 = ____
13. (9 + 8) - ___ = 9	14. (73 + 49) - ____ = 73

(continued on next page)

15. $(57 + 24) - \_\_\_ = 57$

16. $(68 + 25) - \_\_\_ = 25$

17. $(6 + \_\_\_) - 7 = 6$

18. $(93 + \_\_\_) - 15 = 93$

19. $(19 + \_\_\_) - 3 = 19$

20. $(72 + \_\_\_) - 72 = 72$

21. $(9 + \_\_\_) - \_\_\_ = 9$

22. $(34 + \_\_\_) - \_\_\_ = 34$

23. $(53 + \_\_\_) - \_\_\_ = 53$

24. $(117 + \_\_\_) - \_\_\_ = 117$

25. $(\_\_\_ + 5) - 5 = \_\_\_$

26. $(\_\_\_ + 71) - 71 = \_\_\_$

27. $(\_\_\_ + 48) - 48 = \_\_\_$

28. $(\_\_\_ + 5\frac{1}{4}) - 5\frac{1}{4} = \_\_\_$

29. If you add 7 to a number, you can get back the number by subtracting ____ from the sum.

    Example 1. $91 + 7 = 98$ and $98 - \_\_\_ = 91$.

    Example 2. $(43 + 7) - \_\_\_ = 43$.

30. If you add 192 to a number, you can get back the number by ______________ from the sum.

    Example. $(4375 + 192) - \_\_\_ = 4375$.

31. If you want to undo the result of adding 15, subtract ____ from the sum.

    Example. $(869 + 15) - \_\_\_ = 869$.

32. Subtracting ____ undoes what adding 97 did.

33. ______________ undoes what adding 54 did.

34. ______________ undoes what adding 71 did.

C. Fill in the blanks to make true sentences.

1. $(\_\_\_ \times 5) \times \frac{1}{5} = 3$

2. $(\_\_\_ \times 7) \times \frac{1}{7} = 4$

3. $(\_\_\_ \times 9) \times \frac{1}{9} = 2$

4. $(\_\_\_ \times 35) \times \frac{1}{35} = 86$

5. $(\_\_\_ \times \frac{1}{4}) \times 4 = 12$

6. $(\_\_\_ \times \frac{1}{7}) \times 7 = 42$

7. $(\_\_\_ \times \frac{1}{8}) \times 8 = 40$

8. $(\_\_\_ \times \frac{1}{11}) \times 11 = 99$

9. $(6 \times 3) \times \frac{1}{3} =$ ____

10. $(7 \times 2) \times \frac{1}{2} =$ ____

11. $(8 \times 5) \times \frac{1}{5} =$ ____

12. $(173 \times 84) \times \frac{1}{84} =$ ____

13. $(15 \times \frac{1}{3}) \times 3 =$ ____

14. $(36 \times \frac{1}{9}) \times 9 =$ ____

15. $(8 \times \frac{1}{9}) \times 9 =$ ____

16. $(92 \times \frac{1}{79}) \times 79 =$ ____

17. $(6 \times 0.5) \times$ ____ $= 6$

18. $(72 \times 0.125) \times$ ____ $= 72$

19. $(63 \times \frac{1}{7}) \times$ ____ $= 63$

20. $(56 \times \frac{1}{13}) \times$ ____ $= 56$

21. $(51 \times 9) \times$ ____ $= 51$

22. $(38 \times 7) \times$ ____ $= 38$

23. $(85 \times 11) \times$ ____ $= 85$

24. $(77 \times 6) \times$ ____ $= 6$

25. $(29 \times 17) \times$ ____ $= 29$

26. $(583 \times 0) \times$ ____ $= 583$

27. $(81 \times$ ____ $) \times 3 = 81$

28. $(751 \times$ ____ $) \times 19 = 751$

29. $(52 \times$ ____ $) \times \frac{1}{8} = 52$

30. $(847 \times$ ____ $) \times \frac{1}{19} = 847$

31. $(48 \times$ ____ $) \times$ ____ $= 48$

32. $(31 \times$ ____ $) \times$ ____ $= 31$

33. $($ ____ $\times 7) \times \frac{1}{7} =$ ____

34. $($ ____ $\times \frac{1}{8}) \times 8 =$ ____

35. If you multiply a number by 7, you can get back the number by multiplying the product by ____.

Example 1. $5 \times 7 = 35$ and $35 \times$ ____ $= 5$.

Example 2. $(31 \times 7) \times$ ____ $= 31$.

36. If you multiply a number by $\frac{1}{6}$, you can get back the number by multiplying the product by ____.

Example 1. $12 \times \frac{1}{6} = 2$ and $2 \times$ ____ $= 12$.

Example 2. $(54 \times \frac{1}{6}) \times$ ____ $= 54$.

[Note: Since $2 \times 0.5 = 1$, 0.5 is called the reciprocal of 2. Since $0.5 \times 2 = 1$, 2 is called the reciprocal of 0.5. In general, pairs of numbers whose product is 1 are called reciprocals, and each is the reciprocal of the other. What is the reciprocal of 9? Of $\frac{1}{9}$? Of 0.37? Of $\frac{2}{3}$? Of 1? Is there a number of arithmetic which does not have a reciprocal? What is the reciprocal of 0? $0 \times ? = 1$.]

(continued on next page)

37. If you multiply a number by ____, you can get back the number by multiplying the product by the reciprocal of 53.

38. If you multiply a number by 12, you can get back the number by multiplying the product by __________.

39. If you want to undo the result of multiplying by ______, multiply the product by the reciprocal of 17.

<u>Example</u>. $(93 \times \_\_\_\_) \times \frac{1}{17} = 93.$

40. Multiplying by ____ undoes what multiplying by the reciprocal of 17 did.

<u>Example</u>. $(42 \times \frac{1}{17}) \times \_\_\_\_\_ = 42.$

41. Multiplying by __________________ undoes what multiplying by 9 did.

42. Multiplying by _______ undoes what multiplying by the reciprocal of 9 did.

43. Dividing by ______ undoes what multiplying by 7 did.

44. Multiplying by ______ undoes what multiplying by 7 did.

45. Multiplying by the ______________ of ______ undoes what multiplying by 7 did.

46. Dividing by _______ <u>does</u> what multiplying by the reciprocal of _____ does.

1.08 <u>Inverse operations</u>.--Here is a grade school subtraction problem:

$$13 - 4 = ?$$

Everyone knows that the result is 9. But, let's look a bit more deeply into how we know that this is the correct answer.

In Part B of the preceding Exploration Exercises you learned that

subtracting 4 undoes what adding 4 did.

This statement tells us what we mean by 'subtracting 4'. Subtracting 4 is just the operation you carry out to undo what adding 4 did. So, if you wish to subtract 4 from 13, you must try to find the number to which 4 was added to get 13. This number is 9, since 9 + 4 = 13. So, we say that 13 - 4 = 9.

Let us rewrite the subtraction statement:

$$13 - 4 = 9$$

as:

$$(9 + 4) - 4 = 9.$$

Do you see from this last statement that subtracting 4 undoes what adding 4 did? A shorter way of expressing this idea is to say that

subtracting 4 is the inverse of adding 4.

Similarly, we can say that

$$(3582 + 649) - 649 = 3582$$

because

subtracting 649 is the inverse of adding 649.

OPERATIONS

Let us examine a statement such as:

subtracting 4 is the inverse of adding 4

and see what it tells us. It says that 'subtracting 4' and 'the inverse of adding 4' are names of the same thing. What is this thing? 'subtracting 4' names an operation, and so does 'adding 4'. Let's look at these operations.

When you add 4, you start with a number and get to a number. For example, you may start with 3 and end with 7, start with 0 and end with 4, start with 15 and end with 19, start with 937 and end with 937 + 4. You can think of the operation adding 4 as the whole set of such pairs of numbers. Here is a list of just some of these pairs. Would it be possible to list all of them? Could the pair (9, 13)

(3, 7) (0, 4) (15, 19) (937, 937 + 4)

(19, 23) (81, 85) (52, 56) (45, 49)

$(1\frac{2}{3}, 5\frac{2}{3})$ (7.5, 11.5) (9.03, 9.03 + 4) (7, 11)

. . .

be included in this list? How about (40, 45)? How about (11, 7)?

Notice that (11, 7) could not be included in this list because $11 + 4 \neq 7$. But, there is a related operation to which (11, 7) <u>does</u> belong. What is it?

Here is a list of some of the pairs which belong to the operation subtracting 4. [As before, it would not be possible to list all the pairs which belong to the operation.]

(11, 7) (7, 3) (4, 0) (19, 15) (937 + 4, 937)

(23, 19) (85, 81) (56, 52) (49, 45)

$(5\frac{2}{3},\ 1\frac{2}{3})$ (11.5, 7.5) (9.03 + 4, 9.03)

. . .

Examine the two lists we have given. Suppose someone put another pair into the first list. Would that suggest a pair which you could put into the second list? Give three more pairs which you could put into the list for adding 4. Give three which you could put into the list for subtracting 4.

EXERCISES

<u>A</u>. Here is a list of some of the pairs of numbers which belong to a certain operation. Alongside it, make a list of some of the pairs which belong to the <u>inverse</u> of that operation.

$(3\frac{1}{5},\ 11\frac{3}{5})$ $(9\frac{2}{5},\ 17\frac{4}{5})$

$(\frac{4}{5},\ 9\frac{1}{5})$

$(11\frac{1}{10},\ 19\frac{1}{2})$ $(7\frac{1}{3},\ 15\frac{11}{15})$

$(101\frac{1}{4},\ 109\frac{13}{20})$ (35.9, 44.3)

.
.
.

Can you guess what these two operations are called?

B. Make two lists of pairs of numbers which illustrate that

subtracting 7 is the inverse of adding 7.

C. 1. Here is a list of eleven pairs which belong to a certain operation. Can you guess what operation this is?

(7, 35) (2, 10) (1, 5)

(2.3, 11.5) (0, 0) (3, 15)

(5, 25) (9, 45) (1000, 5000) (4, 20)

(6, 30)

. . .

2. Make a list of eleven pairs which belong to the operation dividing by 5.

3. What do your answers to Exercises 1 and 2 suggest about dividing by 5 and the operation referred to in Exercise 1?

4. Make a list of eleven pairs which belong to the operation multiplying by the reciprocal of 5.

5. What do your answers to Exercises 2 and 4 suggest about the operations dividing by 5 and multiplying by the reciprocal of 5?

* * *

Here are two lists of pairs which belong to operations. Note that the second operation is the inverse of the first.

adding 5	subtracting 5
(0, 5) (15, 20)	(5, 0) (20, 15)
(31, 36) $(4\frac{1}{2}, 9\frac{1}{2})$	(36, 31) $(9\frac{1}{2}, 4\frac{1}{2})$
(7, 12) (91, 96)	(12, 7) (96, 91)
(40, 45) (27, 32)	(45, 40) (32, 27)
(22, 27) (10, 15)	(27, 22) (15, 10)
(2.41, 7.41)	(7.41, 2.41)
(36, 41) (96, 101)	(41, 36) (101, 96)
(1004, 1009)	(1009, 1004)
(2, 7) (45, 50)	(7, 2) (50, 45)
⋮	⋮

How could we use them to do a problem? Consider the problem:

$$91 + 5 = ?$$

This is a problem in adding 5. Go to the adding 5 list, and look for the pair whose first number is 91. What is the second number in this pair? Is it the result of adding 5 to 91?

Now, try a subtraction problem:

$$45 - 5 = ?$$

This is a problem in subtracting 5. Go to the subtracting 5 list, and look for the pair whose first number is 45. What is the second number in the pair? Is it what you get when you subtract 5 from 45?

Use these lists to solve the following problems [even though you can do the problems without the lists]:

(1) $15 + 5 = ?$

(2) $2.41 + 5 = ?$

(3) $96 - 5 = ?$

(4) $27 - 5 = ?$

(5) $7 + 5 = ?$

(6) $36 - 5 = ?$

* * *

D. Here is a list of pairs which belong to the operation adding 19.

(3, 22) (5, 24) (7, 26) (2, 21) (0, 19) (26, 45)

(21, 40) (6.2, 25.2) (8.4, 27.4) ($2\frac{1}{2}$, $21\frac{1}{2}$) (19, 38)

(181, 200) (25.2, 44.2) (162, 181) (27.4, 46.4)

. . .

Use just this list to solve the following problems.

1. $5 + 19 = ?$
2. $181 - 19 = ?$
3. $26 - 19 = ?$
4. $27.4 - 19 = ?$

E. Here is a list of pairs which belong to the operation adding 9734.62.

(15578.14, 25312.76) (3205.91, 12940.53)

(5843.52, 15578.14) (12940.53, 22675.15) (6824.37, 16558.99)

(11972.87, 21707.49) (2238.25, 11972.87)

(16558.99, 26293.61)

. . .

Use just this list to solve the following problems.

1. $15578.14 - 9734.62 = ?$
2. $11972.87 - 9734.62 = ?$
3. $16558.99 - 9734.62 = ?$
4. $12940.53 - 9734.62 = ?$

F. Here is a list of pairs which belong to the operation multiplying by 789.

(59, 46551) (0, 0) (743, 586227) (46551, 36728739)

(586227, 462533103) (156, 123084) (1, 789)

(123084, 97113276) (789, 622521)

• • •

Use this list to solve the following problems.

1. $586227 \div 789 = ?$
2. $123084 \times \frac{1}{789} = ?$
3. $789 \div 789 = ?$
4. $46551 \times$ the reciprocal of $789 = ?$

G. Think of the set of all pairs of numbers of arithmetic in which the first number is the same as the second number. What operation is this?

EXPLORATION EXERCISES

A. Guess the number.

1. Add $^+3$ to it, and you get $^+7$.
2. Add $^-4$ to it, and you get $^+2$.
3. Add $^+5$ to it, and you get $^-3$.
4. Add $^-7$ to it, add $^+7$ to the sum, and you get $^+4$.
5. Add $^-9$ to it, add $^+9$ to the sum, and you get $^-1$.
6. Add $^+5$ to it, add $^-5$ to the sum, and you get $^-6$.
7. Add $^-2$ to it, add $^+2$ to the sum, and you get $^-9$.
8. Add $^+3$ to it, add $^-3$ to the sum, and you get 0.
9. Add $^-10$ to it, add $^+10$ to the sum, and you get $^-173$.
10. Add 0 to it, add 0 to the sum, and you get $^+286$.

B. Fill in the blanks to make true sentences.

1. $(\underline{\quad} + {}^{+}2) + {}^{-}2 = {}^{+}5$
2. $(\underline{\quad} + {}^{+}3) + {}^{-}3 = {}^{-}1$
3. $(\underline{\quad} + {}^{-}4) + {}^{+}4 = {}^{-}8$
4. $(\underline{\quad} + {}^{-}8) + {}^{+}8 = {}^{+}4$
5. $(\underline{\quad} + {}^{+}6) + {}^{-}6 = {}^{-}73$
6. $(\underline{\quad} + {}^{-}51) + {}^{+}51 = {}^{+}824$
7. $({}^{+}7 + {}^{-}3) + {}^{+}3 = \underline{\quad}$
8. $({}^{+}12 + {}^{+}5) + {}^{-}5 = \underline{\quad}$
9. $({}^{-}3 + {}^{+}8) + {}^{-}8 = \underline{\quad}$
10. $({}^{-}7 + {}^{-}15) + {}^{+}15 = \underline{\quad}$
11. $({}^{+}9.3 + {}^{-}2.1) + {}^{+}2.1 = \underline{\quad}$
12. $({}^{-}73 + {}^{-}58) + {}^{+}73 = \underline{\quad}$
13. $({}^{+}14 + {}^{-}35) + \underline{\quad} = {}^{+}14$
14. $({}^{-}17 + {}^{+}21) + \underline{\quad} = {}^{-}17$
15. $({}^{-}31 + {}^{+}65) + \underline{\quad} = {}^{-}31$
16. $({}^{+}72 + {}^{+}100) + \underline{\quad} = {}^{+}100$
17. $({}^{+}19 + \underline{\quad}) + \underline{\quad} = {}^{+}19$
18. $({}^{-}35 + \underline{\quad}) + \underline{\quad} = {}^{-}35$
19. $(\underline{\quad} + {}^{+}72) + {}^{-}72 = \underline{\quad}$
20. $(\underline{\quad} + {}^{-}18) + {}^{+}18 = \underline{\quad}$
21. $(\underline{\quad} + {}^{-}57) + {}^{+}57 = \underline{\quad}$
22. $(\underline{\quad} + 0) + 0 = \underline{\quad}$

23. If you add ${}^{+}8$ to a number, you can get back the number by adding ____ to the sum.

    Example 1. ${}^{-}2 + {}^{+}8 = {}^{+}6$ and ${}^{+}6 + \underline{\quad} = {}^{-}2$.

    Example 2. $({}^{-}384 + {}^{+}8) + \underline{\quad} = {}^{-}384$.

24. If you add ${}^{-}5$ to a number, you can get back the number by adding ____ to the sum.

    Example 1. ${}^{-}11 + {}^{-}5 = {}^{-}16$ and ${}^{-}16 + \underline{\quad} = {}^{-}11$.

    Example 2. $({}^{+}384 + {}^{-}5) + \underline{\quad} = {}^{+}384$.

[Note: Since ${}^{+}73 + {}^{-}73 = 0$, ${}^{-}73$ is called the opposite of ${}^{+}73$. Since ${}^{-}73 + {}^{+}73 = 0$, ${}^{+}73$ is called the opposite of ${}^{-}73$. In general, pairs of real numbers whose sum is 0 are called opposites, and each is the opposite of the other. How do we know that ${}^{-}10$ is the opposite of ${}^{+}10$? Because our rule for addition tells us that ${}^{+}10 + {}^{-}10 = 0$. Each real number has an opposite. What is the opposite of 0?]

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25. If you add ____ to a number, you can get back the number by adding <u>the</u> <u>opposite</u> <u>of</u> <u>$^{+}3$</u> to the sum.

26. If you add ____ to a number, you can get back the number by adding <u>the</u> <u>opposite</u> <u>of</u> <u>$^{-}3$</u> to the sum.

27. If you add $^{-}9$ to a number, you can get back the number by adding the opposite of _____ to the sum.

28. If you add $^{+}9$ to a number, you can get back the number by adding the opposite of _____ to the sum.

29. If you want to undo the result of adding _____, add the opposite of $^{-}7$ to the sum.

<u>Example 1</u>. $(^{+}36 + \_\_\_\_) + {}^{+}7 = {}^{+}36$.

<u>Example 2</u>. $(^{+}36 + \_\_\_\_)$ + the opposite of $^{-}7 = {}^{+}36$.

30. Adding _____ undoes what adding $^{-}5$ did.

<u>Example</u>. $(^{+}42 + {}^{-}5) + \_\_\_\_ = {}^{+}42$.

31. Adding the opposite of ____ undoes what adding $^{-}5$ did.

32. Adding the opposite of ____ undoes what adding $^{+}11$ did.

33. Adding the opposite of ____ is the inverse of adding $^{-}17$.

34. Adding the opposite of ____ is the inverse of adding $^{+}7$.

35. Adding ________________ is the inverse of adding $^{-}2$.

36. The inverse of adding $^{-}7$ is adding ________________.

<u>Example</u>. $(^{-}382 + {}^{-}7) + \_\_\_\_\_ = {}^{-}382$.

37. The inverse of adding $^{+}24$ is adding ________________.

<u>Example</u>. $(^{+}15 + {}^{+}24) + \_\_\_\_\_\_ = {}^{+}15$.

38. The inverse of adding a real number is adding __________.

1.09 <u>Subtraction of real numbers</u>.--As with numbers of arithmetic, we shall use the word 'subtracting' in naming the operation which is the inverse of adding a given real number. So, for example, we shall say that

subtracting $^-5$ is the inverse of adding $^-5$

and that

subtracting $^+4$ is the inverse of adding $^+4$.

Now, let us solve a subtraction problem:

$$^+9 - {}^-5 = ?$$

This is a problem in subtracting $^-5$. We can solve this problem by thinking in terms of the operation adding $^-5$ and its inverse.

adding $^-5$

$(^+7, {}^+2)$	$(^+8, {}^+3)$
$(^-2, {}^-7)$	$(^-3, {}^-8)$
$(^+4, {}^-1)$	$(^+10, {}^+5)$
$(^+12, {}^+7)$	$(^+14, {}^+9)$
$(^+27, {}^+22)$	$(^+1, {}^-4)$
$(^-7, {}^-12)$	$(^+9, {}^+4)$
$(^+22, {}^+17)$	$(^+2, {}^-3)$
⋮	

subtracting $^-5$

$(^+2, {}^+7)$	$(^+3, {}^+8)$
$(^-7, {}^-2)$	$(^-8, {}^-3)$
$(^-1, {}^+4)$	$(^+5, {}^+10)$
$(^+7, {}^+12)$	$(^+9, {}^+14)$
$(^+22, {}^+27)$	$(^-4, {}^+1)$
$(^-12, {}^-7)$	$(^+4, {}^+9)$
$(^+17, {}^+22)$	$(^-3, {}^+2)$
⋮	

We go to the list for subtracting $^-5$, and look for the pair whose first number is $^+9$. Its second number is $^+14$. So,

$$^+9 - {}^-5 = {}^+14.$$

Check these subtraction problems.

(a) $^+22 - {}^-5 = {}^+27$ (b) $^-7 - {}^-5 = {}^-2$

(c) $^-3 - {}^-5 = {}^+2$ (d) $^+4 - {}^-5 = {}^+9$

EXERCISES

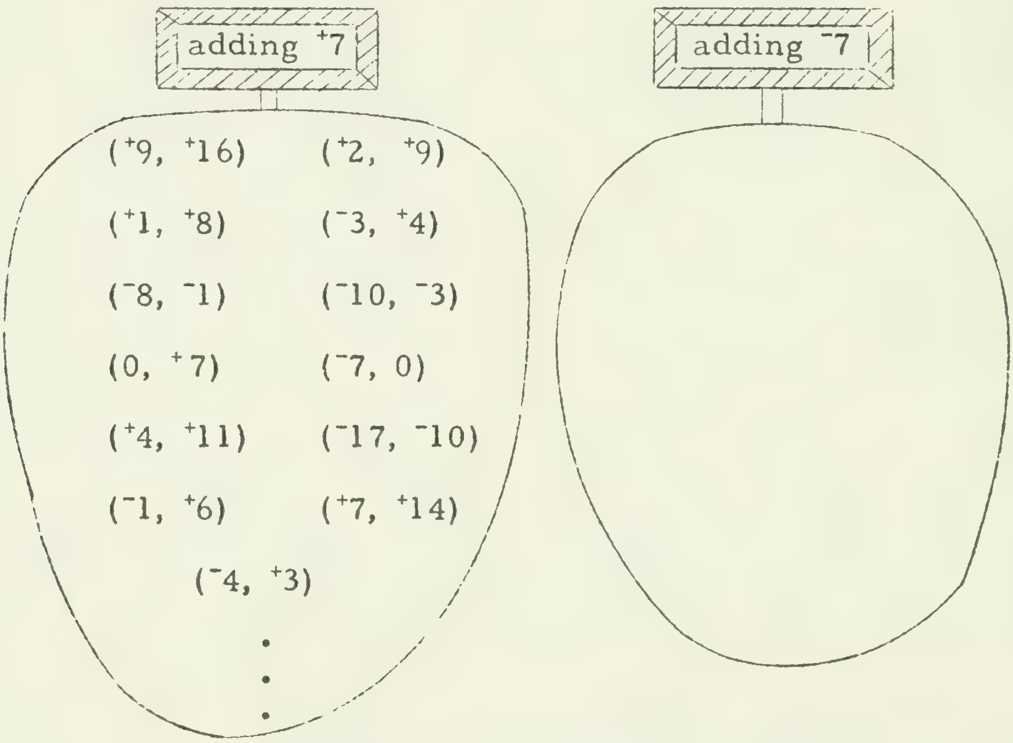
A. 1. Make a list of ten pairs which belong to the operation adding $^{-}3$.

2. Use these ten pairs to make a list of pairs which belong to the inverse of adding $^{-}3$, that is, to the operation subtracting $^{-}3$.

3. Make a list of ten pairs which belong to the operation adding $^{+}3$.

4. Examine the lists you get in Exercises 2 and 3. What do they suggest about the operations of subtracting $^{-}3$ and adding $^{+}3$? Could you have predicted this without making the lists?

B. 1. Here is a list of pairs which belong to the operation adding $^{+}7$. Use it to construct a list of pairs which belong to the operation adding $^{-}7$.



2. Solve each problem by using the list for adding $^{+}7$. Then, solve each problem again by using the list for adding $^{-}7$.

(a) $^{+}4 - {}^{+}7 = ?$ (b) $0 - {}^{+}7 = ?$

(c) $^{-}1 - {}^{+}7 = ?$ (d) $^{+}7 - {}^{+}7 = ?$

3. Now solve these problems all of which involve <u>subtracting $^{+}7$</u>.

(a) $^{+}12 - {}^{+}7 = ?$ (b) $^{+}15 - {}^{+}7 = ?$

(c) $^{+}100 - {}^{+}7 = ?$ (d) $^{-}7 - {}^{+}7 = ?$

(e) $^{-}21 - {}^{+}7 = ?$ (f) $^{+}43 - {}^{+}7 = ?$

(g) $^{-}100 - {}^{+}7 = ?$ (h) $^{-}15 - {}^{+}7 = ?$

* * *

The problem:

$$^{-}9 - {}^{+}8 = ?$$

involves subtracting $^{+}8$. Subtracting $^{+}8$ is the inverse of adding $^{+}8$. And, as you have seen, the inverse of adding $^{+}8$ is adding $^{-}8$, that is,

the inverse of adding $^{+}8$ is <u>adding the opposite of</u> $^{+}8$.

So,

subtracting $^{+}8$ is <u>the same as</u> adding the opposite of $^{+}8$.

Hence, the subtraction problem:

$$^{-}9 - {}^{+}8 = ?$$

can be <u>converted</u> into the addition problem:

$$^{-}9 + {}^{-}8 = ?$$

Therefore,

$$^{-}9 - {}^{+}8 = {}^{-}9 + {}^{-}8 = {}^{-}17.$$

* * *

<u>C</u>. Solve these subtraction problems.

<u>Sample</u>. $^{+}8 - {}^{-}3 = ?$

<u>Solution</u>. Since subtracting $^{-}3$ and adding $^{+}3$ are both the inverse of adding $^{-}3$, subtracting $^{-}3$ is the same as adding $^{+}3$. So,

$$^{+}8 - {}^{-}3 = {}^{+}8 + {}^{+}3 = {}^{+}11.$$

(continued on next page)

1. $^{+}12 - {}^{-}3 = ?$	2. $^{-}5 - {}^{+}4 = ?$	3. $^{-}11 - {}^{+}12 = ?$
4. $^{-}6 - {}^{-}9 = ?$	5. $^{+}8 - {}^{+}10 = ?$	6. $8 - 10 = ?$
7. $15 - 3 = ?$	8. $15 - {}^{-}3 = ?$	9. $^{-}15 - 3 = ?$

D. Simplify as quickly as possible.

1. $^{+}3 - {}^{+}2$	2. $^{+}12 - {}^{+}13$	3. $^{+}4 - {}^{+}17$
4. $^{+}8 - {}^{+}8$	5. $^{+}3 - {}^{+}8$	6. $4 - 17$
7. $^{+}5 - {}^{-}2$	8. $^{+}12 - {}^{-}10$	9. $^{+}12 - {}^{-}12$
10. $^{-}3 - {}^{+}4$	11. $^{-}3 - 4$	12. $^{-}5 - {}^{+}7$
13. $^{-}7 - {}^{-}4$	14. $^{-}9 - {}^{-}11$	15. $^{-}12 - {}^{-}12$
16. $^{-}14 - {}^{-}7$	17. $^{-}14 - 6$	18. $6 - {}^{-}14$
19. $8 - 2$	20. $8 - 1$	21. $8 - 0$
22. $8 - {}^{-}1$	23. $8 - {}^{-}2$	24. $8 - {}^{-}4$
25. $^{-}5 - {}^{-}3$	26. $^{-}5 - {}^{-}2$	27. $^{-}5 - {}^{-}1$
28. $^{-}5 - 0$	29. $^{-}5 - 1$	30. $^{-}5 - 2$
31. $8 - {}^{-}3$	32. $5 - {}^{-}4$	33. $12 - {}^{-}8$
34. $7 - 9$	35. $11 - 7$	36. $13 - {}^{-}5$
37. $^{-}17 - {}^{-}2$	38. $7 - {}^{-}3$	39. $9 - 18$
40. $^{-}5 - 6$	41. $^{-}5 - {}^{-}7$	42. $^{-}7 - {}^{-}5$

[More exercises are in Part I, Supplementary Exercises.]

E. 1. Suppose '9' and '12' are numerals for numbers of arithmetic. Which of the following is a numeral?

(a) $12 - 9$ (b) $9 - 12$

2. Suppose '9' and '12' are numerals for real numbers. Which of the following is a numeral?

(a) $12 - 9$ (b) $9 - 12$

F. Since subtracting a real number is precisely the same thing as adding its opposite [the principle for subtraction], every subtraction problem can be converted into an addition problem.

Sample. $5 - {}^-3 + {}^-8 + 7 - {}^+2$

Solution. $5 - {}^-3 + {}^-8 + 7 - {}^+2$

$= 5 + {}^+3 + {}^-8 + 7 + {}^-2$.

At this point you can simplify either by working from left to right:

$$\begin{aligned} & 5 + {}^+3 + {}^-8 + 7 + {}^-2 \\ = \ & 8 + {}^-8 + 7 + {}^-2 \\ = \ & 0 + 7 + {}^-2 \\ = \ & 7 + {}^-2 \\ = \ & 5, \end{aligned}$$

or by using the commutative and associative principles for addition of real numbers to change the order:

$$\begin{aligned} & 5 + {}^+3 + {}^-8 + 7 + {}^-2 \\ = \ & (5 + 3 + 7) + ({}^-8 + {}^-2) \\ = \ & 15 + {}^-10 \\ = \ & 5. \end{aligned}$$

Simplify.

1. ${}^-3 + 8 - {}^-2 - {}^+7 + {}^-3$
2. ${}^-5 + {}^-3 - {}^+7 + 9 + {}^-16$
3. ${}^+2 + {}^-4 + {}^-3 + {}^-5 - {}^-6$
4. $8 - {}^-5 + {}^-3 - {}^-6 - {}^-7 - {}^+8$
5. $3 - 15 + {}^-2 - 17 + {}^-1$
6. ${}^-1 + {}^-5 - 6 - 3 - 10 + {}^-2$
7. ${}^-2 + {}^-3 - {}^-5 - {}^-5 + 10$
8. ${}^-4 - {}^-6 + {}^-3 - 8 - {}^-9 + {}^-1$
9. $11 - 15 + 7 - 9 - 3 + 1$
10. $1 - {}^-2 - 6 - {}^-9 - {}^-3 - 7$

(continued on next page)

11. ${}^-9 - ({}^-2 + {}^+5)$ 12. ${}^-3 - ({}^-4 - {}^-6)$

13. $({}^-1 + {}^-3) - ({}^-5 - {}^-8)$ 14. $(6 - 12) - (4 - 9)$

15. ${}^-2 - ({}^-8 + {}^+2) - ({}^-3 + {}^-2) - ({}^-5 + {}^-8) + ({}^-7 - {}^-9)$

[More exercises are in *Part J, Supplementary Exercises.*]

1.10 *Opposites.* --In studying the system of numbers of arithmetic and the system of real numbers, we have noticed certain similarities. For example, we have seen that the nonnegative numbers act like the numbers of arithmetic with respect to addition. What are some other similarities?

There is, also, an important difference between the two systems. It concerns the operation of subtraction. Can you tell what this difference is?

If you pick two numbers of arithmetic, a first number and a second number, can you subtract the first number from the second number? Can you subtract 9 from 11? Can you subtract 6 from 2? Notice that there are cases in which you cannot subtract the first selected number from the second. So, we say that subtraction is not always possible in the system of numbers of arithmetic.

Is subtraction always possible in the system of real numbers? Can you subtract ${}^+9$ from ${}^+11$? ${}^+6$ from ${}^+2$? ${}^-5$ from ${}^-8$? To subtract a real number is to add its opposite. Since addition is always possible, and since each real number has an opposite, subtraction is always possible in the real number system.

An important fact about the real numbers is that, for each real number, there is a real number [the opposite of the first] which when added to the first gives the sum 0. This fact is expressed by *the principle of opposites*--a number plus its opposite is 0. You will see in Unit 2 that it follows from the principle of opposites together with other principles that if the sum of a first number and a second number is 0, the second number is *the* opposite of the first.

What real number is the opposite of ${}^-4$? It is ${}^+4$, because ${}^-4 + {}^+4 = 0$. What is the opposite of ${}^+4$? [What principle tells you that if a first number is the opposite of a second then the second is the opposite of the first?]

THE OPERATION OPPOSITING

Finding the opposite of a real number is an operation just as adding $^+4$ or multiplying by $^-2$ are operations. Here are some pairs of real numbers which belong to the operation <u>oppositing</u>.

oppositing

$(^+5, {}^-5)$ $(^-4, {}^+4)$ $(0, 0)$ $(7, {}^-7)$ $(^-9, 9)$

$(100 - 2,\ 2 - 100)$ $(^-5, {}^+5)$ $(^+4, {}^-4)$ $(^-7, 7)$ $(9, {}^-9)$

$(^+3 - {}^-2,\ {}^-2 - {}^+3)$ . . . $\left(\frac{^+3}{4}, \frac{^-3}{4}\right)$

List some pairs which belong to the <u>inverse of oppositing</u>. What can you say about oppositing and its inverse?

Just as we have signs for other operations [×, +, -, ÷], we should like to have a sign for oppositing. We might use a '☆'. Then '☆ $^+4$' would mean the opposite of $^+4$. So, we would have statements like:

$$☆\,{}^+4 = {}^-4,$$

$$☆\,{}^-11 = {}^+11,$$

$$^+4 - {}^-3 = {}^+4 + ☆\,{}^-3,$$

$$0 - {}^+5 = ☆\,{}^+5,$$

$$0 - {}^-12 = ☆\,{}^-12,$$

$$0 - (^+3 - {}^-8) = ☆\,(^+3 - {}^-8).$$

The last three examples suggest the notation that most people use for oppositing. It is just to write a minus sign. So, following this practice, we shall write '$-\,^+4$' [read as 'the opposite of positive four'], to mean the opposite of $^+4$, and '$-\,^-11$' when we mean the opposite of $^-11$. Similarly, the expression:

$$-\,(^+3 - {}^-8)$$

which is a name for the opposite of $(^{+}3 - {}^{-}8)$ can be simplified to:

$$-(^{+}11)$$

which is finally simplified to:

$$^{-}11.$$

EXERCISES

A. Use this new notation to write a name for the opposite of each given number.

1. $^{+}8 - {}^{-}7$ [Answer: $-(^{+}8 - {}^{-}7)$]

2. $^{-}3 + {}^{+}5$ 3. $^{+}5 - {}^{-}6$ 4. $^{-}2 \times {}^{-}3 + {}^{-}5$

5. $6 - 2$ 6. $9 - 15$ 7. $3 - 5 + 6 \times (5 + 3)$

B. Read aloud first, and then simplify.

Sample. $-(12 - 9) + (9 - 12)$

Solution. Read as 'the opposite of the difference of 9 from 12, plus the difference of 12 from 9'.

$$\begin{aligned} & -(12 - 9) + (9 - 12) \\ = \; & -{}^{+}3 + {}^{-}3 \\ = \; & ^{-}3 + {}^{-}3 \\ = \; & ^{-}6. \end{aligned}$$

1. $-{}^{+}8$ 2. $-{}^{-}15$ 3. $-(^{+}5 - {}^{-}7)$

4. $-(^{+}2 \times {}^{-}3)$ 5. $-(55 - 30)$ 6. $-55 + 30$

7. $-(^{-}3 \times {}^{-}7)$ 8. $-(10 - 12)$ 9. $-{}^{+}10 \div 12$

10. $-{}^{+}10 - 12$ 11. $--{}^{+}7$ 12. $-(5 - 7) + (5 - 7)$

13. $---{}^{-}3$ 14. $-{}^{-}7 + -{}^{+}7$ 15. $--(^{-}3 - -{}^{+}3)$

16. $-(93 - 97)$ 17. $--(84 - 89)$ 18. $--(100 - 101)$

19. $-(^{+}9 + {}^{-}7)$ 20. $-(^{-}9 + {}^{+}7)$ 21. $-(^{-}3 + {}^{-}6)$

22. $-(983 + 729 - 604) + (983 + 729 - 604)$

23. $(3572 - 4871) + (4871 - 3572)$

* * *

Consider the numbers

$$^{-}8 - {}^{+}4 \qquad \text{and} \qquad {}^{+}8 - {}^{-}4.$$

Is the second number, $^{+}8 - {}^{-}4$, the opposite of the first, $^{-}8 - {}^{+}4$? One way to find the answer to this question is to add the second number to the first. If the sum is 0, the answer is 'yes'. If the sum is not 0, the answer is 'no'. Let's try it.

$(^{-}8 - {}^{+}4) + ({}^{+}8 - {}^{-}4) = ({}^{-}8 + {}^{-}4) + ({}^{+}8 + {}^{+}4)$ [principle for subtraction; $^{-}4 = -{}^{+}4$; $^{+}4 = -{}^{-}4$]

$(^{-}8 + {}^{-}4) + ({}^{+}8 + {}^{+}4) = [({}^{-}8 + {}^{+}8) + {}^{-}4] + {}^{+}4$ [apa, cpa]

$[(^{-}8 + {}^{+}8) + {}^{-}4] + {}^{+}4 = [0 + {}^{-}4] + {}^{+}4$ [principle of opposites; $^{+}8 = -{}^{-}8$]

$[0 + {}^{-}4] + {}^{+}4 = 0 + ({}^{-}4 + {}^{+}4)$ [apa]

$0 + ({}^{-}4 + {}^{+}4) = 0 + 0$ [principle of opposites; $^{+}4 = -{}^{-}4$]

$0 + 0 = 0$ [principle for adding 0]

So, since

$$(^{-}8 - {}^{+}4) + ({}^{+}8 - {}^{-}4) = 0,$$

it follows that $^{+}8 - {}^{-}4$ is the opposite of $^{-}8 - {}^{+}4$. That is, that

$$^{+}8 - {}^{-}4 = -({}^{-}8 - {}^{+}4).$$

[Is $^{-}8 - {}^{+}4$ the opposite of $^{+}8 - {}^{-}4$?]

Consider a second example. Is it the case that

$$-({}^{-}98 + 35) = 98 + {}^{-}35?$$

To find the answer to this question, we can proceed as before and find out whether $({}^{-}98 + 35) + (98 + {}^{-}35) = 0$.

$({}^{-}98 + 35) + (98 + {}^{-}35) = ({}^{-}98 + 98) + (35 + {}^{-}35)$ [Why?]

$({}^{-}98 + 98) + (35 + {}^{-}35) = 0$ [Why?]

So, $(98 + {}^{-}35) = -({}^{-}98 + 35)$.

* * *

C. For each exercise, use the "adding method" and the principle of opposites to check each sentence.

1. $-(8 + 5) = {}^-8 + {}^-5$ [Hint: Show that $(8 + 5) + ({}^-8 + {}^-5) = 0$.]
2. $-(95 + {}^-27) = {}^-95 + 27$
3. $-(38 - 16) = {}^-38 + 16$
4. $-({}^-57 - {}^-9) = 57 - 9$
5. $-(24 - 30) = 30 - 24$
6. $-({}^-17 - 15) = 15 - {}^-17$
7. $-(25 + 20) = -25 + -20$
8. $-(57 - 12) = -57 + 12$
9. $-(38 - 10) = 10 - 38$
10. $-(53 \times {}^-28) = {}^-53 \times {}^-28$
11. $-74 = 74 \times {}^-1$
☆12. $-(38 + {}^-57 - {}^-76) = -38 - {}^-57 + {}^-76$
☆13. $-[(725 - 631) - (497 - 985)] = (631 - 725) - (985 - 497)$

D. Multiple-choice. [There may be more than one right answer.]

1. $12 - 3 = ?$

   (a) $3 + {}^-12$ (b) ${}^-3 + 12$ (c) $-(12 - 3)$ (d) $-(3 - 12)$

2. $5 + 4 - 10 = ?$

   (a) $5 + 6$ (b) $9 - 10$ (c) $5 - (10 - 4)$ (d) $5 + (4 - 10)$

3. $5 - 4 - 10 = ?$

   (a) $5 - (4 - 10)$ (b) $5 - (4 + 10)$

   (c) $5 - (10 - 4)$ (d) $-(10 + 4 - 5)$

4. $3 \times {}^-2$ is the opposite of ?

   (a) ${}^-3 \times 2$ (b) $-(3 \times {}^-2)$ (c) $-({}^-3 \times 2)$ (d) $-(3 \times 2)$

5. $978 \times 357 = ?$

(a) ${}^{-}978 \times {}^{-}357$ (b) $-(978 \times {}^{-}357)$ (c) $-({}^{-}978 \times 357)$

6. $-(596 - 984)$ is the opposite of ?

(a) ${}^{-}984 + 596$ (b) $-596 + 984$ (c) ${}^{-}596 + {}^{+}984$ (d) $596 - 984$

7. $-(19 + 11) = ?$

(a) $19 - 11$ (b) $-19 + -11$ (c) $11 + 19$ (d) ${}^{-}19 + {}^{-}11$

8. $-({}^{-}19 + {}^{-}11) = ?$

(a) $-{}^{-}19 + -{}^{-}11$ (b) ${}^{+}19 + {}^{+}11$ (c) ${}^{+}19 - {}^{-}11$ (d) ${}^{-}11 - {}^{-}19$

9. $-({}^{+}3 + {}^{+}5) = ?$

(a) ${}^{-}3 + {}^{-}5$ (b) $-{}^{+}3 + -{}^{+}5$

(c) ${}^{+}3 \times {}^{-}1 + {}^{+}5 \times {}^{-}1$ (d) $({}^{+}3 + {}^{+}5) \times {}^{-}1$

10. $-({}^{-}3 + {}^{+}5) = ?$

(a) ${}^{+}3 + {}^{-}5$ (b) $-{}^{-}3 + -{}^{+}5$

(c) ${}^{-}3 \times {}^{-}1 + {}^{+}5 \times {}^{-}1$ (d) $({}^{-}3 + {}^{+}5) \times {}^{-}1$

11. $-({}^{+}3 - {}^{-}5) = ?$

(a) ${}^{-}3 + {}^{-}5$ (b) $-{}^{+}3 - -{}^{-}5$

(c) ${}^{+}3 \times {}^{-}1 - {}^{-}5 \times {}^{-}1$ (d) $({}^{+}3 - {}^{-}5) \times {}^{-}1$

12. A first number is the opposite of a second number

(a) if the sum of the numbers is 0.

(b) if the opposite of the first number is the opposite of the second number.

(c) if the first number is the product of the second number by ${}^{-}1$.

(d) if the second number is the product of the first number by ${}^{-}1$.

(continued on next page)

13. The opposite of the sum of a first number and a second number is

    (a) the sum of the opposites of the numbers.

    (b) the first number minus the second number.

    (c) the second number minus the first number.

    (d) the opposite of the first number, minus the second number.

    (e) the product of the sum by $^{-}1$.

NEW NAMES FOR NEGATIVE NUMBERS

Earlier in this unit we agreed that we could shorten such names as '$^{+}9$' and '$^{+}304$' by leaving out the raised plus signs. So, we could write '9' and '304' when we intended the real numbers $^{+}9$ and $^{+}304$. Now, to each positive real number there corresponds a real number which is its opposite. And we get a name for this opposite by writing an oppositing sign to the left of a numeral for the positive number. Hence, -9 is the opposite of the real number 9. But the opposite of the positive number 9 is a negative number. Hence, a name for this negative number is '-9'. And this is the number which we have been calling '$^{-}9$'.

For most of our purposes, we shall use such names as:

$$9, \quad 35, \quad 6\frac{1}{4}, \quad 15.83$$

for positive numbers, and such names as:

$$-7, \quad -58, \quad -3\frac{1}{5}, \quad -49.7$$

for negative numbers. In other cases where we want to stress the notion of oppositing, or the fact that the nonnegative real numbers are different from the numbers of arithmetic, we shall return to our use of names like:

$$^{+}9, \quad ^{-}7, \quad ^{+}35, \quad ^{-}58, \quad ^{+}6\frac{1}{4}, \quad ^{-}3\frac{1}{5}, \quad ^{+}15.83, \quad ^{-}49.7.$$

[Here is a case in which you would want to use a raised minus sign in naming a negative number: Without using the words 'opposite' and 'negative', write a sentence which states that the opposite of seven is negative seven.]

Using the oppositing sign to name negative numbers, instead of the raised minus sign, means that we are now going to use the same sign in three ways,

(1) when naming a negative number,

(2) when naming the opposite of a real number,

and (3) when indicating a subtraction problem.

You will seldom make a mistake because of this ambiguity for you will always be able to tell when meaning (3) is intended, and since the opposite of a positive number is a negative number, confusing (1) with (2) will make no difference. Here is an expression in which the minus sign is used with different meanings:

$$-7 - - -8.$$

You can think of this as subtracting the opposite of the opposite of 8 from the opposite of 7

$$-{}^{+}7 - - -{}^{+}8,$$

in which case the expression simplifies to '$^{-}15$'. Or, you can think of it as subtracting the opposite of $^{-}8$ from $^{-}7$

$$^{-}7 - - {}^{-}8,$$

in which case, the expression again simplifies to '$^{-}15$'. Can you give two more ways of thinking of this expression?

Notice how the expression is to be read.

$$\underbrace{-7}_{\substack{\text{negative 7}\\ \text{or}\\ \text{the opposite of 7}}} \quad \underset{\text{minus}}{-} \quad \underset{\text{the opposite of}}{-} \quad \underbrace{-8}_{\substack{\text{negative 8}\\ \text{or}\\ \text{the opposite of 8}}}$$

MORE NAMES FOR POSITIVE NUMBERS

In many books [this one, too] you will see numerals like '+7', '$+3\frac{1}{2}$', and '+(4 - 3)', in which the plus sign is used in the same position as the oppositing sign. One interpretation of such a numeral is that the writer wants to emphasize that he is talking about positive numbers. [We have done this when we have written, say, '$^{+}7$' instead of the shorter '7'.] Another interpretation is that just as the minus sign in '−7' may refer to the operation oppositing, the plus sign in '+3' refers to the operation "sameing". For example,

$$+{}^{+}3 = {}^{+}3, \quad +{}^{-}8 = {}^{-}8, \quad +-3 = -3, \quad \text{and} \quad +(5 - 9) = 5 - 9.$$

When the plus sign is used in naming a positive number as it is in '+7', you could read it as 'positive', or not pronounce it at all. When it is used as a sameing sign, as it must be in '+(3 - 5)' [why 'must'?], read it as 'plus', or don't read it at all.

We have said that oppositing is the operation which takes you from a real number to its opposite. Can you describe the operation sameing? Recall the principles for the real numbers. Do any of these principles tell you about an operation which is really sameing? Describe the inverse of sameing.

EXERCISES

A. You have seen that there are several ways of naming real numbers. It is important that you become familiar with all of these ways, and that you develop skill in simplifying expressions which contain the various kinds of numerals. The expressions given below should be read and simplified.

Sample 1. $-8 - 6 + -3$

Solution. Read as 'the opposite of 8, minus 6, plus the opposite of 3', or as 'negative 8, minus 6, plus negative 3'.

Simplify, converting the subtraction to addition-of-the-opposite:

$$-8 - 6 + -3 = -8 + -6 + -3$$
$$= -14 + -3 = -17.$$

<u>Sample 2</u>. $^+8 - {}^-5 + {}^-6 - {}^+12$

<u>Solution</u>. Read as 'positive 8, minus negative 5, plus negative 6, minus positive 12'.

Simplify.

$$\begin{aligned} ^+8 - {}^-5 + {}^-6 - {}^+12 &= {}^+8 + {}^+5 + {}^-6 + {}^-12 \\ &= {}^+13 + {}^-6 + {}^-12 \\ &= {}^+7 + {}^-12 \\ &= {}^-5. \end{aligned}$$

<u>Sample 3</u>. $2 - 5 + 6 - 3 + 9 - 6 - 5$

<u>Solution</u>. Read as '2, minus 5, plus 6, minus 3, plus 9, minus 6, minus 5'.

Simplify.

$$\begin{aligned} & 2 - 5 + 6 - 3 + 9 - 6 - 5 \\ &= 2 + {}^-5 + 6 + {}^-3 + 9 + {}^-6 + {}^-5 \\ &= {}^-3 + 6 + {}^-3 + 9 + {}^-6 + {}^-5 \\ &= 3 + {}^-3 + 9 + {}^-6 + {}^-5 \\ &= 0 + 9 + {}^-6 + {}^-5 \\ &= 9 + {}^-6 + {}^-5 \\ &= 3 + {}^-5 \\ &= {}^-2. \end{aligned}$$

Here is a second method of simplifying. [Note that the associative and commutative principles for addition are used in this second method.]

We see that

$$\begin{aligned} & 2 - 5 + 6 - 3 + 9 - 6 - 5 \\ &= 2 + {}^-5 + 6 + {}^-3 + 9 + {}^-6 + {}^-5. \end{aligned}$$

So, the given expression:

$$2 - 5 + 6 - 3 + 9 - 6 - 5$$

can be thought of as naming the <u>sum</u> of positive and negative numbers. The positive numbers are 2, 6, and 9, and the negative numbers are $^-5$, $^-3$, $^-6$, and $^-5$.

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Add the positive numbers, add the negative numbers, and then add the two sums.

$$\begin{aligned} & 2 - 5 + 6 - 3 + 9 - 6 - 5 \\ = \ & (2 + 6 + 9) + (-5 + -3 + -6 + -5) \\ = \ & 17 + -19 \\ = \ & -2. \end{aligned}$$

[Note: In doing the following simplifications you may wish to put in more steps than you feel you really need. It is a good idea to do this at the beginning. Then, as you do more problems, you will discover short cuts which will enable you to do many problems without writing more than one or two steps.]

1. $9 - 3 - 7$
2. $-5 + 6 - 11$
3. $-12 - 4 + 19$
4. $-6 + 5 + 9$
5. $+7 - 3 - 4$
6. $+8 + 12 + 17$
7. $+4 - +7 + -3$
8. $+5 - -3 + -4$
9. $+5 - -6 - -5$

B. Simplify.

1. $+12 - +6 - -7 + -8 + -13 - -7$
2. $-3 + -6 - -4 + -7 - -7 + -13$
3. $3 + -7 + 3 - -7 - 5 + 12 + -3$
4. $-4 + 0 + -8 - -3 + 11 + -17 + 16$
5. $7 - 3 + -3 - +8 + 17 - -1 - -6 + 4$
6. $-10 + 17 + -3 - -10 - -7 - -3 - -1$
7. $-10 - -10 + -10 + -3 - -3 + 0 + +2 - +2$
8. $-19 + -3 - -4 + -6 + -17 + -4 - -3 - -19$
9. $0 - 7 + -3 + 7 - -8 + 9 - -10$
10. $3 - 5 + 6 - 7 + 9 + 8 - 3$
11. $0 + 5 - 6 + 8 - 0 + 9 - 5$

12. 2 - 7 + 8 - 6 + 4 + 3

13. 1 - 1 + 2 - 2 + 3 - 6 + 7

14. 10 - 8 + 7 - 5 + 6 + 8

15. 5 - 15 - 20 + 18 + 2 - 8

16. 4 + 7 - 3 - 8 + 9 - 5

17. 6 + 4 - 3 + 12 + 10 - 16

18. $5\frac{1}{2} + 9 - 2\frac{1}{2} - 7\frac{1}{2}$

19. $6\frac{1}{3} - 2\frac{1}{4} - 5\frac{1}{2} + 3\frac{1}{6} - 2\frac{2}{3}$

20. 4.83 - 10 - 3.8 + 7.9

21. 16.75 - 11.3 + 40.72 - 61.007

[More exercises are in Part K, Supplementary Exercises.]

C. Examine each of the following expressions, and tell whether it stands for a positive number, a negative number, or 0.

1. −$^{+}$5 2. −7 3. +4 4. +$^{-}$5

5. +(3 - 2) 6. +(7 - 9) 7. −(8 - 3)

8. −(4 - 7) 9. −−−5 10. +−−−5

D. Simplify.

Sample 1. (3 - 8) × (7 - 11) - (5 - 12) × (6 - 10)

Solution. (3 - 8) × (7 - 11) - (5 - 12) × (6 - 10)

= (−5 × −4) - (−7 × −4)

= 20 - 28

= −8.

Sample 2. 6 - 3 × (4 - 7)

Solution. 6 - 3 × (4 - 7)

= 6 - (3 × −3)

= 6 - −9

= 15.

1. 5 + 7 × (3 - 5)

2. −8 + −5 × (−2 + −3)

3. −9 × (5 + −2) + 7

4. −3 × (−7 - 8) + −2

5. 6 × (3 - −8) - 2 × (5 - −5)

6. (−2 - 3) × (−7 - 5) + (−3 + −8) × (−2 + 3)

(continued on next page)

7. $(5 - 8) \times (9 - 12) - (6 - 13) \times (7 - 9)$

8. $(4 - 3) \times (2 - 7) - (12 - 15) \times (8 - 5)$

9. $(6 - 1) \times (7 - 2) - (12 + 1) \times (3 - 3)$

10. $7 - 2 \times (5 - 8) - 3 \times (4 - 2) - 5 \times (6 - 7)$

11. $-2 - 6 - (3 + 5) - (7 + 2) - (8 - 3) - (-9 - 2)$

12. $+6 + -2 - 3 - 8 - 7 \times (5 - 2 - 1) + 2 \times (-3 - 6 - 2)$

13. $-2 \times [5 + -(3 - 4)] - 5 + [(7 + 3) - 6 \times (2 + 8)]$

14. $[(8 - 2) - (7 - 5)] \times +4 - [(2 - 7) - 7 \times (3 - 5)] \times 3$

15. $[(9 - 5) + (7 - 24)] \times 83 + [(24 - 7) + (5 - 9)] \times 83$

16. $(18 - 27) \times (53 - 41) + (27 - 18) \times (53 - 41)$

17. $17 - \{5 - [2 - 3 \times (4 + 1)] - 6 \times (7 - 5) + 21\}$

18. $-7 - 3 \times \{4 + 2 \times (3 - 5) - 3 \times [7 - 2 \times (4 - 10) + 5] - 7\}$

19. $5 \times (3 + -9) - 7 \times [-6 - 3 \times (2 - 3) + 4 \times (3 + 4) - 2]$

[More exercises are in Part L, Supplementary Exercises.]

1.11 Division of real numbers. --In grade school you learned that to do the division problem:

$$24 \div 6 = ?$$

you had to search for a number which when multiplied by 6 gave 24. This number is 4 because $4 \times 6 = 24$. Look at the following sentence:

$$(4 \times 6) \div 6 = 4.$$

Does it illustrate the idea that dividing by 6 is the inverse of multiplying by 6?

On the following page, at the left, is a list of pairs which belong to the operation multiplying by $^-3$. The list on the right is made by reversing each pair in the list on the left.

multiplying by $^{-}3$

$(^{+}9, ^{-}27)$ $(^{+}4, ^{-}12)$

$(^{-}3, ^{+}9)$ $(^{+}1, ^{-}3)$

$(0, 0)$ $(^{+}15, ^{-}45)$ $(^{-}1, ^{+}3)$

$(^{-}7, ^{+}21)$ $(^{+}5.1, ^{-}15.3)$

$(^{+}6, ^{-}18)$ $(^{-}5, ^{+}15)$

$(^{-}18, ^{+}54)$ $(^{-}12, ^{+}36)$

$(^{+}21, ^{-}63)$

.
.
.

$(^{-}27, ^{+}9)$ $(^{-}12, ^{+}4)$

$(^{+}9, ^{-}3)$ $(^{-}3, ^{+}1)$

$(0, 0)$ $(^{-}45, ^{+}15)$ $(^{+}3, ^{-}1)$

$(^{+}21, ^{-}7)$ $(^{-}15.3, ^{+}5.1)$

$(^{-}18, ^{+}6)$ $(^{+}15, ^{-}5)$

$(^{+}54, ^{-}18)$ $(^{+}36, ^{-}12)$

$(^{-}63, ^{+}21)$

.
.
.

Can you give a name for the operation to which the pairs in the right list belong? This operation is the inverse of multiplying by $^{-}3$.

Use the lists to solve the following problems in division.

(1) $^{+}15 \div {}^{-}3 = ?$ (2) $^{+}9 \div {}^{-}3 = ?$

(3) $^{-}18 \div {}^{-}3 = ?$ (4) $^{-}12 \div {}^{-}3 = ?$

(5) $^{+}21 \div {}^{-}3 = ?$ (6) $0 \div {}^{-}3 = ?$

Now, suppose you wanted to solve the division problem:

$$^{-}63 \div {}^{+}7 = ?$$

To solve this problem you could use a list for the operation dividing by $^{+}7$, or a list for the operation multiplying by $^{+}7$. But, even without such lists, you could still use your knowledge of multiplication to solve this division problem. You could imagine yourself searching through a long multiplying by $^{+}7$ list for a pair in which the second number was $^{-}63$. The first number in the pair gives you the answer to the division problem. What is the answer?

Do these division problems.

1. $^{+}18 \div {}^{+}3 = ?$ 2. $^{+}18 \div {}^{-}3 = ?$ 3. $^{-}18 \div {}^{+}3 = ?$

4. $^{-}18 \div {}^{-}3 = ?$ 5. $^{-}14 \div {}^{+}2 = ?$ 6. $^{-}2 \div {}^{-}1 = ?$

(continued on next page)

7. $^{-}9 \div {}^{+}3 = ?$	8. $^{-}12 \div {}^{-}4 = ?$	9. $^{+}27 \div {}^{-}3 = ?$
10. $18 \div 6 = ?$	11. $18 \div -6 = ?$	12. $24 \div -3 = ?$
13. $-24 \div -3 = ?$	14. $24 \div 3 = ?$	15. $-24 \div 3 = ?$
16. $36 \div -4 = ?$	17. $0 \div 5 = ?$	18. $0 \div -3 = ?$

WAYS OF NAMING A QUOTIENT

The quotient of a first number (dividend) by a second number (divisor) may be named by putting a divide-by sign between numerals for the numbers, the dividend numeral being placed on the left. So, the quotient of 8 by -2 is named by:

$$8 \div -2.$$

A simpler name for this quotient is:

$$-4.$$

In grade school you learned that a fraction can also be used to name a quotient. Thus, the quotient of 8 by -2 is named by:

$$8/-2 \text{ or by: } \frac{8}{-2}.$$

The part of the fraction which names the dividend is called <u>the numerator of the fraction</u>, and the part which names the divisor is called <u>the denominator of the fraction</u>. In the fraction '$\frac{8}{-2}$', the numerator is '8' and the denominator is '-2'.

EXERCISES

<u>A</u>. Simplify.

1. $12 \div 3$	2. $-17 \div 1$	3. $-6 \div -2$
4. $8 \div -2$	5. $10 \div -1$	6. $-7 \div -1$
7. $0 \div -3$	8. $9 \div 3$	9. $16 \div -4$
10. $17 \div -1$	11. $-27 \div -3$	12. $9 \div -9$
13. $\frac{-16}{8}$	14. $\frac{-21}{-7}$	15. $\frac{-33}{3}$

16. $0/{-17}$
17. $34/{-17}$
18. $-18/{-6}$
19. $\frac{18}{-3}$
20. $\frac{18}{3}$
21. $\frac{-18}{3}$
22. $\frac{4}{1}$
23. $\frac{-4}{-1}$
24. $\frac{-4}{1}$
25. $-\frac{6}{-2}$
26. $-\frac{-9}{-3}$
27. $-\frac{-12}{4}$
28. $+4/+1$
29. $4/+1$
30. $6/{-3}$
31. $\frac{-7}{1}$
32. $\frac{0}{-8}$
33. $\frac{-15}{3}$
34. $^{+}1 \div {}^{+}2$
35. $^{+}1 \div {}^{-}3$
36. $^{+}1 \div \frac{^{-}2}{3}$

[More exercises are in Part M, Supplementary Exercises.]

B. 1. Construct a list of ten pairs which belong to the operation multiplying by 0.

2. Make another list of pairs by reversing the pairs in the first list.

3. Use the lists to try to solve these problems.

(a) $6 \div 0 = ?$ (b) $\frac{-8}{0} = ?$ (c) $0 \div 0 = ?$

1.12 Comparing numbers.--Who has more stamps, Richard with 2350 or Al with 1820? This is an easy question. Richard does, because 2350 is a larger number than 1820. And, how do you know 2350 is larger than 1820? You may have decided this by remembering that in counting 1, 2, 3, etc., 1820 comes before 2350.

Another way of telling that 2350 is larger than 1820 is to recognize that you can get 2350 by adding a number [not 0] to 1820. And, as long as we are talking about numbers of arithmetic, this is a perfectly good method to use. To decide which of two numbers is the greater, just decide which of the two numbers you would have to add to in order to get the other. The one you have to add to is the smaller, and the other is the larger. [If you find that you can get one of the numbers of arithmetic by adding 0 to the other, how *many* numbers were you comparing in the first place?]

EXERCISES

A. True or false?

1. 9 is greater than 7
2. 3 is less than 17
3. 6 is greater than 0
4. 19 is less than 38 ÷ 2
5. 11.1 is greater than 11.09
6. 7.38 is greater than 7.379

B. Each of the following exercises contains a pair of numbers of arithmetic. Compare the numbers in the pair, and write your result as a sentence.

Sample. (15, 17)

Solution. I can add a number of arithmetic (not 0) to 15 to get 17. So, I'll write:

15 is less than 17.

[I could have written '17 is greater than 15'.]

1. (9, 4)
2. (16, 61)
3. (10002, 100002)
4. $(3\frac{4}{5}, 3\frac{3}{4})$
5. $(9\frac{1}{8}, 9\frac{1}{7})$
6. (0.10002, 0.100002)

*

[Note: Let's agree to abbreviate 'is greater than' by '>' and 'is less than' by '<'.]

*

7. (21, 17)
8. (93, 39)
9. (74, 73.5)
10. $(\frac{3}{17}, \frac{5}{19})$
11. $(\frac{9}{24}, \frac{30}{80})$
12. (.304, .208)

C. Fill in the blank with a numeral for a number of arithmetic to make a true sentence, and then read the sentence aloud.

1. 19 < _____
2. 34 > _____
3. _____ > 55.1
4. $6\frac{1}{8}$ < _____
5. $\frac{9}{73}$ > _____
6. _____ < .0003

COMPARING REAL NUMBERS

How shall we proceed in comparing real numbers? Is $^+5 < {}^+17$? Is $^+4 > {}^-2$? Is $^-3 < {}^-2$? Is $^-100 > {}^+1$? There are some clues around which can help us understand why mathematicians do decide as they do about which of two real numbers is the larger.

Consider the sentence:

$$4 < 19.$$

Is this a sentence about numbers of arithmetic, or is it a sentence about real numbers? Actually, we can't tell, because the numerals '4' and '19' are ambiguous. Up to now, this ambiguity did not bother us because no matter what interpretation we used, sentences such as '$5 + 7 = 12$' and '$6 \times 4 = 24$' made sense. So, in order to keep this freedom of interpretation, let's agree that if we see the sentence:

$$4 < 19,$$

it could be telling us either that the number 4 of arithmetic is less than the number 19 of arithmetic, or that $^+4$ is less than $^+19$. So, which is smaller, $^+7$ or $^+20$? $^+15$ or $^+2$? $^+1000$ or 0?

Let's review how we answer questions like these for numbers of arithmetic. All we do is find out which number you have to add to to get the other. The one you add to is the smaller.

But, does this simple test work for real numbers? Take the pair of real numbers $^+2$ and $^+15$. Is there a number you can add to $^+2$ to get $^+15$? Is there a number you can add to $^+15$ to get $^+2$? The answer to <u>both</u> questions is 'yes'. So, the simple adding test is no help. How could we change the test so that it would help? The accepted answer is: find out which number you would have to add a <u>positive</u> number to to get the other; the one you have to add a positive number to is the smaller. Use this test to check the following sentences:

$$^+5 < {}^+17, \qquad {}^+101 > {}^+100, \qquad 0 < {}^+6.9.$$

Now we have a test for deciding which is the larger of two non-negative numbers. And this test is such that when we ask the question 'Is $4 < 19$?', we get the same answer whether we are thinking of numbers of arithmetic or of real numbers.

What about dealing with the rest of the real numbers? Once again, the accepted answer is that the same test is to be used for <u>all</u> pairs of real numbers. The number of the pair to which you can add a positive number to get the other is the smaller. For example, ${}^{-}3 < {}^{+}7$ because if you add ${}^{+}10$ to ${}^{-}3$, you get ${}^{+}7$. The sentence '${}^{-}12 < {}^{-}1$' is true because we can add ${}^{+}11$ to ${}^{-}12$ to get ${}^{-}1$. Use this test of adding a positive number to check each of the following true sentences.

${}^{-}5 < {}^{+}12$ ${}^{-}11 > {}^{-}13$ ${}^{-}100 < {}^{+}1$ ${}^{-}73 < 0$

EXERCISES

<u>A</u>. Use one of the signs '$>$', '$<$', and '$=$', and write a true sentence which compares the given real numbers of each pair.

<u>Sample</u>. $({}^{-}15, {}^{-}19)$

<u>Solution</u>. To which of these numbers can I add a positive number to get the other? Since

$${}^{-}19 + {}^{+}4 = {}^{-}15,$$

${}^{-}19$ is the smaller. So, I write:

$${}^{-}15 > {}^{-}19.$$

[Could you write something else?]

1. $({}^{+}8, {}^{+}11)$	2. $({}^{+}3, {}^{-}1)$	3. $({}^{-}10, {}^{+}7)$	4. $({}^{-}5, {}^{-}30)$
5. $({}^{+}15, {}^{+}3)$	6. $(-7, -4)$	7. $(-98, 2)$	8. $(5, 2)$
9. $(5, 1)$	10. $(5, 0)$	11. $(5, -1)$	12. $(5, -2)$
13. $(5, -5)$	14. $(5, -7)$	15. $(-7, -7)$	16. $(-7, -6)$
17. $(-7, -8)$	18. $(-7, -9)$	19. $(-20, 19)$	20. $(-19, 20)$

<u>B</u>. Fill in the blanks to make true sentences.

1. $-19 <$ _____	2. $19 <$ _____	3. $19 >$ _____
4. $-23 >$ _____	5. $0 >$ _____	6. _____ > 0
7. _____ < -30	8. _____ > -30	9. _____ $< -\frac{1}{7}$

1.13 *The number line.* --Many people think of the numbers of arithmetic as being "lined up in order", starting with the smallest number, 0, and with larger numbers to the "right" of smaller numbers.

(1) Draw a picture in the space above to show this arrangement. Make dots on the picture for the numbers 0, $1\frac{1}{2}$, and 2.

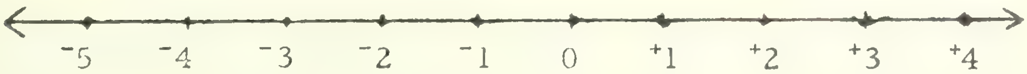
(2) Is the dot for $1\frac{1}{2}$ between the dots for 0 and 2?

(3) Why should the dot for $1\frac{1}{2}$ be between the dots for 0 and 2?

(4) Make a dot for the number 5. How does your picture show that $2 < 5$?

(5) Make a dot for the number 8. How does your picture show that $8 > 1$?

One of the advantages of our agreement for comparing real numbers is that we can also think of all the real numbers as lined up in order.



(6) How does this picture show that $2 > -4$?

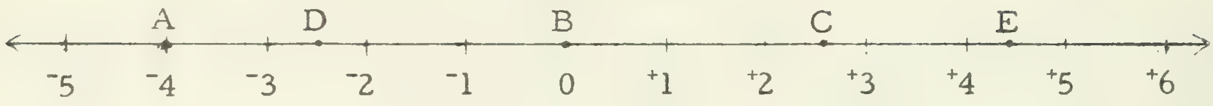
(7) How does this picture show that $-2 < -1$?

(8) Make a dot on the picture for $-\frac{3}{4}$. Why should this dot be placed between the dots for 0 and -1?

In comparing real numbers it is helpful to think of them as being lined up. So, we often call the ordered set of real numbers *the line of real numbers*, or just *the number line*, for short. When you draw a picture like the one just above question (6), you have *a picture (of part) of the number line*.

EXERCISES

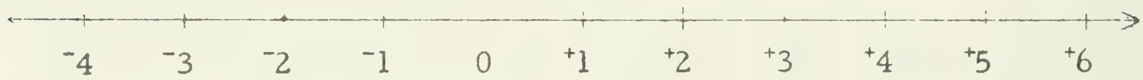
A. Here is a picture of the number line.



Give the real number which corresponds with each labeled dot.

A: B: C: D: E:

B. Here is another picture of the number line. Mark dots on the picture which correspond with the listed real numbers.



A: 5 B: 2 C: -3 D: $\frac{1}{2}$ E: $-2\frac{1}{3}$ F: -3.2 H: $-\frac{1}{4}$

C. A mental picture of the number line makes it easy for you to compare real numbers. Write a comparison sentence about each pair.

1. $(-3, +17)$
2. $(+5.5, +3)$
3. $(-6, -5)$
4. $(-152, -2,176)$
5. $(+.0012, -.0138)$
6. $(+.001, +.0001)$
7. $(-6.382, +\frac{1}{2})$
8. $(\frac{11}{3}, \frac{21}{6})$
9. $(-1,428, +.0052)$
10. $(-.00016, -43,213)$

[More exercises are in Part N, Supplementary Exercises.]

D. True or false? [Just as '$\neq$' is used for 'is not equal to', so are '$\nless$' used for 'is not less than' and '$\ngtr$' for 'is not greater than'.]

1. $5 \ngtr 10$
2. $4 \nless 0$
3. $-10 \neq -20$
4. $-10 \neq -10$
5. $.5 \ngtr .75$
6. $2 \ngtr 2$

7. $5 \ngtr 10 \div 2$ 8. $-3 \ngtr 0$ 9. $-13 \nless 5/3$

10. $5 = 5$ 11. $5 < 5$ 12. $5 \ngtr 5$

*

13. Do '$\ngtr$' and '$<$' tell you the same thing? Compare Exercises 11 and 12.

* * *

Suppose each of two students, Alice and Rachel, picks a real number and each whispers her choice to Ned. Ned tells you that Alice's number is not less than Rachel's. Can you conclude that Alice's is greater than Rachel's?

When you write the comparison sign '$\nless$' between numerals and have a true sentence, the numeral on the left can name a number which is <u>greater</u> <u>than</u> <u>or</u> <u>equal</u> <u>to</u> the number named by the numeral on the right. So, sometimes, instead of using '$\nless$' we use '$\geq$', a combination of '$>$' and '$=$'. Similarly, we sometimes use '$\leq$' instead of '$\ngtr$'. [Read '$\geq$' as 'is greater than <u>or</u> equal to' and '$\leq$' as 'is less than <u>or</u> equal to'.]

So, for example, the sentence:

(1) $5 \ngtr 6$

says the same thing as does the sentence:

(2) $5 < 6$ or $5 = 6$,

and (1) and (2) say the same thing as does:

(3) $5 \leq 6$.

* * *

<u>E</u>. True or false?

1. $5 > -6$ 2. $10 < -26$ 3. $4 \leq 4$

4. $\frac{1}{2} \geq \frac{1}{2}$ 5. $-3 \leq -\frac{21}{2}$ 6. $\frac{1}{5} \nless \frac{1}{6}$

7. $1.53 \leq 1.053$ 8. $-1.542 \geq +0.001$

9. $-\frac{11}{7} \leq -\frac{21}{14}$ 10. $972 - 846 \geq -(972 - 846)$

F. For each of the following exercises, make as many true sentences as you can by inserting the signs '$=$', '$\neq$', '$<$', '$\not<$', '$>$', '$\not>$', '$\leq$', and '$\geq$'.

Sample. 6 4

Solution. $6 > 4$, $6 \neq 4$, $6 \geq 4$, $6 \not< 4$

1. $5 \quad 3$	2. $-4 \quad -4$	3. $6 \quad -3$
4. $-10 \quad -9$	5. $-\frac{10}{3} \quad 2$	6. $-\frac{1}{782} \quad 0$
7. $0 \quad 0$	8. $\frac{3}{7} \quad \frac{-3}{-7}$	9. $\frac{8-2}{3-7} \quad \frac{2-8}{3-7}$

*

10. Suppose we abbreviated:

is less than or is equal to or is greater than

by:

$$\underset{>}{\leq} .$$

In which of the Exercises 1-9 above could you use this sign to make a true sentence? Do you think anybody uses a sign like this?

[More exercises are in Part O, Supplementary Exercises.]

* * *

None of the pictures of the number line which we have drawn looked like this.

This picture would be perfectly all right if we were interested only in using it to check a sentence like '$^{+}3 > {}^{-}1$'. But, we can draw our pictures so that they also enable us to check a sentence like '$4 - 2 = {}^{-}1 - {}^{-}3$' just by looking at the picture. The picture above doesn't help in this case. Why not?

The picture didn't help because the distance between the dots for 2 and 4 is not the same as the distance between the dots for $^{-}3$ and $^{-}1$. Pictures of the number line such that sentences like '$4 - 2 = {}^{-}1 - {}^{-}3$' can be checked by comparing distances between dots are said to be drawn with a uniform scale. When you did the exercises in Parts A and B on page 1 - 100, you probably took it for granted that the pictures were drawn with a uniform scale.

THE ABSOLUTE VALUE OPERATION

You have seen that it is helpful to think of the set of real numbers as a line, each real number being a point of the line. When thinking about ordinary lines, we have a notion about what we mean by 'distance between points'. Can we develop such a notion for the number line? That is, can we make sense out of 'the distance between real numbers'?

Think of a picture of the number line which has been drawn with a uniform scale. The length of the segment between the dots for $^{-}4$ and $^{-}1$ is 3 units, no matter what unit was used in drawing the picture. Whatever unit was used in drawing the picture, the distance (with respect to this unit) between the dots is 3. So, let us agree that the distance between the real numbers $^{-}4$ and $^{-}1$ is 3. [As you saw in our earlier discussion of trips, distances are numbers of arithmetic. What is the distance between $^{-}1$ and $^{-}4$?]

EXERCISES

A. For each of the listed pairs of real numbers, give the number of arithmetic which is the distance between the real numbers.

1. $({}^{+}7, {}^{+}10)$ 2. $({}^{+}10, {}^{+}7)$ 3. $({}^{-}5, {}^{+}5)$

4. $({}^{-}7, {}^{-}12)$ 5. $({}^{+}12, {}^{+}7)$ 6. $({}^{-}8, {}^{+}24)$

(continued on next page)

7. $(^-10, ^+35)$ 8. $(^-72, ^-72)$ 9. $(^-38, ^-59)$

10. $(^-724, ^+589)$ 11. $(^-57, ^-804)$ 12. $(^+72, ^+596)$

*

13. Ruth and Rachel each pick a real number. To find the distance between these real numbers, Paul suggests that Ruth subtract her number from Rachel's. Ned protests, and says that Rachel should subtract her number from Ruth's. Arthur, who understands that a distance is a number of arithmetic, says that both are right as far as they go, but each needs to take one more step.

What is that step?

* * *

You have seen in the exercises in Part A that you can find the distance between real numbers, say, $^-5$ and $^-9$, by first subtracting either from the other.

$$^-5 - {}^-9 = {}^+4 \qquad \text{or} \qquad {}^-9 - {}^-5 = {}^-4$$

Then, the distance between the real numbers is the number of arithmetic which corresponds with both of these differences.

4 is the number of arithmetic which corresponds with $^+4$

and

4 is the number of arithmetic which corresponds with $^-4$.

So,

the distance between $^-5$ and $^-9$ is 4.

We call the number of arithmetic which corresponds with a real number <u>the absolute value of the real number</u>. For example,

the absolute value of $^-7$ is 7,
the absolute value of $^+3$ is 3,
the absolute value of 0 is 0,
the absolute value of -4 is 4,
and the absolute value of $(9 - 15)$ is 6.

So, the distance between two real numbers is the absolute value of the difference of either one from the other.

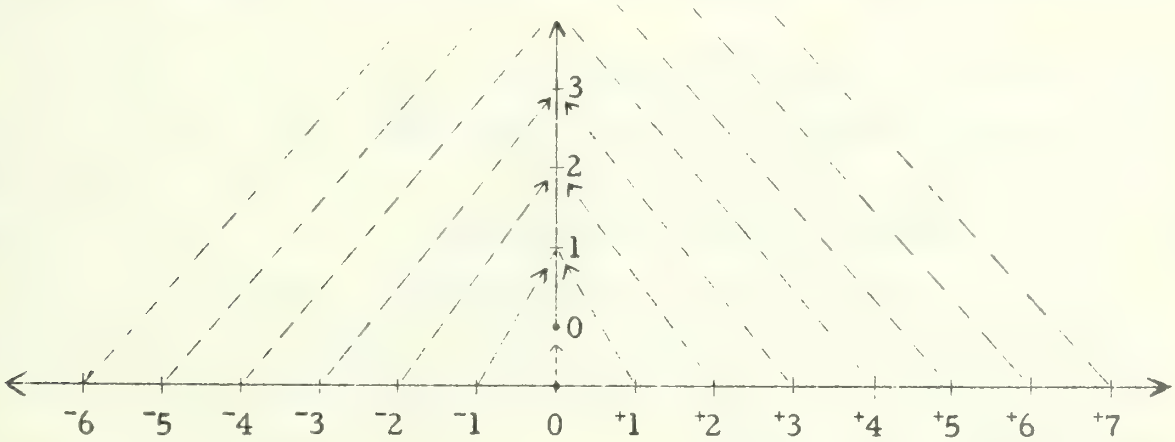
What is the absolute value of $^{-}27$? What is the distance between $^{-}27$ and 0? What is the distance between 0 and $^{-}27$?

What is the distance between $^{+}400$ and $^{+}973$? What is the absolute value of ($^{+}400$ - $^{+}973$)? Of ($^{+}973$ - $^{+}400$)?

What is the absolute value of ($^{+}634$ - $^{-}66$)? Of ($^{-}5 \times {}^{+}4$)? Of $^{+}73$? Of $^{-}48$? Of $^{-}100$? Of 0?

Notice that absolute valuing is an operation. It takes you from each real number to a single corresponding number of arithmetic. We say that it is an operation <u>on</u> the set of real numbers <u>to</u> the set of numbers of arithmetic. Up to now, we have used a list of some of the pairs in an operation in order to "visualize" it. Make such a list [about ten pairs] for absolute valuing.

Here is another way to visualize absolute valuing.



(1) What two real numbers each have absolute value 2?

(2) What two real numbers each have absolute value 3?

(3) What two real numbers each have absolute value 527?

(4) Does any real number have absolute value —3?

(5) Does any real number have absolute value 0?

As in the case of other operations, it is convenient to have a sign for absolute valuing. The standard sign consists of two vertical bars. Thus,

'$|{}^{-}5|$' means the absolute value of ${}^{-}5$,

and

'$|{}^{+}17|$' means the absolute value of ${}^{+}17$.

[We pronounce '$|\ldots|$' as 'the absolute value of ...'.]

* * *

B. Simplify.

Sample. $|{}^{-}3| + |{}^{+}7|$

Solution. $|{}^{-}3| + |{}^{+}7| = 3 + 7 = 10$

1. $|{}^{+}7| + |{}^{-}2|$
2. $|{}^{-}5| + |{}^{+}12|$
3. $|{}^{-}8| + |{}^{-}4|$
4. $|{}^{-}9| + |{}^{-}11|$
5. $|{}^{+}2| + |{}^{+}10|$
6. $|{}^{-}71| + |{}^{+}71|$
7. $|{}^{+}21| - |{}^{-}3|$
8. $|{}^{-}15| - |{}^{+}15|$
9. $|{}^{-}103| - |{}^{+}100|$
10. $|{}^{+}5| \times |{}^{-}3|$
11. $|{}^{-}2| \times |{}^{-}17|$
12. $|{}^{+}5| \times |{}^{-}3|$
13. $|{}^{+}6 + {}^{+}3|$
14. $|{}^{+}9 - {}^{+}2|$
15. $|{}^{+}2 - {}^{+}9|$
16. $|{}^{-}5 + {}^{-}7|$
17. $|{}^{-}5 + {}^{+}5|$
18. $|{}^{-}102 - {}^{+}117|$
19. $|{}^{+}4 - {}^{-}3| + |{}^{-}5 - {}^{-}7| - |{}^{+}4 - {}^{-}2|$
20. $|{}^{-}2 + {}^{-}3| \times |{}^{+}4 - {}^{-}5| - |{}^{+}6 - {}^{+}2| \times |{}^{+}5 - {}^{-}5|$

C. Fill in the blanks to make true sentences.

1. $7 + |\_\_\_\_| = 10$
2. $|\_\_\_\_| + 4 = 9$
3. $8 - |\_\_\_\_| = 3$
4. $9 \times |\_\_\_\_| = 36$
5. $|\_\_\_\_ + {}^{+}5| = 10$
6. $|\_\_\_\_ + {}^{-}7| = 5$
7. $|\_\_\_\_ - {}^{+}4| = 10$
8. $|\_\_\_\_ - {}^{+}3| = 0$

D. Complete each to a true sentence by inserting one of the signs '=', '<', or '>'.

1. $|{}^{+}5| \quad |{}^{+}3|$
2. $|{}^{-}5| \quad |{}^{+}3|$
3. $|{}^{+}7| \quad |{}^{+}2|$
4. $|-{}^{+}7| \quad |-{}^{+}2|$
5. $|0| \quad |{}^{-}7|$
6. $|0| \quad 7$
7. $|{}^{+}\frac{6}{10}| \quad |{}^{-}\frac{2}{5}|$
8. $|{}^{-}\frac{1}{2}| \quad |{}^{+}\frac{1}{3}|$
9. $|{}^{-}\frac{8}{12}| \quad |{}^{+}\frac{2}{3}|$

10. $|{}^+7 - {}^+5| + |{}^+5 - {}^+2| \quad |{}^+7 - {}^+2|$

11. $|{}^+4 - {}^+6| + |{}^+6 - {}^+3| \quad |{}^+4 - {}^+3|$

12. $|{}^+4 - {}^-2| + |{}^-2 - {}^+8| \quad |{}^+4 - {}^+8|$

13. $|{}^-2 - {}^+1| + |{}^+1 - {}^+3| \quad |{}^-2 - {}^+3|$

14. $|{}^+2| \times |{}^-3| \quad |{}^+2 \times {}^-3|$

15. $|{}^-3| \times |{}^-4| \quad |{}^-3 \times {}^-4|$

16. $|{}^+7 - {}^-3| \quad |{}^+7| - |{}^-3|$

17. $|{}^+9 + {}^-2| \quad |{}^+9| + |{}^-2|$

DOES ABSOLUTE VALUING HAVE AN INVERSE?

(I) I am thinking of a number. If I add ${}^+2$ to it, I get ${}^+12$. What number am I thinking of?

(II) I am thinking of a number. The absolute value of this number is 7. What number am I thinking of?

Do you see a difference between these problems? You can tell what number was thought of in the first problem because the operation adding ${}^+2$ has an inverse. In the second problem, although you know that the number must be either ${}^+7$ or ${}^-7$, you can't tell which. The operation absolute valuing does not have an inverse. Let's look into what this means.

To begin with, we ought to be clear on what an operation is. The sets of pairs which we have been calling 'operations' have this important property--none of them contains two pairs that have the same first number [but different second numbers]. That is, an operation operates on a number to produce a unique result. [When the operation adding ${}^+2$ operates on ${}^-8$, the result is ${}^-6$, and nothing else; when the operation absolute valuing operates on ${}^-4$, the result is 4, and nothing else.] This notion of uniqueness of result is the essence of the idea of operation. So, let us agree that a set of pairs is an operation just if no two pairs in the set have the same first number.

Now, from any operation you can get a second set of pairs by reversing each pair which belongs to the given operation. For example, when you reverse each pair in the operation adding ${}^+7$, you get a set of pairs. If you think a moment, you will see that this second set of pairs is an operation. In fact, it is the operation we call 'subtracting ${}^+7$'.

But, is it always the case [as here] that when one reverses the pairs in an operation, the new set of pairs is also an operation? Let's try it with the operation absolute valuing. Here is a list of some of the pairs in the new set.

$(7, {}^-7)$ $(7, {}^+7)$ $(0, 0)$ $(5, {}^+5)$

$(5, {}^-5)$

. . .

Is this set of pairs an operation? No, because it contains pairs with the same first number but different second numbers. Since this set of reversed pairs is not an operation, we say that absolute valuing does not have an inverse. Can you think of another operation which does not have an inverse? In general, an operation has an inverse just if the set of its reversed pairs is an operation.

Let's look again at the set of pairs we get by reversing the pairs in absolute valuing. Even though this set of pairs is not itself an operation, we can "split" the set into two sets each of which <u>is</u> an operation.

$(7, {}^-7)$ $(5, {}^-5)$ $(6, {}^-6)$	$(7, {}^+7)$ $(5, {}^+5)$ $(6, {}^+6)$
$(104, {}^-104)$ $(1, {}^-1)$ $(0, 0)$	$(104, {}^+104)$ $(1, {}^+1)$ $(0, 0)$
. . .	. . .

Notice that the operation listed on the left is an operation on the set of numbers of arithmetic. So is the one listed on the right. [If (0, 0) were not included in one of these sets, the set would still be an operation, but not on the set of numbers of arithmetic. Why?] When the first operation operates on a number of arithmetic, the result is the corresponding nonpositive real number. What is the result of operating on numbers of arithmetic with the second operation? Do you recognize these operations? You learned about them when we first mentioned real numbers. These operations do not have standard names, but we have been using two signs almost as though they were signs for the operations. These are the signs '$^-$' and '$^+$'. If you <u>apply the operation</u> $^-$ to the number 7 of arithmetic, you get the corresponding nonpositive real number $^-7$. If you apply the operation $^+$ to the number 58 of arithmetic, you get the corresponding nonnegative real number $^+58$.

Notice that (0, 0) belongs to each of these operations. To use '$^-$' and '$^+$' as signs for the operations, we must define '$^-0$' and '$^+0$' to be numerals for the real number 0. [Warning: Even though we are claiming that $^-0 = 0$ and $^+0 = 0$, 0 is still neither negative nor positive. But, it is both nonpositive and nonnegative.]

EXERCISES

A. For each number listed, tell by checking in the appropriate column whether it is a number of arithmetic, a positive real number, a negative real number, or the real number 0. [Since throughout this section on absolute valuing it has been essential to distinguish between numbers of arithmetic and real numbers, we have not used the convention according to which a numeral for a number of arithmetic is used to name the corresponding nonnegative real number. We continue this policy in the table.]

	Number	Number of Arithmetic	Positive Real Number	Negative Real Number	Real Number 0
1.	$^+(7 + 3)$				
2.	$^+8 - {}^+10$				
3.	$\lvert {}^-173 \rvert$				
4.	$45 - 45$				
5.	$^+45 - {}^+45$				
6.	$^+\lvert {}^-15 \rvert$				
7.	$^-\lvert {}^+7 \div {}^-5 \rvert$				
8.	$\lvert {}^+10 + {}^-10 \rvert$				
9.	$\lvert {}^-3 \rvert - \lvert {}^+3 \rvert$				
10.	$^+\lvert {}^-2 \times {}^+5 \rvert$				
11.	$^+\lvert {}^-\lvert {}^+5 \rvert \rvert$				

B. Each of the following marks looks like a numeral but isn't because it doesn't stand for a number. Explain why.

Sample 1. $|3|$

Solution. Absolute valuing is an operation which is applied only to real numbers. Since '3' stands for a number of arithmetic [recall that we are not using ambiguous numerals in these exercises], '$|3|$' is nonsense.

Sample 2. $^{+}(^{-}3)$

Solution. The operation $^{+}$ applies only to numbers of arithmetic.

1. $|12|$ 2. $^{-}(^{+}5)$ 3. $^{+}(^{+}2)$ 4. $||^{-}2||$

5. $^{+}(^{+}3 - {}^{+}4)$ 6. $^{+}(3 - 4)$ 7. $-|3 - 4|$

* * *

Some of the expressions in Part B do make sense if they are interpreted according to the convention that a numeral for a number of arithmetic stands for the corresponding nonnegative real number. [Even with this convention, there are still some expressions in Part B which do not make sense.] In Exercise 1, if we regard '12' as standing for $^{+}12$ then '$|12|$' is a numeral for a number of arithmetic. But, since this is the case, we can use the convention again and regard '$|12|$' as standing for $^{+}12$. So, if you see '$|12|$' in a place where it is intended to make sense, you will know that '12' is being used as a name for $^{+}12$. But, you will have to look further in order to decide whether '$|12|$' is being used as a name for the number 12 of arithmetic, or as a name for $^{+}12$. For example, if you come upon the sentence:

$$|12| - 3 = {}^{+}9$$

and you believe that it is intended to make sense, you will interpret '$|12|$' and '3' as names for nonnegative real numbers. But, if you see:

$$|12| - 3 = 9,$$

then, without additional information, all you can be sure of is that '12' stands for $^{+}12$, and that either '$|12|$', '3', and '9' all stand for numbers of arithmetic, or all stand for nonnegative real numbers.

Two other expressions in Part B which can be interpreted to make sense are those in Exercises 4 and 7. Tell how to make sense out of them.

MISCELLANEOUS EXERCISES

A. Simplify.

1. $+3 + -2$	2. $-3 - +6$	3. $-3 - -8$
4. $6 + -1$	5. $-12 + -8$	6. $-16 - 5$
7. $27 - -6$	8. $5 - -3$	9. $-5 + 7$
10. $-15 + 3$	11. $+1.3 - +1.6$	12. $-5 + 3$
13. $5.6 + -2.3$	14. $-7.8 + 2.4$	15. $10.7 - -3.4$
16. $-3.5 + -2.9$	17. $+5 - -3 + -7$	18. $-2 - -4 - +3$
19. $7 + 9 - -2$	20. $24 + -6 - -1$	21. $-13 + 8 - -9$
22. $1.2 + -1.7 + -7.8$	23. 0.5×10	24. -3×-12
25. $-7 \times +9$	26. 11×-6	27. 6×12
28. $-2 \times -3 \times +7$	29. $11 \times -3 \times 4$	30. $0 \times -3 \times -8$
31. $-12 \div -3$	32. $18 \div -2$	33. $-24 \div +8$
34. $-150 \div 25$	35. $0 \div -5$	36. $0 \div +250$
37. $-27 \div 3$	38. $32 \div -8$	39. $+48 \div -12$
	40. $198 \div -3$	

B. True or false?

1. $\lvert -4 \rvert + \lvert +4 \rvert = 0$	2. $\lvert +4 \rvert + \lvert -4 \rvert = \lvert -8 \rvert$
3. $\lvert -3 \rvert - \lvert +3 \rvert = 0$	4. $\lvert -2 \rvert - -3 = 5$
5. $-4 + -2 > -7$	6. $\lvert -1 \rvert + -3 - \lvert -4 \rvert = -8$
7. $\lvert +6 \rvert + \lvert -6 \rvert < \lvert +6 \rvert + \lvert +6 \rvert$	8. $\lvert +5 \rvert + \lvert +5 \rvert \geq \lvert +5 \rvert + \lvert -5 \rvert$
9. $3 + \lvert -2 \rvert - \lvert -1 \rvert = 4$	10. $-7 + \lvert -2 \rvert + 0 = -5$
11. $-10 = -7 + \lvert -3 \rvert$	12. $\lvert -5 \rvert \geq \lvert -6 \rvert$
13. $+10 = -11 + 1$	14. $3 - 2 < 2 - 3$
15. $\lvert +3 \rvert - {}^-\lvert +2 \rvert = 3 + 2$	16. $\lvert 4 \rvert - \lvert -2 \rvert \leq 4 - -2$
17. $\lvert +1 \rvert - \lvert -2 \rvert \leq +1 - -2$	18. $\lvert -2 \rvert \times \lvert -3 \rvert = -2 \times -3$

(continued on next page)

19. $-2 \times +3 \nless -2 \times +3$

20. $|+2| \times |+3| = +2 \times +3$

21. $|4 - 7| = {}^{-}|7 - 4|$

22. $|+2 - -13| = |-13 - +2|$

23. $|-5 - -3| = |5 - 3|$

24. $|-15 - 2| = |2 + 15|$

C. When we think of the real numbers as points of a line, the number line, we can also think of trips on the number line. We have already decided how to find the distance between two real numbers, so we can also use real numbers to measure trips on the number line once we have chosen a positive direction. It helps in understanding many problems to translate them into problems concerning trips on the number line. For this purpose it is most convenient to choose the direction from 0 to ${}^{+}1$ as the positive direction. [For example, the measure of a trip from ${}^{+}7$ to ${}^{+}1$ is ${}^{-}6$.] We shall use this choice in each of the following exercises.

1. What is the measure of each of the following trips?

   (a) from 0 to ${}^{+}1$
   (b) from ${}^{+}6$ to ${}^{+}15$
   (c) from ${}^{+}15$ to ${}^{+}6$
   (d) from ${}^{-}3$ to ${}^{+}7$
   (e) from ${}^{-}6$ to ${}^{-}8$
   (f) from ${}^{+}5.7$ to ${}^{+}38.2$
   (g) from ${}^{-}7.6$ to ${}^{-}5.4$
   (h) from ${}^{-}127$ to ${}^{-}239$

2. [Draw a picture of the number line and label some of its points to help you with this problem.] A trip from point M to point N is measured by +2, a trip from N to P is measured by -6, and a trip from P to D is measured by -2. If the point N is +5, what real number is point D?

3. A trip from W to Q is measured by +9, and a trip from Q to A is measured by +7. If point A is +4, what real numbers are the points W and Q?

4. The measure of a trip

   from C to R is 14, from R to S is -32,
   from S to A is $+10$, and from A to C is $+8$.

   Give the real numbers which are the points A, C, and S

   (a) if point R is 3. (b) if R is -3.

   (c) if R is 0. (d) if R is $\frac{1}{2}$.

   (e) if R is -105.2. (f) if R is 3986.7.

5. A trip from A to Z is measured by -5, and a trip from Z to C is measured by $+9$. If point A is -6, what is point C?

D. Each of the following problems involves changes. [In Exercise 1 the change is in financial status, in Exercise 2 the change is in point standing, etc.] Choose a direction of change for the positive direction, and use real numbers to measure changes.

1. Bill is in business and loses \$2.00 on one day and makes a profit of \$4.75 on the second day. What is the change in his financial status over the two-day period?

2. A player has 5 points at the beginning of a round, loses 3 points during that round, gains 7 points the next round, loses 12 points the next round, and gains 1 point the final round. What is the change in his point standing from the beginning of the first round mentioned to the end of the final round? What was his point standing at the end of the final round?

3. If the temperature is $+15°$ Fahrenheit and it drops $27°$, what is the temperature at the end of the drop?

[Note: In doing these problems does it help if you think of "trips" along the number line?]

4. Let us say that the main floor in a department store corresponds with the real number 0, that the floor 3 levels below corresponds with the real number +3, and the floor 2 levels above the main floor corresponds with -2. What real number corresponds with the floor at which the elevator stops after moving 11 floors up from floor +1?

5. The highest temperature on January 3 in Chicago was 10° above zero. On January 4 the highest temperature was 2° higher than on January 3, and on January 5 it was 5° lower than January 4. What was the highest temperature on January 5?

6. At the end of one year, a firm had a balance of $-10,700, and at the end of the next year, the balance was $15,400. How much better off was the firm at the end of the second year?

7. The temperature at 7 p.m. on a certain day in New York City was 7°. By 3 a.m. of the next day the temperature had dropped to −9°. What was the change in temperature from 7 p.m. to 3 a.m. the next day?

8. During one day 12 gallons of water was pumped out of a tank, and that night 17 gallons was pumped into the tank. During the next day 30 gallons was pumped out, and 42 gallons was pumped into the tank that night.

   (a) What is the change in volume of water in the tank from the morning of the first day to the morning of the third day?

   (b) How much water was in the tank on the morning of the first day?

   (c) What is the least amount of water which could have been in the tank on the morning of the first day?

   (d) How much water was there in the tank on the morning of the first day if there were 50 gallons in it on the morning of the third day?

E. Guess the number.

1. A certain number is added to -5, and the result is 16.
2. -5 is subtracted from a certain number, and the result is 15.
3. A certain number is divided by 6, and the result is -18.
4. A certain number is added to its reciprocal, and the result is 2.
5. If I add a certain number to its reciprocal, the sum is -2.
6. If I add the opposite of a certain number to its reciprocal the sum is 0.
7. If I add a certain number to $+1$, I get -1.

F. Rearrange this list into columns with all the numerals for the same number in one column.

$\frac{1}{2}$ $\qquad -\frac{6}{2}$ $\qquad \frac{31}{62}$ $\qquad -\frac{-8}{-16}$ $\qquad 1$

150% of 2 $\qquad +\frac{11}{3}$ $\qquad \frac{12}{12}$ $\qquad 0.5$ $\qquad \frac{22}{6}$

$\frac{-.0042}{-.0042}$ $\qquad 350\%$ of $\frac{1}{7}$ $\qquad \frac{5}{11} \times \frac{121}{15}$ $\qquad -3$

$\frac{40 - -4}{12}$ $\qquad -\frac{-2}{4}$ $\qquad -1 - -4$ $\qquad 20\%$ of $(100\%$ of $5)$

$\frac{-4}{-8}$ $\qquad \frac{9}{3} \times (-1)$ $\qquad +3\frac{2}{3}$ $\qquad \frac{1}{3} + \frac{1}{6}$

$-2 \times -\frac{1}{2}$ $\qquad \frac{3+7}{6+4}$ $\qquad \dfrac{1}{38 \times \frac{1}{19}}$ $\qquad \frac{16}{32} + 0$

G. Fill in the blanks to make true sentences.

Sample. (9, ____) belongs to the operation adding 5.

Solution. Since 9 + 5 = 14, we can make a true sentence by writing a '14' in the blank.

1. (3, ____) belongs to adding 7.
2. (−8, ____) belongs to adding 6.
3. (−5, ____) belongs to multiplying by −6.
4. (−7, ____) belongs to subtracting −7.
5. (____, 11) belongs to adding 9
6. (____, 36) belongs to multiplying by −9.
7. (8, ____) belongs to the inverse of adding 2.
8. (9, ____) belongs to the inverse of adding ⁻3.
9. (−12, ____) belongs to oppositing.
10. (____, −5) belongs to sameing.
11. (−5, ____) belongs to squaring. [To square a number is to multiply it by itself.]
12. (+3, ____) belongs to squaring.
13. (____, +36) belongs to squaring.
14. (−27, ____) belongs to dividing by −3.
15. (−27, ____) belongs to multiplying by the reciprocal of −3.
16. (−3, ____) belongs to adding the opposite of −6.
17. (7, −7) belongs to multiplying by ____ .
18. (7, −7) belongs to dividing by ____ .
19. (7, −7) belongs to adding ____ .
20. (7, −7) belongs to ____ .
21. (⁻7, ____) belongs to absolute valuing.

H. Solve these problems.

1. The average of two numbers is 12. One of the numbers is 30. What is the other number?

2. Four numbers average -7. What is their sum?

3. Two numbers average -22. One number is -13. What is the other number?

4. Three numbers average $+14$. One number is -26; a second number is $+25$. What is the third number?

5. Two numbers average -25; one number is $+17$. What is the other number?

6. Two numbers average -33; one number is -20. What is the other number?

7. There are four numbers whose average is -9. What is their sum?

8. Two numbers average $+27$; one number is -51. What is the other number?

9. Find the average for each set of numbers.

   (a) $\{3.5, 6.09, 2.37, 100.12, 9.75\}$
   (b) $\{3.89, 12.1, 7.14, 8.0\}$
   (c) $\{+5, -14\}$
   (d) $\{+3, -101, +105, -3\}$
   (e) $\{-\frac{3}{8}, +\frac{1}{2}, -\frac{3}{4}, -\frac{15}{24}\}$

10. Fill in the blank so that the average for each set is -2.

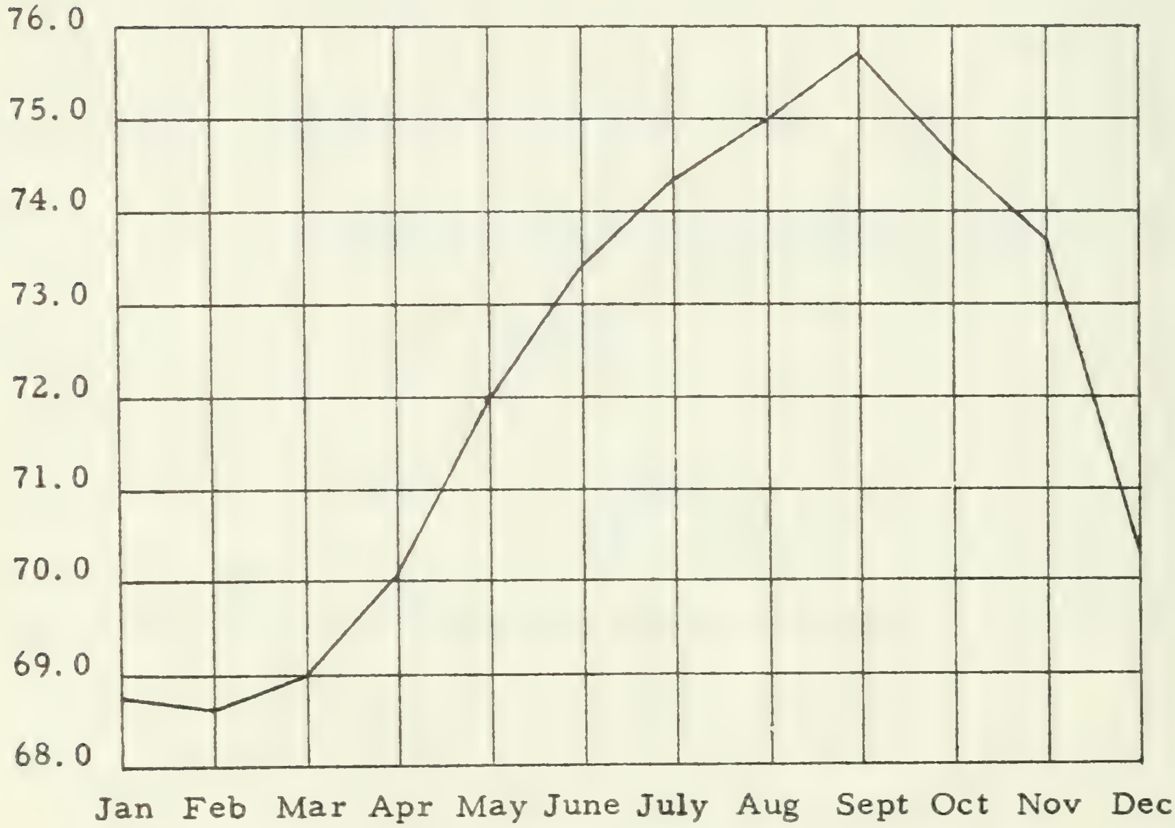
   (a) $\{-6, -9, +12, \_\_\_\_\}$
   (b) $\{-9, \_\_\_\_, 9\}$
   (c) $\{\frac{1}{2}, \_\_\_\_, -\frac{1}{3}, -\frac{3}{4}\}$
   (d) $\{-3.0, -3.4, \_\_\_\_, +2.0, +6.4\}$
   (e) $\{2, -2, -1, 0, \_\_\_\_, 3, 1\}$

11. The average of three numbers is 47. One of the numbers is 67. What is the sum of the other two?

I. Below is a chart which gives the mean (average) temperature for each month of a certain year in Hawaii. Use the chart and complete the following table. Then answer the questions which are at the top of the next page.

Month	Mean Temperature	Month	Mean Temperature	Month	Mean Temperature
Jan.	68.8°	May		Sept.	
Feb.		June		Oct.	
March		July		Nov.	
April		Aug.		Dec.	

Monthly Mean Temperatures in Hawaii



1. Find the mean of the mean monthly temperatures for the year.

2. Subtract the mean temperature for each month from the mean for the year.

3. Add all the differences obtained in Exercise 2.

4. Did you get 0 for Exercise 3?

5. Suppose you subtract the mean of a set of numbers from each of the numbers. The resulting differences are called deviations from the mean. Do you think that, for each set of numbers, the sum of the deviations from the mean is 0?

J. The bar chart on the next page shows mean January temperatures in Fahrenheit degrees computed over a period of years at some of the Alaska weather stations.

1. Give the mean of the January temperatures at the following weather stations: Ketchikan, Valdez, Susitna, Ruby, Tanana, and Dawson.

2. According to the graph, how many degrees colder was Anchorage than Seward?

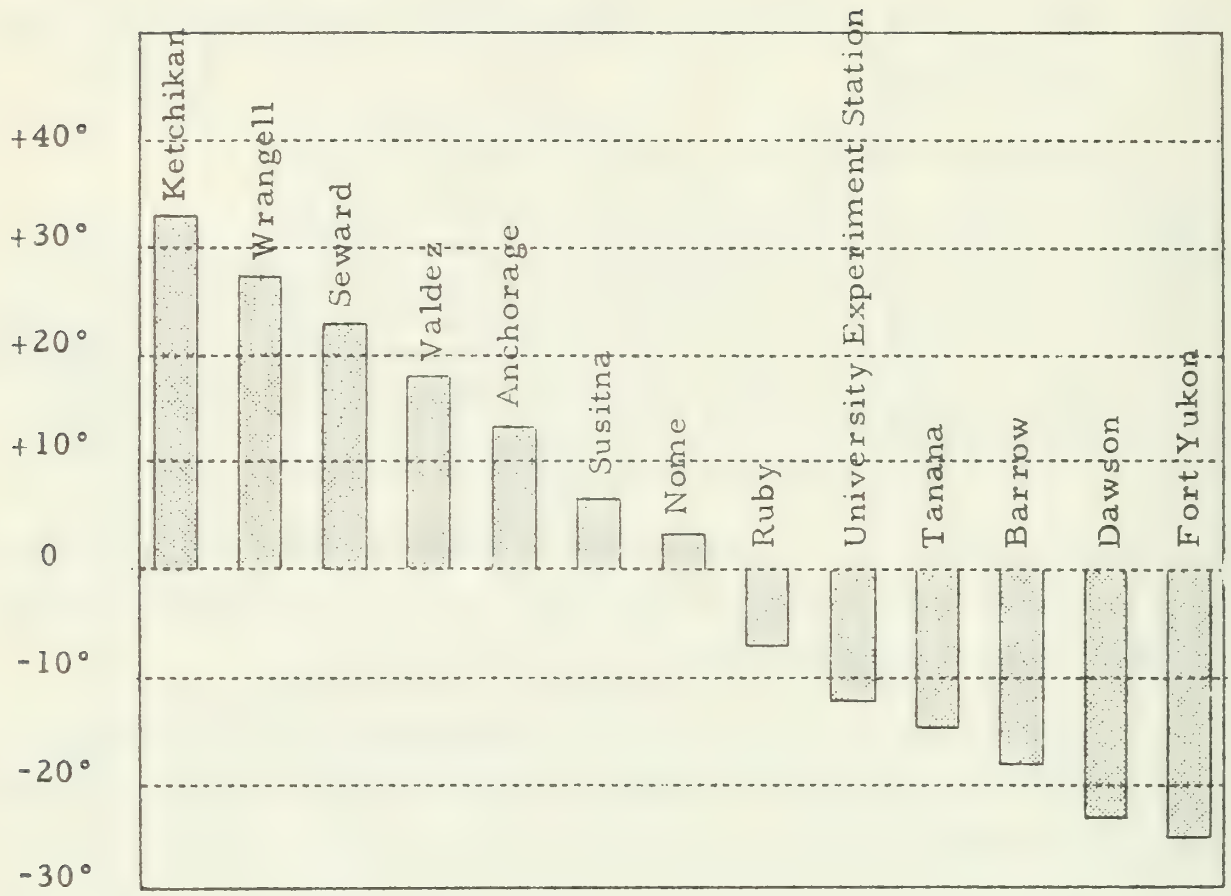
3. What station had the temperature closest to 0? What was this temperature?

4. How many degrees warmer was Ruby than Barrow?

5. What was the warmest station?

6. How many stations had their mean January temperature above freezing? How many below freezing?

7. The mean temperature for the thirteen stations is +2.8°. What practical use could be made of this fact?



Fahrenheit Temperatures at
Various Alaskan Weather Stations

K. Punctuate to make sense.

1. Ray was making a list of the class. First he wrote Jim. Beside Jim he wrote 1. Then he wrote Helen and beside Helen he wrote 3. He looked at the 3 and realized that he had made an error. He erased the 3 and this time put 2 beside Helen. After writing 13 names he wrote 14. When he finished the list he had 23 names and 23 numerals.

2. Harry in the sentence in which Chester had had had had had had had had had had had had had had had had the approval of the examiners.

TEST

I. Simplify.

1. $-4 + {}^{-}1$
2. $7 + {}^{-}5$
3. $10 + 2$
4. $-9 + 0$
5. $0 + 3$
6. $2 + {}^{-}13$
7. $-9 - 4$
8. $7 - {}^{-}3$
9. $+2 - {}^{+}7$
10. $317 - {}^{-}289$
11. $11.2 - 16 \times {}^{-}\frac{1}{2}$
12. $87\frac{1}{2}\%$ of ${}^{-}16$
13. $13 \times {}^{-}3 + 4 \times {}^{-}3$
14. $(9 \times {}^{-}2 + {}^{-}3 \times {}^{-}6) \div (5 - 8)$
15. $-\frac{3}{2} \times \left(1 - \frac{3}{2} \times \frac{2}{3}\right)$
16. ${}^{-}7 \times 2 \times {}^{-}1$
17. $-2 \times [-3 \times 5]$
18. $5 \times \{-2 \times [-3 + 8]\}$
19. $-3 \times [-5 \times -2 + (-1 + 7)]$
20. $\dfrac{{}^{-}12 \times 3 + 8 \div {}^{-}2}{10 \times {}^{-}\frac{1}{2}}$

II. Use single quotes to punctuate the following paragraph so that it makes sense.

The book was on the table but it wasn't on a table. If the book were on a table that was on a paper on the table then you couldn't see that a table was on the paper on the table.

III. You can convert the sentence:

$$5 \wr 4 + 5 \wr 9 = 5 \wr (4 + 9)$$

into a true one by replacing each '$\wr$' in it by which of the following symbols?

(a) $+$ (b) $\times$ (c) $\div$ (d) $-$ (e) $/$

IV. Which of the following is a name for $-8 + 13$?

(a) $8 + {}^-13$ (b) $8 + 13$ (c) $-8 + {}^-13$

(d) $13 - 8$ (e) $-(-8 + 13)$

V. Which of the following is an instance of the commutative principle for multiplication?

(a) $10 \times \left(\frac{1}{4} \times \frac{4}{5}\right) = \left(10 \times \frac{1}{4}\right) \times \frac{4}{5}$

(b) $\frac{1}{4} \times -\frac{1}{7} = -\frac{1}{4} \times \frac{1}{7}$ (c) ${}^+7 + {}^-2 = {}^-2 + {}^+7$

(d) $\frac{1}{2} \times -\frac{3}{7} = -\frac{3}{7} \times \frac{1}{2}$ (e) ${}^-12 + (8 \times {}^-3) = ({}^-12 + {}^-3) \times 8$

VI. Which of the following is an instance of the associative principle for addition?

(a) $(-8 + -9) + 2 = -8 + (-9 + 2)$

(b) ${}^+2 + (-8 + {}^-9) = {}^+2 + ({}^-9 + -8)$

(c) $(-8 + 2) + {}^-9 = (-8 + {}^-9) + 2$

(d) $2 \times (-8 + {}^-9) = 2 \times {}^-8 + 2 \times -9$

(e) $(-8 \times {}^-9) \times 1 = -8 \times {}^-9$

VII. Simplify.

$$ {}^+2 \times ({}^+3 \times {}^-2 + {}^-5 \times {}^+4) + {}^-10 $$

(a) ${}^-42$ (b) ${}^+42$ (c) ${}^-62$ (d) ${}^-72$ (e) ${}^-18$

VIII. Between the two numerals given in these exercises insert one of the symbols '>', '<' or '=' so that the resulting sentence is true.

1. $.52 \qquad .498$
2. ${}^-3.45 \qquad {}^-3.72$
3. $\frac{1}{7} \qquad \frac{2}{9}$
4. ${}^-756.3 \qquad {}^-784.5$
5. $.0025 \qquad \frac{1}{400}$
6. $.032 \qquad {}^-.033$
7. $0 \qquad -\frac{4}{5}$
8. $\frac{2}{11} \qquad 0$
9. $\frac{7 + 3}{28 - 8} \qquad \frac{167}{334}$
10. $\frac{1}{.4} \qquad \frac{1}{.5}$

IX. True or false?

1. $-1000 > -2$
2. $-10 > 8$
3. $-\frac{21}{71} > -\frac{20}{71}$
4. $-\frac{14}{3} < -5$
5. $0.016 > 0.0016$
6. $-0.016 > -0.0016$
7. $1\frac{1}{2} < \frac{5}{4}$
8. $\frac{-6}{-3} = 2$
9. $-\frac{16}{3} > 1$
10. $-1425 < 0$
11. $+|{}^{+}7| \not< |{}^{-}3|$
12. $|-10001| > |{}^{-}10002|$
13. ${}^{-}|5.47| \leq |5.469|$
14. $|8 - 13.5| \not> -|2.466|$
15. $|-1278.543| = -|1278.543|$
16. ${}^{-}|\frac{37}{15}| \leq {}^{-}|2.466|$
17. $\frac{231}{157} \not< \frac{231}{258}$
18. $\frac{1}{.8} < \frac{1}{.79}$
19. $|87| \geq 92$
20. $.0079 \not> -|.00791|$

X.
1. List 5 pairs of real numbers which belong to the operation adding -7.
2. List 5 pairs of real numbers which belong to the operation which is the inverse of subtracting 3.
3. List 5 pairs of real numbers which belong to the operation dividing by -4.
4. List 5 pairs of real numbers which belong to the operation oppositing.
5. List 5 pairs of real numbers which belong to the operation which is the inverse of dividing by -2.

XI. Which operations are the same as multiplying by -3?

(a) dividing by -3
(b) multiplying by the reciprocal of -3
(c) dividing by the reciprocal of -3
(d) the inverse of multiplying by -3
(e) the inverse of dividing by -3

XII. Which operations are the same as subtracting 6?

(a) adding the reciprocal of 6

(b) the inverse of subtracting 6

(c) adding the opposite of 6

(d) the inverse of subtracting -6

(e) the inverse of adding -6

XIII. Each of the following sentences is a consequence of one of the principles you studied in this unit. Below them are the names of these principles, each being preceded by a letter. In the blank at the left of each statement write the letter corresponding to the principle which is illustrated.

Sample: I 0. $5 + -5 = 0$ [Note: The letter 'I' is placed alongside the statement because it illustrates the principle of opposites.]

____ 1. $3 + (4 + 7) = (3 + 4) + 7$

____ 2. $(5 + 0) + 7 = 5 + 7$

____ 3. $4 + (7 \times 9) = 4 + (9 \times 7)$

____ 4. $(6 \times 2) + (4 \times 2) = (6 + 4) \times 2$

____ 5. $(3 \times 4) \times 1 = 3 \times 4$

____ 6. $[(8 \times 13) \times 7] + 5 = [8 \times (13 \times 7)] + 5$

____ 7. $^-7 + -^-7 = 0$

____ 8. $3 + (8 + 5) = (8 + 5) + 3$

____ 9. $6 \times (3 \times 0) = 6 \times 0$

____ 10. $(^-587 \times 169) \times \frac{1}{169} = {}^-587 \times (169 \times \frac{1}{169})$

____ 11. $(7 \times 0) + 5 = (0 \times 7) + 5$

____ 12. $2 \times [(8 + 4) + 3] = 2 \times [3 + (8 + 4)]$

____ 13. $9 - 5 = 9 + -5$

A. Commutative principle for addition

B. Commutative principle for multiplication

C. Associative principle for addition

D. Associative principle for multiplication

E. Distributive principle for multiplication over addition

F. Principle for adding 0

G. Principle for multiplying by 0

H. Principle for multiplying by 1

I. Principle of opposites

J. Principle for subtraction

XIV. 1. In a football game, the team from Zilchville High got possession of the ball and gained 5 yards on the first down. Then they lost 12 yards, gained 2 yards, and lost 3 on successive plays. Did they keep possession of the ball?

2. Below is a record of Mr. Sellars' bank account for one week. Determine his balance at the end of the week.

Monday -- Balance on hand: $1297.58; deposit: $415.00;
checks paid: $56.75, $32.19, $77.95.

Tuesday -- Checks paid: $41.68, $8.92, $13.12, $87.78.

Wednesday -- Deposits: $219.37, $682.46; checks paid:
$486.39, $17.62.

Thursday -- Checks paid: $63.97, $39.76, $102.96.

Friday -- Deposits: $57.81, $43.55.

SUPPLEMENTARY EXERCISES

A. Use single quotes in punctuating each of the following paragraphs in order to make sense out of it.

1. Marika, who just arrived in this country from Greece, went to the First National Store to buy a box of Kleenex. She walked up one aisle and down another until she saw a stack of boxes, each one of which had a Kleenex on it. She looked for the price and finally found 2/29 on the end of a box. She wondered if this meant that you could buy 29 boxes for 2 dollars or if it meant that 2 boxes cost 29 dollars. Neither possibility seemed very reasonable to her. She tried crossing out the 2s, but she wasn't sure whether she should get 0/9 or 1/19. She also thought about dividing both 2 and 29 by 2; since she wasn't sure, she decided to ask the clerk.

2. John was 8 years old. At his birthday party he had a cake with a small 8 on it. He wrote ate on a piece of paper and put the paper beside the cake. He put ate by the 8, but he didn't have 88. He had 8 ate. He could find the sum of 8 and 8 but he couldn't find the sum of 8 and ate. So, he ate the 8. But, he didn't eat the ate. The ate was left because it hadn't been eaten yet. John tried to eat the ate, but the ate was too big to be eaten. John took the e off the ate [it didn't hurt] and put it in front of the at. Then it spelled eat. So he did. He ate the eat. It was awful.

3. Mary is quite confused. She is Mary but yet she is not Mary. She said, "If my name is Mary then I must be Mary. But you say I am not Mary. If I am not Mary, why does everyone call me Mary?" I told Mary that if she wrote Mary on a piece of paper, that was Mary but not Mary. The reason for this is that Mary is Mary but Mary is not Mary. Do you think Mary will ever understand this? I hope so. I know that Mary doesn't understand this because it can't.

B. Simplify.

1. ${}^{+}8 + {}^{+}3$
2. ${}^{-}7 + {}^{+}9$
3. ${}^{-}3 + {}^{+}6$
4. ${}^{-}2 + {}^{-}9$
5. ${}^{+}5 + {}^{-}7$
6. ${}^{-}8 + {}^{+}7$
7. ${}^{-}4 + {}^{-}10$
8. ${}^{+}12 + {}^{+}3$
9. ${}^{-}5 + {}^{-}9$
10. ${}^{-}12 + {}^{+}4$
11. ${}^{-}17 + {}^{-}5$
12. ${}^{+}18 + {}^{-}3$
13. ${}^{-}17 + {}^{+}21$
14. ${}^{+}33 + {}^{-}15$
15. ${}^{-}51 + {}^{+}4$
16. ${}^{-}62 + {}^{-}8$
17. ${}^{-}5 + {}^{-}50$
18. ${}^{+}3 + {}^{+}17$
19. $({}^{+}2 + {}^{+}3) + {}^{-}4$
20. $({}^{-}5 + {}^{-}8) + {}^{+}5$
21. $({}^{-}51 + {}^{+}5) + {}^{-}7$
22. $({}^{-}7 + {}^{+}9) + {}^{-}17$
23. $({}^{+}41 + {}^{-}10) + {}^{+}15$
24. $({}^{-}5 + {}^{-}11) + {}^{+}27$
25. ${}^{+}6 + ({}^{-}3 + {}^{+}7)$
26. ${}^{+}8 + ({}^{+}9 + {}^{-}12)$
27. ${}^{-}12 + ({}^{-}3 + {}^{-}8)$
28. ${}^{+}10 + ({}^{-}4 + {}^{-}5)$
29. $({}^{+}4 + {}^{-}7) + ({}^{-}5 + {}^{-}12)$
30. $({}^{-}6 + {}^{+}9) + ({}^{-}9 + {}^{+}12)$
31. $({}^{-}71 + {}^{+}40) + ({}^{-}35 + {}^{+}49)$
32. $({}^{+}124 + {}^{-}72) + ({}^{-}584 + {}^{-}16)$
33. ${}^{+}\frac{2}{3} + {}^{+}\frac{7}{3}$
34. ${}^{-}\frac{5}{12} + {}^{-}\frac{9}{12}$
35. ${}^{+}\frac{5}{9} + {}^{-}\frac{2}{9}$
36. ${}^{+}\frac{2}{7} + {}^{-}\frac{3}{5}$
37. ${}^{-}\frac{1}{3} + {}^{+}\frac{2}{5}$
38. ${}^{+}\frac{5}{8} + {}^{-}\frac{3}{4}$
39. $\left({}^{+}\frac{1}{5} + {}^{+}\frac{2}{5}\right) + {}^{+}\frac{8}{5}$
40. $\left({}^{-}\frac{1}{3} + {}^{+}\frac{1}{2}\right) + {}^{+}\frac{3}{4}$
41. $\left({}^{-}\frac{2}{7} + {}^{-}\frac{1}{3}\right) + {}^{+}\frac{4}{21}$
42. $\left({}^{+}\frac{8}{3} + {}^{+}\frac{10}{3}\right) + {}^{-}\frac{5}{6}$
43. ${}^{+}4.03 + {}^{+}8.29$
44. ${}^{-}5.21 + {}^{+}7.83$
45. ${}^{-}7.83 + {}^{-}9.24$
46. ${}^{-}2.1 + {}^{+}1.73$
47. ${}^{+}3.08 + {}^{-}7.153$
48. ${}^{-}5.9 + {}^{+}6.83$
49. $({}^{+}5.93 + {}^{-}7.12) + {}^{+}6.21$
50. $({}^{-}5.08 + {}^{+}0.35) + {}^{-}2$

C. Answer each of the following questions.

1. On the New York Stock Exchange, a certain kind of stock whose par value is \$50 (per share) was listed on Monday at $62\frac{1}{2}$, on Tuesday at $61\frac{7}{8}$, on Wednesday at $61\frac{1}{2}$, on Thursday at $61\frac{1}{8}$, and on Friday at $61\frac{3}{4}$.

   (a) Use real numbers to list the changes from each day to the next day.

   (b) Mr. Brockman sold 25 shares of this stock on Tuesday, and Mr. Stockert sold 25 shares of the same kind of stock on Friday. Mr. Brockman received how much more money than Mr. Stockert?

2. The table below gives the T. V. weatherman's report of the high, the low, and the normal mean temperatures in a certain city for the first week of June.

Day	High	Low	Mean	Normal Mean
Monday	87	62		72
Tuesday	85	57		70
Wednesday	80	49		68
Thursday	83	53		65
Friday	78	51		66
Saturday	75	48		68
Sunday	77	56		67

   (a) Compute the mean temperature for each day. [Weathermen compute the mean temperature by averaging the high and low temperatures. This does not give you the exact mean but does come close.]

   (b) Use real numbers to indicate the difference of the mean from the normal mean for each day.

(c) Use real numbers to indicate the changes in the "High" readings from Monday through Sunday.

(d) Use real numbers to indicate the changes in the "Low" readings from day to day throughout the week.

3. The temperature chart of a certain hospital patient showed these readings.

	Morning	Noon	Evening
Wednesday	99.6	99.9	103.4
Thursday	99.1	100.2	102.1
Friday	99.9	100.1	100.9
Saturday	99.0	100.2	101.3
Sunday	99.2	99.6	100.2

(a) Use real numbers to indicate changes in the morning temperature readings for the successive days.

(b) Use real numbers to measure the changes in the evening temperatures for the successive days.

(c) Find the average temperature reading for each day.

4. Bruce was playing a game with Randy. On the first play Bruce gained 10 points. On successive plays he lost 6, gained 8, gained 3, lost 7, gained 9, lost 4. What was his standing after the final play?

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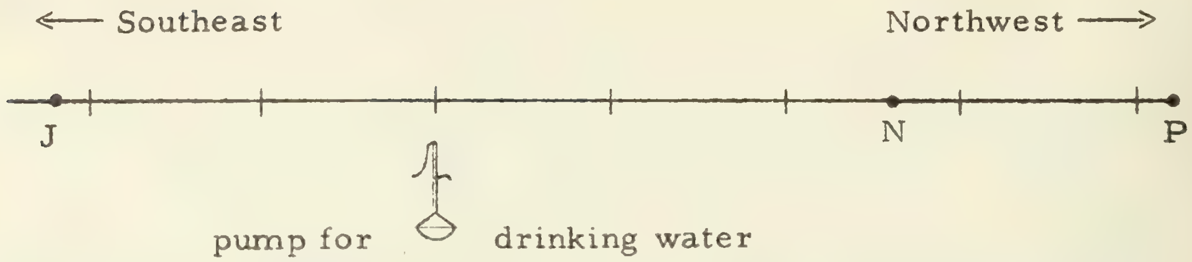
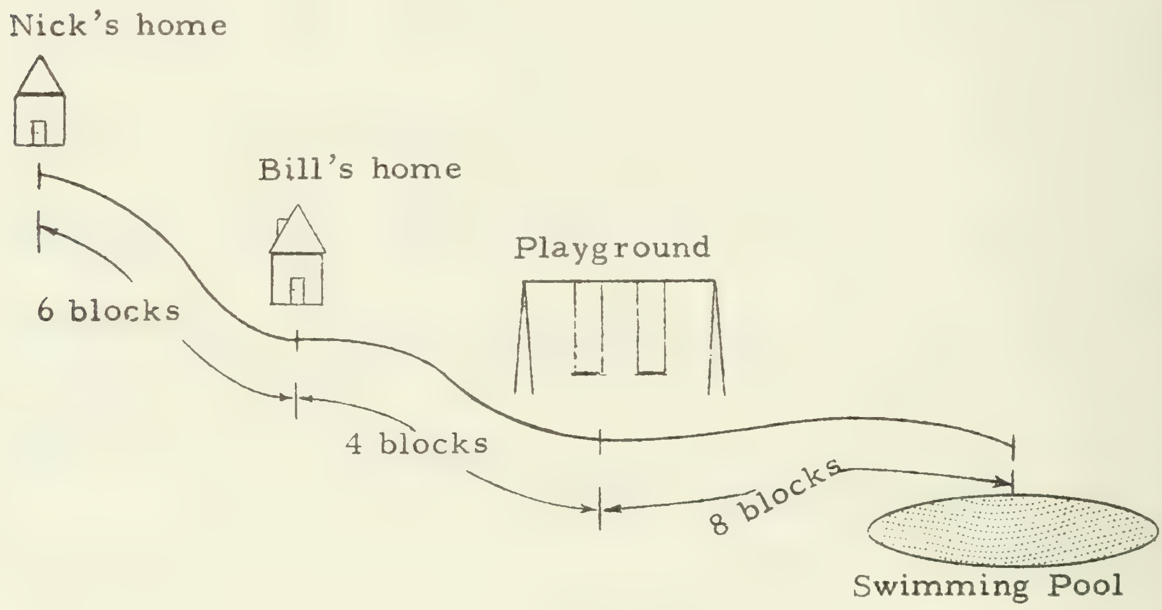
5.



On a hiker's trail, John starts from P which is 21 units northwest of the spot where pure drinking water may be obtained, and walks toward the pump. Bob starts at J (which is 11 units southeast of the pump) at the same time, and also hikes toward it. They pass each other at N, which is 13 units northwest of the pump, and continue on their way. However, Bob becomes thirsty and turns back at P to hike to the pump, and then on to J. Does Bob overtake John before he arrives at the pump? How far from the pump is John when Bob overtakes him? [The boys hike at steady rates.]

6.



Bill leaves home at 10:00 a.m. to walk to the playground which is 4 blocks away. His friend Nick leaves his home at 10:00 a.m. to visit Bill. When Nick arrives at Bill's home and finds he isn't there, he has a hunch that Bill may

have gone to the playground, so he goes on toward it. Meanwhile, Bill hasn't found anything of interest at the playground, so he has started on to the swimming pool, which is 8 blocks farther. When Nick can't see Bill at the playground, he turns around and starts home; however, when he gets there he still wants to talk to Bill, and decides to return to Bill's home. But Bill is on his way toward Nick's home, since he saw none of his friends at the pool. If Bill and Nick have walked at the same rate (and you disregard any time lost while they look around at playground and pool), can you discover how far from Nick's home the two boys finally meet?

7. Ann starts from Sally's home and bicycles north. At the same time, Carol starts from her own home, which is 22 blocks south of Sally's, and is also bicycling north. Ann travels only 4 blocks when she turns around and starts south. Carol is 18 blocks north of Sally's home when she turns back; at that time Ann is midway between the point where Carol turns back and Carol's home. When Carol gets home and does not see Ann, she starts north again and travels until she meets her. How far from Sally's home do they meet? In what direction from Sally's home is the meeting point?

8. Jane and Betty both start hiking at the same time, and both travel in an easterly direction. Jane starts at M, where a sign post pointing west reads '4 miles to Fish Hook'. Betty starts at Q; the sign board there points east and reads, '8 miles to Fish Hook'.

   Betty passes Jane at the R signboard which points west and reads '7 miles to Fish Hook'. She continues hiking east until she reaches a certain apple tree along the road. She turns here (after hastily picking an apple to munch on the way home!) and starts back to Q. On the way she meets Jane at the W signboard. It points west and reads, '10 miles to Fish Hook'.

   If both Betty and Jane are hiking at steady rates, what is the distance of the apple tree from the village of Fish Hook? What direction is the tree from the village?

D. Simplify.

1. $^{+}5 + ^{-}5$
2. $0 + ^{+}10$
3. $^{-}2\frac{1}{2} + ^{+}2\frac{1}{3}$
4. $^{+}3\frac{2}{3} + ^{+}4\frac{1}{2}$
5. $^{+}11\frac{1}{8} + ^{-}10\frac{3}{4}$
6. $^{+}1 + ^{-}.05$
7. $^{+}1001 + ^{-}101$
8. $^{-}1,237,248 + ^{+}1,237,248$
9. $\left(0 + ^{+}8\frac{5}{16}\right) + ^{-}7\frac{9}{16}$
10. $^{+}3\frac{7}{16} + ^{+}4\frac{1}{4}$
11. $^{+}6.35 + ^{-}8\frac{2}{5}$
12. $^{-}1,000,000 + ^{-}1,000,000$
13. $^{-}2,000,000 + ^{+}2,000,000$
14. $^{-}7 + ^{+}8$
15. $^{-}8 + ^{+}7$
16. $(^{-}1 + ^{+}1) + (^{-}1 + ^{+}1)$
17. $^{-}423 + ^{+}398$
18. $^{-}9.8 + ^{-}7.6$
19. $^{-}2 + ^{-}3.16$
20. $(^{-}9 + ^{-}9) + ^{-}9$
21. $(^{+}28 + ^{+}29) + ^{+}25$
22. $^{+}1,098,762 + ^{-}1$
23. $^{-}17,098 + ^{+}17,097$
24. $^{+}18,607,487 + ^{-}18,607,487$
25. $(^{+}6.5 + ^{-}6.5) + (^{+}7.5 + ^{-}7.5)$
26. $^{+}7 + ^{-}3\frac{2}{7}$
27. $^{+}18.01 + ^{+}1.6$
28. $(^{-}3 + ^{-}8) + ^{+}7$
29. $^{-}3\frac{1}{2} + ^{+}2\frac{7}{8}$
30. $^{-}8\frac{1}{3} + ^{+}9\frac{1}{16}$
31. $^{-}7\frac{4}{7} + ^{-}9\frac{2}{5}$
32. $^{+}.03 + \frac{^{-}1}{4}$
33. $\frac{^{-}1}{9} + ^{+}4\frac{1}{11}$
34. $(^{+}9 + ^{-}5) + ^{-}23$
35. $(^{-}8 + ^{+}3) + ^{-}12$
36. $^{+}5 + ^{+}0.02$
37. $^{-}.05 + ^{-}.28$
38. $^{+}.312 + ^{-}.213$
39. $^{-}.008 + ^{-}.082$
40. $(^{+}5 + ^{+}2) + (^{-}3 + ^{-}4)$
41. $(^{-}10 + ^{+}2) + (^{-}12 + ^{-}3)$
42. $(^{-}10 + ^{-}7) + (^{+}3 + ^{-}2)$
43. $(^{+}12 + ^{-}32) + ^{-}12$
44. $\left(\frac{^{+}1}{2} + \frac{^{-}1}{4}\right) + \frac{^{-}1}{8}$
45. $^{-}18 + (^{+}10 + ^{-}12)$
46. $(^{-}25 + ^{+}5) + (^{-}4 + ^{+}9)$
47. $(^{-}17 + ^{-}8) + (^{-}5 + ^{+}30)$
48. $(^{-}97 + ^{-}3) + (^{+}25 + ^{-}75)$
49. $(^{+}302 + ^{-}201) + (^{-}11 + ^{+}10)$
50. $(^{+}76 + ^{-}16) + (^{-}20 + ^{+}10)$

E. Simplify.

1. $^-7 \times {}^+4$	2. $^+7 \times {}^-4$	3. $^+7 \times {}^+4$
4. $^-7 \times {}^-4$	5. $^+8 \times {}^-5\frac{1}{2}$	6. $^-8 \times {}^-5\frac{1}{2}$
7. $^-7\frac{1}{2} \times {}^-3\frac{1}{3}$	8. $^-8 \times 0$	9. $^-\frac{5}{4} \times {}^-\frac{4}{5}$
10. $^-6 \times {}^+4\frac{1}{3}$	11. $^+10 \times {}^+\frac{1}{8}$	12. $^+\frac{1}{2} \times {}^+2\frac{1}{3}$
13. $^+9\frac{1}{3} \times {}^-2\frac{1}{2}$	14. $^-2\frac{1}{2} \times {}^-9\frac{1}{3}$	15. $^+2\frac{1}{2} \times {}^-8\frac{1}{4}$
16. $^+5 \times {}^+2$	17. $^+3 \times {}^+4$	18. $^-7 \times {}^+3$
19. $^-4 \times {}^-9$	20. $^+5 \times {}^-3$	21. $^+8 \times {}^-11$
22. $^-12 \times {}^+9$	23. $^-4 \times {}^+6$	24. $^-3 \times 0$
25. $0 \times {}^+5$	26. $^-7 \times {}^-7$	27. $^-4 \times {}^-3$
28. $^-5 \times {}^+2$	29. $0 \times {}^-6$	30. $^+4 \times 0$
31. $(^-5 \times {}^-4) \times {}^+4$	32. $(^-2 \times {}^-9) \times {}^-4$	33. $^-7 \times (^+8 \times {}^-7)$
34. $(^+6 \times {}^-10) \times {}^-4$	35. $(^+6 \times {}^-10) \times {}^+4$	36. $(^-8 \times {}^-12) \times {}^+20$
37. $^+3 \times (^-4 \times {}^-5)$	38. $(^-2 \times {}^-3) \times {}^-8$	39. $^-10 \times (^-20 \times 0)$
40. $^-\frac{1}{2} \times (^-\frac{1}{2} \times {}^+1)$	41. $(^-5 \times {}^-5) \times {}^-5$	42. $(^-40 \times {}^+4) \times {}^-2$
43. $^+30 \times (^-2 \times {}^-4)$	44. $(^-35 \times {}^-\frac{1}{5}) \times {}^+7$	45. $^-4 \times (^-1 \times {}^-\frac{1}{4})$
46. $^-5 \times (^+10 \times {}^-\frac{1}{2})$	47. $(^-10 \times 0) \times {}^-8$	48. $(^-7 \times {}^-9) \times {}^+\frac{1}{3}$

49. $^+\frac{1}{4} \times (^-8 \times {}^-12)$

50. $(^-12 \times {}^+\frac{1}{3}) \times {}^-9$

F. Simplify.

1. $1 + 2 \times 7$	2. $1 + 7 \times 2$	3. $15 - 3 \times 2$
4. $12 + 4 \div 2$	5. $6 \times 5 + 3$	6. $11 \times 4 - 9$

7. $7 + 8 - 3 - 2 + 12$

8. $12 \times 2 \div 4 + 1 \times 3 + 6$

9. $3 \times 2 \times (5 \times 12)$

10. $3 \times 6 \times 5 + (1 + 3) \times 4$

11. $13 \times (2 \times 4) \div 8 + 3 \times 5 \times 2 \div 10$

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12. $\frac{12}{5} \times \frac{10}{13} \times \frac{5}{2}$

13. $44 \times 2 \div 8 \times 6 \div 3 \times 4\frac{1}{2}$

14. $36 \div \frac{1}{3} \times \frac{1}{6} + 48 \times \frac{5}{6} + 2$

15. $^{-}2 \times (^{-}8 + {}^{-}7) \times {}^{-}8$

16. $(5 + {}^{-}4) \times {}^{-}3 + 8$

17. $^{-}7 \times {}^{-}3 \times {}^{-}4 \times \frac{5}{6}$

18. $8 \times {}^{-}3 \div {}^{-}2 \times {}^{-}3$

19. $13 \div {}^{-}26 \times \frac{1}{2} + {}^{-}8$

20. $10 + {}^{-}8 + [7 + {}^{-}3 + {}^{-}2] + {}^{-}9 + 12$

21. $^{-}3 \times {}^{-}8 + 2$

22. $^{-}7 \times 3 + 4 \times {}^{-}5$

23. $5 \times {}^{-}7 + 5 \times {}^{-}3$

24. $^{-}2 \times 5 + 5 \times 6$

25. $^{-}82 + (^{-}3 \div 82)$

26. $^{-}2 \times (^{-}20 \div 20)$

27. $12 \times {}^{-}3 \times {}^{-}1 \times {}^{-}2$

28. $^{-}3 \times 0 + {}^{-}4 \times {}^{-}1$

29. $(^{-}18 + {}^{-}12) \times (^{-}5 \times 6)$

30. $(48 \div {}^{-}16 + {}^{-}32) \times (8 \times {}^{-}9)$

G. Each of the following sentences is an instance of one of the principles for the numbers of arithmetic. Tell which principle.

1. $8 \times 8 + 2 \times 8 = (8 + 2) \times 8$

2. $1 \times (5 + 2) = (5 + 2) \times 1$

3. $(3 + 4) + 5 = 3 + (4 + 5)$

4. $73 + 0 = 73$

5. $18 + 32 = 32 + 18$

6. $392 \times 1 = 392$

7. $618 \times 0 = 0$

8. $17 \times (8 + 3) = 17 \times 8 + 17 \times 3$

9. $(16 \times 15) \times 4 = 16 \times (15 \times 4)$

10. $397 \times 18 = 18 \times 397$

11. $798.3 + 0 = 798.3$

12. $62 + 13 + 7 = 62 + (13 + 7)$

13. $3\frac{1}{2} + 5\frac{1}{4} = 5\frac{1}{4} + 3\frac{1}{2}$

14. $357.25 \times 0 = 0$

15. $.7361 = .7361 \times 1$

16. $14 \times 5 \times 18 = 14 \times (5 \times 18)$

17. $\frac{2}{5} \times \frac{3}{7} = \frac{3}{7} \times \frac{2}{5}$

18. $373.8 = 373.8 + 0$

19. $16 \times 5\frac{1}{4} = (16 \times 5) + (16 \times \frac{1}{4})$

20. $7.23 + .77 = .77 + 7.23$

21. $\frac{2}{3} \times \frac{5}{6} \times \frac{3}{5} = \frac{2}{3} \times \left(\frac{5}{6} \times \frac{3}{5}\right)$

22. $0 = 19.3 \times 0$

23. $\frac{13}{17} \times 1 = \frac{13}{17}$

24. $\left(\frac{1}{7} + \frac{3}{14}\right) + \frac{11}{42} = \frac{1}{7} + \left(\frac{3}{14} + \frac{11}{42}\right)$

25. $57.3 = 57.3 + 0$

26. $.32 + .73 = .73 + .32$

27. $35 \times 7 + 35 \times \frac{1}{5} = 35 \times 7\frac{1}{5}$

28. $.4 \times (.8 \times 1.5) = (.4 \times .8) \times 1.5$

29. $93\frac{1}{7} \times 0 = 0$

30. $475.8 = 475.8 \times 1$

31. $\left(10\frac{1}{2} + 3\frac{3}{4}\right) + 5\frac{1}{4} = 10\frac{1}{2} + \left(3\frac{3}{4} + 5\frac{1}{4}\right)$

32. $.92 \times .34 \times 5 = .92 \times (.34 \times 5)$

33. $(.90 + .02) \times 1.7 = (.90 \times 1.7) + (.02 \times 1.7)$

34. $27\frac{3}{5} + 0 = 27\frac{3}{5}$

35. $187 \times 37 = 37 \times 187$

36. $43\frac{1}{3} \times 1 = 43\frac{1}{3}$

37. $\frac{5}{6} \times \left(\frac{3}{7} \times \frac{2}{3}\right) = \frac{5}{6} \times \frac{3}{7} \times \frac{2}{3}$

38. $\frac{9}{15} + \frac{3}{8} = \frac{3}{8} + \frac{9}{15}$

39. $0 = 1297.8 \times 0$

40. $827 + (73 + 769) = (827 + 73) + 769$

41. $25\frac{1}{4} \times 12 = 25 \times 12 + \frac{1}{4} \times 12$

42. $\frac{3}{14} \times \frac{2}{21} = \frac{2}{21} \times \frac{3}{14}$

43. $\frac{9}{17} = \frac{9}{17} \times 1$

44. $\frac{9}{17} = \frac{9}{17} + 0$

H. Simplify. Do as much of the computation as possible without writing.

1. $2 \times \{[(5 \times 3) + (5 \times 2)] + [(5 \times 5) + (5 \times 2)]\}$

2. $10 \times \{[50 \times 6] + [50 \times 7] + [(50 \times 5) + (50 \times 2)]\}$

3. $.5 \times \{[(62 \times 8) + (62 \times 4)] + [(62 \times 5) + (62 \times 3)]\}$

4. $\frac{1}{3} \times \{[(5 \times \frac{1}{2}) + (\frac{1}{2} \times 7)] + [(340 \times \frac{1}{2}) + (\frac{1}{2} \times 8)]\}$

5. $(8 \times 40) \times 25$

6. $5 \times (2 \times 7)$

7. $(5 \times 7) + (7 \times 8)$

8. $(15 \times 7) + (7 \times 15)$

9. 125×0

10. $75 \times 0 \times 9$

11. $(8 + 75) + 25$

12. $752 + (48 + 1,762)$

13. $(45 \times 5) \times 2$

14. $\frac{8}{9} \times (9 \times 222)$

15. $(36 \times 17) + (36 \times 3)$

16. $(18 \times 27) + (3 \times 18)$

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17. $(51 \times 1{,}763) + (51 \times 237)$

18. $17\frac{1}{3} \times 12$

19. 102% of 35

20. $\frac{72}{100} \times 200$

21. $\left(\frac{7}{12} + \frac{5}{6}\right) \times \frac{6}{5}$

22. $3\frac{1}{10} \times 8$

23. $(972.75 \times 37) + (37 \times 27.25) + 490$

24. $(29 \times 51) + (62 \times 51) + (9 \times 51)$

25. $\frac{1}{2} \times \{[33 \times 18] + [18 \times 7] + [18 \times 60]\} + 100$

I. Simplify.

1.	$^-8 - {}^+3$	2.	$^-8 - {}^-3$	3.	$^+8 - {}^-3$
4.	$^+3 - 0$	5.	$0 - {}^+3$	6.	$^+8 - {}^+3$
7.	$^-6 - {}^+14$	8.	$^-7 - {}^+8$	9.	$^+9 - {}^-2$
10.	$0 - {}^-4$	11.	$^-6 - {}^-7$	12.	$^-13 - {}^-13$
13.	$2 - {}^-2$	14.	$7 - {}^+2$	15.	$^+9 - {}^+15$
16.	$^-17 - {}^-8$	17.	$^-23 - 31$	18.	$17 - 29$
19.	$^-4 - 9$	20.	$9 - {}^-4$	21.	$^-36 - {}^-43$
22.	$^-5 - {}^-8$	23.	$^-8 - {}^-5$	24.	$5 - 8$
25.	$8 - 5$	26.	$0 - 47$	27.	$17 - 17$
28.	$638 - 635$	29.	$^-638 - {}^-635$	30.	$635 - 638$
31.	$7 - {}^-31$	32.	$^-5 - {}^-21$	33.	$^+6 - {}^-31$
34.	$8 - 6$	35.	$10 - 18$	36.	$0 - \frac{1}{8}$
37.	$^-2 - {}^-13$	38.	$^+8 - {}^-6$	39.	$^+10 - {}^+28$
40.	$^-15 - 0$	41.	$^+5 - {}^+5$	42.	$^+16 - {}^+14$
43.	$^+\frac{1}{2} - {}^+\frac{1}{3}$	44.	$^+2 - {}^+13$	45.	$^+5 - {}^-5$
46.	$^+16 - {}^-14$	47.	$^+3 - {}^+1\frac{1}{4}$	48.	$^-2 - {}^+13$
49.	$^-5 - {}^+5$	50.	$^+10 - {}^+8$	51.	$^+3 - 0$
52.	$^+2 - {}^-13$	53.	$^-5 - {}^-5$	54.	$^-16 - {}^+14$
55.	$^-10 - {}^+28$	56.	$^-.3 - {}^+.5$	57.	$^+.02 - {}^-.09$

58. ${}^{+}103 - {}^{-}52$

59. ${}^{-}92 - {}^{-}37$

60. ${}^{-}102 - {}^{+}724$

61. ${}^{+}8.5 - {}^{+}3.7$

62. ${}^{-}9.6 - {}^{+}11.3$

63. ${}^{+}17.1 - {}^{-}19.3$

64. $983 - 1475$

65. $683 - 729$

66. $75.3 - 82.7$

67. ${}^{+}\frac{2}{3} - {}^{+}\frac{1}{6}$

68. ${}^{-}\frac{3}{4} - {}^{-}\frac{7}{8}$

69. ${}^{+}\frac{3}{5} - {}^{-}\frac{2}{3}$

70. ${}^{-}\frac{5}{7} - {}^{+}\frac{8}{9}$

71. ${}^{+}\frac{3}{7} - {}^{+}\frac{3}{8}$

72. ${}^{-}\frac{1}{16} - {}^{-}\frac{3}{8}$

73. ${}^{+}6\frac{1}{5} - {}^{-}3\frac{1}{5}$

74. ${}^{-}5\frac{2}{3} - {}^{+}7\frac{1}{3}$

75. ${}^{-}6\frac{1}{6} - {}^{-}7\frac{2}{3}$

76. ${}^{-}17\frac{1}{5} - {}^{+}3\frac{1}{5}$

77. ${}^{-}8\frac{1}{5} - {}^{-}9\frac{1}{3}$

78. ${}^{-}12\frac{1}{2} - {}^{+}6.4$

<u>J</u>. Simplify.

1. ${}^{-}8 + {}^{-}3 - {}^{+}6$

2. ${}^{-}4 - {}^{-}3 - {}^{+}6$

3. ${}^{+}7 - {}^{-}3 + {}^{-}9$

4. ${}^{+}5 - {}^{-}2 + {}^{-}5$

5. ${}^{-}6 - {}^{+}7 - {}^{-}8 + {}^{-}3$

6. ${}^{+}10 - {}^{-}3 - {}^{+}7 - {}^{-}7$

7. ${}^{+}11 + {}^{-}12 - {}^{-}13 + {}^{+}9$

8. ${}^{-}4 - {}^{-}5 - 12 - 15$

9. $15 - 6 - 3 - 7 - 2$

10. $19 + 5 - 3 - 2 - 23$

11. $({}^{+}4 - {}^{-}3) + ({}^{-}5 + {}^{-}2)$

12. $({}^{-}7 - {}^{+}7) + ({}^{-}8 - {}^{-}3)$

13. $({}^{+}7 - {}^{-}2) - ({}^{-}3 - {}^{-}4)$

14. $({}^{+}10 - {}^{-}3) - ({}^{-}5 + {}^{-}7)$

15. $(9 - 2) - (8 - 3)$

16. $(12 - 14) + (7 - 8)$

17. $(5 - 12) - (9 - 11)$

18. $(6 - 61) - (17 - 43)$

19. $({}^{-}6.3 + {}^{+}1.4) - ({}^{-}4.8 + {}^{+}7.7) + ({}^{-}5.4 + {}^{+}3.2)$

20. ${}^{-}\frac{1}{2} + \left({}^{-}\frac{3}{4} - {}^{+}\frac{1}{4}\right) + \left({}^{+}3\frac{1}{4} + {}^{-}5\frac{1}{2}\right)$

<u>K</u>. Simplify.

1. $8 - 3 + 12$

2. $7 - 18 - 12$

3. $-5 + 9 - 6$

4. $-3 - 8 - 15$

5. $-6 + 9 + 2$

6. $-5 + 3 - 2$

7. $-12 - 3 + 15$

8. $-61 + 11 - 35$

9. $-10 - 4 - 40$

10. $-17 + 3 - 25$

11. $10 - 11 - 13$

12. $-9 - 8 + 7$

(continued on next page)

13. $-7 + 9 + 12 - 8 - 5 - 10 + 15$

14. $+12 + 8 - 9 - 6 - 5 + 7 + 3 - 8$

15. $-25 - 5 - 20 + 42 + 5 - 6 + 4$

16. $+70 - 20 - 7 - 6 + 3 + 4 - 14 - 27$

17. $+13 + 3 - 5 - 3 + 2 - 5 - 6$

18. $-27 - 13 + 32 + 4 - 9 - 3 + 5$

19. $-12 - 19 - 3 + 6 - 2 + 4 + 12 + 8$

20. $-13 - 2 + 9 + 8 + 7 - 10 - 3 + 7$

21. $+19 + 3 - 10 - 8 + 6 - 11 - 4 + 7$

22. $+17 - 4 - 3 + 6 - 8 - 12 + 5$

23. $-23 - 4 - 3 + 12 - 2 + 8 + 14 - 2$

24. $-31 + 27 - 3 - 8 + 2 + 14$

25. $+18 + 9 - 5 - 11 - 8 + 7 - 12$

26. $+17 - 8 - 11 + 9 - 12 - 5 + 7$

27. $-12 + 7 + 16 - 5 + 9 - 8 - 11$

28. $+7 - 8 - 12 + 15 - 5 - 11 + 9$

29. $-11 - 9 + 6 + 14 + 10 - 12 - 4$

30. $-5 + 9 - 8 + 7 + 13 - 11 - 12$

31. $+17 + 4 - 8 - 9 - 3 - 9$

32. $-23 + 19 - 3 + 10 - 5 - 7$

33. $-\frac{1}{2} + \frac{3}{4} - \frac{1}{4} + 3\frac{1}{4} - 5\frac{1}{2}$

34. $+.5 + .3 - .8 - .7 - .2 + 1.2$

35. $-7\frac{1}{2} + 2\frac{1}{4} + 1\frac{1}{2} - 8\frac{1}{4} - 3\frac{3}{4}$

36. $+.12 + .11 - .10 - .02 + .03 - .13 - .01$

37. $-6.3 + 1.4 + 4.8 - 7.7 + 3.2 - 5.4$

L. Simplify.

1. $-3 \times (5 - 1)$
2. $-7 \times (8 - 10)$
3. $-5 \times (7 - 2)$
4. $5 \times (4 - 8)$
5. $6 \times (7 + 2)$
6. $-3 \times (8 - -5)$
7. $4 + (9 - 12) \times 5$
8. $5 + 3 \times (15 - 21)$
9. $6 - 2 \times (10 - 5)$
10. $11 - 7 \times (2 - 9)$
11. $-9 - 3 \times (8 - 11)$
12. $-12 - 3 \times (4 - 2)$
13. $(2 - 8) \times (7 - 3)$
14. $(12 - 5) \times (7 - 5)$
15. $(6 + 3) \times (6 - 3)$
16. $(5 - 9) \times (5 + 9)$
17. $-3 + (7 - 5) + (6 - 8) + (2 - 5) + (7 - 11)$
18. $-9 - (4 - 7) + (12 - 3) - (6 - 8) - (15 - 7)$
19. $-5 + 2 \times (8 - 3) + 5 \times (6 - 2) + 4 \times (8 - 17)$
20. $-11 - 3 \times (4 - 1) - 2 \times (9 - 8) - 3 \times (9 + 2)$
21. $(9 - 3) \times (7 - 5) + (6 - 1) \times (11 - 2)$
22. $(5 - 7) \times (11 - 3) + (12 - 2) \times (15 - 9)$
23. $(6 + 4) \times (7 - 2) - (8 - 15) \times (9 - 20)$
24. $(12 - 7) \times (9 - 18) - (8 + 2) \times (15 - 17)$
25. $[5 + 3 \times (8 - 4)] \times [9 - 4 \times (8 - 6)]$
26. $[-6 - 2 \times (7 - 10)] \times [21 + 5 \times (6 - 10)]$
27. $11 - [6 + 2 \times (7 - 3)] + [5 - 3 \times (6 - 10)]$
28. $-19 + [-3 - 9 \times (11 - 12)] - [-15 - 5 \times (2 - 8)]$
29. $15 - 2 \times [3 + 4 \times (7 - 2)] - 5 \times [8 - 2 \times (5 + 9)]$
30. $-83 - 2 \times [-4 - 7 \times (9 - 2)] - 3 \times [-2 - (9 - 8)]$
31. $100 - 6 \times \{3 - [5 - 2 \times (7 - 3)] + [-2 - 3 \times (7 - 2)]\}$
32. $[17 - (8 - 3) \times (4 - 7)] \times [18 - (7 - 9) \times (8 - 12)]$
33. $[51 - (6 + 5) \times (7 - 2)] \times [-17 - (5 - 9) \times (8 - 3)]$
34. $-26 \times (57 - 82) + -31 \times (57 - 82) - 43 \times (57 - 82)$

M. Simplify.

1. $^{+}7 \div {}^{+}2$	2. $^{-}18 \div {}^{-}3$	3. $^{-}21 \div {}^{+}7$
4. $^{+}42 \div {}^{-}7$	5. $^{+}8 \div {}^{-}6$	6. $^{-}8 \div {}^{-}6$
7. $^{-}6 \div {}^{+}8$	8. $^{-}6 \div {}^{-}8$	9. $^{+}10 \div {}^{-}20$
10. $^{-}20 \div {}^{+}10$	11. $^{-}20 \div {}^{-}10$	12. $^{+}10 \div {}^{+}15$
13. $^{+}15 \div {}^{-}3$	14. $^{-}15 \div {}^{+}3$	15. $^{+}16 \div {}^{-}32$
16. $^{+}26 \div {}^{-}13$	17. $^{-}34 \div {}^{-}17$	18. $^{+}27 \div {}^{-}54$
19. $^{-}.08 \div {}^{+}4$	20. $^{-}.106 \div {}^{-}.3$	21. $^{+}.873 \div {}^{-}.09$
22. $^{-}1,000,005 \div {}^{+}15$	23. $^{+}9873234 \div {}^{-}1234$	24. $^{+}225 \div {}^{-}15$
25. $^{-}196 \div {}^{+}14$	26. $^{+}169 \div {}^{-}13$	27. $^{-}256 \div {}^{-}16$
28. $^{+}289 \div {}^{-}17$	29. $^{+}324 \div {}^{+}18$	30. $^{-}361 \div {}^{-}19$

31. $\frac{^{+}72}{^{-}8}$	32. $\frac{^{-}54}{^{+}2}$	33. $\frac{^{-}38}{^{-}19}$	34. $98/14$
35. $132/12$	36. $^{-}108/9$	37. $56/^{-}8$	38. $^{-}84/12$
39. $^{-}91/^{-}13$	40. $\frac{128}{^{-}4}$	41. $\frac{^{-}147}{^{-}3}$	42. $\frac{^{-}156}{4}$
43. $\frac{999,999}{^{-}33}$	44. $\frac{^{-}98762}{^{-}23}$	45. $625/^{-}25$	46. $\frac{63}{^{-}9}$
	47. $\frac{^{-}171}{9}$	48. $\frac{^{-}112}{^{-}8}$	

N. Write a comparison sentence for each of the pairs of numbers listed.

1. $(^{+}5,\ ^{+}3)$	2. $(^{+}7,\ ^{-}4)$	3. $(^{-}5,\ ^{+}4)$
4. $(^{-}7,\ ^{-}12)$	5. $(^{-}20,\ ^{-}15)$	6. $(^{+}4,\ ^{+}17)$
7. $(^{-}17,\ ^{-}16)$	8. $(^{-}16,\ ^{-}17)$	9. $(^{-}16,\ ^{+}16)$
10. $(^{+}16,\ ^{-}16)$	11. $(^{+}16,\ ^{-}17)$	12. $(^{-}4,\ ^{-}4.1)$

13. $(^{+}5 + {}^{-}7,\ ^{-}5 + {}^{+}7)$ 14. $(^{-}2 \times {}^{-}3,\ 2 \times 3)$

15. $(2 + 3\frac{1}{2},\ 7 - \frac{1}{2})$ 16. $(9 - 17,\ 17 - 9)$

17. $(5 \times (3 - 5),\ ^{-}5 \times (5 - 3))$ 18. $(-3 - 1\frac{1}{4},\ 5 - 9\frac{3}{4})$

O. Insert an ' > ', an ' < ', or an ' = ' to get a true statement.

1. $\frac{1}{0.9}$ $\frac{1}{0.8}$
2. $7\frac{3}{10}$ $4\frac{4}{5}$
3. $\frac{1}{6}$ $\frac{1}{7}$
4. -8 -8
5. 7 -9
6. 148 -14
7. $\frac{1}{373}$ $\frac{1}{377}$
8. $-5\frac{3}{8}$ $4\frac{1}{4}$
9. 342 $^-342$
10. $-\frac{1}{6}$ $-\frac{1}{11}$
11. $^-1,999,999$ $^-1,999,997$
12. $-\frac{20}{31}$ $-\frac{20}{32}$
13. $.082$ $.820$
14. $^-1.001$ $^-1.0001$

*

True or false?

15. $-99 > -97$
16. $8 > 8.1$
17. $-\frac{17}{4} = -4.25$
18. $-1.2 \leq 0$
19. $7 \not< 8$
20. $-10 \not> 0$
21. $-8 \not> -8$
22. $-8 \not> 8$
23. $8 \not> 8$
24. $8 \not> -8$
25. $(^-7 + 7) \neq 0$
26. $.097 \geq \frac{12}{125}$
27. $\frac{3}{5} < .6001$
28. $-\frac{1}{7} > -.142$
29. $.0028 \not< \frac{7}{2500}$
30. $-5 \geq -2$
31. $5 \not> 2$
32. $5 \neq -5$
33. $-5 \not< -5$
34. $2.3 \not> 2.4$
35. $-\frac{1}{3.8} \geq \frac{1}{5.4}$
36. $-\frac{1}{200} \leq -\frac{1}{2000}$
37. $-13 = 13$
38. $-.0008 \not< .008$
39. $-\frac{.198}{2} \geq -\frac{.196}{2}$
40. $33\frac{1}{3} > 33.33$
41. $^-.082 < {}^-.0082$
42. $^+.091 > {}^+.901$
43. $-\frac{1}{5} \geq -\frac{1}{6}$
44. $\frac{10000}{29786} > \frac{9999}{29787}$
45. $\frac{.084}{.2} < \frac{.087}{.3}$
46. $\frac{1}{.08} \leq \frac{1}{.09}$

HIGH SCHOOL MATHEMATICS

Unit 2.

GENERALIZATIONS AND ALGEBRAIC MANIPULATION

UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS

MAX BEBERMAN, *Director*

HERBERT E. VAUGHAN, *Editor*

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TABLE OF CONTENTS

<u>True</u> <u>or</u> <u>False</u>.--Mr. Jones who teaches mathematics in Zabranchburg Junior High School has an interesting way of preparing True-False tests. First, he duplicates one sheet of items for each student in his class. Then he uses a paper punch to make holes at various spots on these pages.

Turn to page 2-B to see the first page of Mr. Jones' test.

neig

n ea

writ

or it

t sh

for

Name ______________________

Class ______________________

Date ______________________

TRUE - FALSE TEST

Instructions: Write 'T' in the space to the left of an item if the statement is true. Write 'F' in this space if the statement is false.

______ 1. $3 + 7 = 10$

______ 2. $8 - 5 = 12$

______ 3. $5 + \bullet = 17$

______ 4. $4 + -3 = \bullet$

______ 5. $\bullet \times 4 = 24$

______ 6. $\bullet \div 3 = 15$

______ 7. $\bullet \times 5 = 40$

______ 8. $-2 \times \bullet = -8$

______ 9. $\bullet + 9 = 19$

______ 10. $\bullet \times 0 = 0$

Next, he duplicates several different second pages. Each second page can be slid under the first page, and has numerals in positions matching the holes in the first page. When a student takes this test, Mr. Jones gives him a copy of the first page and a copy of one of the second pages. Then the student fastens the pages together and is ready to work on the test.

CLASS EXERCISE

A. Make up your own second page for the True-False test on page 2-B. Choose your numerals for the second page so that about half of the items on the test are false. Now, take the test and record your answers ['T' or 'F'] in a column on another sheet of paper.

B. Exchange second pages with your neighbor. Take this new test and record your answers in a column alongside the answers to the first test.

C. Repeat Part B by exchanging the second page you now have for that of another neighbor.

D. Look at your three columns of answers.

1. Is there a 'T' for item 1 in each column? Should every student in the class have written 'T' for item 1?

2. Is there an 'F' for item 2 in each column? Should every student have written 'F' for item 2?

3. What answer should every student have for item 3? Explain.

4. For what items on the test should every student have the same answer?

5. Explain why it is unlikely for every student to have the same answer for, say, item 8?

TRUE OR FALSE???!!!

Mr. Edwards, principal of Zabranchburg Junior High, took over Mr. Jones' class one day when Mr. Jones was ill. Mr. Jones had sent instructions for Mr. Edwards to give the class the True-False test which was in the top drawer of the desk. But, Mr. Edwards was not told to give out the second page, also. At the beginning of the class period Mr. Edwards distributed the first page of the test to the class, and told them he would collect papers at the end of five minutes. In the meantime,

NO TALKING!

AND NO QUESTIONS!

Of course, no student had any trouble telling that the first statement was true and the second statement was false. Item 3 puzzled the students. They wished they could ask Mr. Edwards for the second page, but they remembered:

NO TALKING! NO QUESTIONS!

Why were the students puzzled? Why could they answer item 2 but <u>not</u> item 6, for example? Think carefully about <u>why</u> the students were unable to answer most of the items.

EXERCISES

Below are several exercises each having blanks. For each exercise find a number such that when a numeral for it is put in <u>all</u> of the blanks in that exercise, the sentences in the exercise become true. In some exercises there may be another number that will do the job. [In some there may not be any.]

<u>Sample</u>.

◯ is an odd number. If I add 3 to ◯, the sum is 8.

<u>Solution</u>. Pick a number at random, say, 9. Write a simple name for it in each blank.

9 is an odd number. If I add 3 to 9, the sum is 8.

The last sentence is false, so try another number. Keep trying until you get true sentences.

Try 5.

5 is a number. If I add 3 to 5, the sum is 8.

For 5, both sentences become true.

1. ◯ is an even number. If I multiply ◯ by 7, I get 42.

2. ◯ is a real number. If I subtract -3 from ◯, I get $+4$.

3. ◯ is a number. If I add 2 to ◯, the sum is 1.

4. If I divide the number ◯ by -3, the quotient is 8.

5. ◯ is an even number. If I divide ◯ by 4, I get 5.

6. If I add ◯ to ◯, I get 6, and if I add ◯ to 6, I get 9.

(continued on next page)

7. If I add $\bigcirc$ to $\bigcirc$, the sum is $\bigcirc$. [Do you have a numeral for the same number in all three blanks?]

8. $\bigcirc$ is a real number. If I multiply $\bigcirc$ by $\bigcirc$, I get 100. [There are two numbers which will work!]

9. $\bigcirc$ is a positive number. If I multiply $\bigcirc$ by $\bigcirc$, the product is 81.

10. If I multiply $\bigcirc$ by $\bigcirc$, the product is 25, and if I add $\bigcirc$ to 7, the sum is 2.

11. If I multiply $\bigcirc$ by 2 and add 4, I get 17.

12. If I add $\bigcirc$ to 8 and divide by 3, I get 8.

13. If I add $\bigcirc$ to 7, I get $2 \times \bigcirc$.

14. If I subtract 5 from $\bigcirc$ and multiply by 5, I get 0.

15. If I add $\bigcirc$ to 12, I get $0 \times \bigcirc$.

16. If I multiply $\bigcirc$ by 4, the product is $\bigcirc$.

17. If I add $\bigcirc$ to 4, the sum is $\bigcirc$.

18. $\bigcirc$ is an odd number, $\bigcirc > 2$, and $\bigcirc < 9$.

19. The absolute value of $\bigcirc$ is 7.

20. The sum of $\bigcirc$ and 2 is the sum of 2 and $\bigcirc$.

21. $\bigcirc \times 3 = 3 \times \bigcirc$.   22. $\bigcirc \times \bigcirc = \bigcirc$.

23. A name of $\bigcirc$ is a two-digit numeral for a positive whole number. If the digits in the numeral are reversed, the result is a two-digit numeral which is still a name of $\bigcirc$.

SAMMY'S PROBLEM

Sammy, a student in Mr. Jones' class, heard his older brother say that algebra was a lot different from arithmetic. "In algebra you do problems with letters as well as with numbers." Sammy was puzzled by his brother's remark because he didn't know, for example, how to add letters and numbers. He wondered what a + b could be. Was it c? How could you add 3 and x? Since Sammy was hardly an expert in mathematics, he supposed that a few of the "whizzes" in his class might know. But, he didn't want to approach them directly with his problem.

The next day he thought he found a chance to learn some arithmetic with letters. Mr. Jones gave out the first page of a new True-False test. He told the students that he was not going to hand out second pages this time but, instead, he wanted each student to make up his own second page [as you did on page 2-C] and hand it to his neighbor. In this way, each student made a test for his neighbor. Can you guess the kind of second page Sammy wrote?

Sammy exchanged second pages with Fred. [Fred's nickname was 'The Brain'.] Fred answered the first two items and then stopped in bewilderment. Turn the page to see his test.

Name Fred

Class Math. I

Date October 20

TRUE - FALSE TEST

Instructions: Write a 'T' in the space to the left of an item if the statement is true. Write 'F' in this space if the statement is false.

F 1. $4 + 12 = 17$

T 2. $9 \times 6 = 54$

___ 3. $3 + x = 2$

___ 4. $5 \times b = 25$

___ 5. $8 - 9 = -1$

___ 6. $7 \times x = 15$

___ 7. $w + 2 = 6$

___ 8. $3 \times 7 = 34$

___ 9. $8 + y = 10$

___ 10. $m \times 7 = 34$

Fred raised his hand and said, "Mr. Jones, I don't know how to do this test." Mr. Jones was very surprised, and so was the rest of the class, for there never seemed to be a problem that Fred couldn't solve. Sammy thought to himself that now he would never find out how to do arithmetic with letters. Fred said that Sammy had put letters instead of numerals on the second page. He said, "It's impossible to tell whether '$3 + x = 2$' is true or false because . . ." Suddenly, Fred stopped talking. He seemed to be thinking very hard. Then he said,

> "I know. The thing wrong with this test is just what was wrong with the test when Mr. Edwards was here."

Fred was right. Can you tell why?

2.01 Sentences.--When Mr. Edwards gave out only the first page of the True-False test, the students felt that most of the questions were silly. How could you ask "True or false?" about a sentence with a hole in it? A sentence such as:

$$9 + \bigcirc = 15$$

is neither true nor false. The sentence:

$$9 + 6 = 15$$

is true because '9 + 6' and '15' are numerals for the same number, and the sentence:

$$9 + 7 = 15$$

is false because '9 + 7' and '15' are numerals for different numbers. But, since a hole is not a numeral,

$$\text{'}9 + \bigcirc\text{'} \text{ is not a numeral,}$$

and so, the sentence with the hole in it is neither true nor false.

You can convert a sentence which has a hole in it into a sentence which is either true or false by putting a numeral in the hole.

EXERCISES

A. Each of the following exercises contains a sentence with one or more "holes" in it. Frames like:

$$\bigcirc \quad \square \quad \hexagon \quad \boxed{}$$

are used to show you where the holes are. Your job is to put a numeral in each hole in the sentence as instructed, and then tell whether the new sentence you get is true or false.

Sample 1. (a) Write a '7' in each frame.

$$\boxed{} + \boxed{} + 3 = \boxed{} - 2$$

(b) Write a '−5' in each frame.

$$\boxed{} + \boxed{} + 3 = \boxed{} - 2$$

(c) Write a '5' in each frame.

$$\boxed{} + \boxed{} + 3 = \boxed{} - 2$$

Solution.

(a) $\boxed{7} + \boxed{7} + 3 = \boxed{7} - 2$

This sentence is false because
$7 + 7 + 3$ is 17, $7 - 2$ is 5, and $17 \neq 5$.

(b) $\boxed{-5} + \boxed{-5} + 3 = \boxed{-5} - 2$

Since $-5 + -5 + 3 = -7$ and $-5 - 2 = -7$, this sentence is true.

(c) $\underbrace{\boxed{5} + \boxed{5} + 3}_{13} = \underbrace{\boxed{5} - 2}_{3}$

Since $13 \neq 3$, this sentence is false.

1. (a) Write a '3' in each frame.

$$2 \times \boxed{} + 5 = 7 \times \boxed{} - 5$$

(b) Write a '−2' in each frame.

$$2 \times \boxed{} + 5 = 7 \times \boxed{} - 5$$

(c) Write a '2' in each frame.

$$2 \times \boxed{} + 5 = 7 \times \boxed{} - 5$$

2. (a) Write a '4' in each frame.

$$3 \times \bigcirc + 2 \times \bigcirc = 5 \times \bigcirc$$

(b) Write a '0' in each frame.

$$3 \times \bigcirc + 2 \times \bigcirc = 5 \times \bigcirc$$

(c) Write a '−5' in each frame.

$$3 \times \bigcirc + 2 \times \bigcirc = 5 \times \bigcirc$$

3. (a) Write a '−2' in each frame.

$$9 \times \bigcirc + 7 = 4 - 2 \times \bigcirc$$

(b) Write a '0' in each frame.

$$9 \times \bigcirc + 7 = 4 - 2 \times \bigcirc$$

(c) Write a '1' in each frame.

$$9 \times \bigcirc + 7 = 4 - 2 \times \bigcirc$$

4. (a) '−1' in each frame.

$$\square + 8 = 6 - \square$$

(b) '6' in each frame.

$$\square + 8 = 6 - \square$$

(c) '2' in each frame.

$$\square + 8 = 6 - \square$$

5. (a) '$\frac{16}{3}$' in each frame.

$$\bigcirc + 3 \times \bigcirc = 25$$

(b) '$\frac{25}{4}$' in each frame.

$$\bigcirc + 3 \times \bigcirc = 25$$

(c) '8' in each frame.

$$\bigcirc + 3 \times \bigcirc = 25$$

(continued on next page)

6. (a) '3' in each frame.

$$\bigcirc + 2 = 2 + \bigcirc$$

(b) '$\frac{5}{2}$' in each frame.

$$\bigcirc + 2 = 2 + \bigcirc$$

(c) '−12' in each frame.

$$\bigcirc + 2 = 2 + \bigcirc$$

7. (a) '3' in each frame.

$$2 \times \square + 2 = 2 \times \square - 2$$

(b) '−2' in each frame.

$$2 \times \square + 2 = 2 \times \square - 2$$

(c) '8' in each frame.

$$2 \times \square + 2 = 2 \times \square - 2$$

8. (a) '3' in each frame.

$$3 \times ⬡ > 2 \times ⬡ + 1$$

(b) '0' in each frame.

$$3 \times ⬡ > 2 \times ⬡ + 1$$

(c) '−2' in each frame.

$$3 \times ⬡ > 2 \times ⬡ + 1$$

9. (a) '5' in each frame.

$$|\bigcirc - 2| = 2 - \bigcirc$$

(b) '−3' in each frame.

$$|\bigcirc - 2| = 2 - \bigcirc$$

(c) '2' in each frame.

$$|\bigcirc - 2| = 2 - \bigcirc$$

10. (a) '4' in each frame.

$$5 \times \square + 1 < 6 \times \square$$

(b) '−3' in each frame.

$$5 \times \square + 1 < 6 \times \square$$

(c) '1' in each frame.

$$5 \times \square + 1 < 6 \times \square$$

11. (a) '2' in each frame.

$$7 - \bigcirc < {}^{+}|\bigcirc - 7|$$

(b) '−3' in each frame.

$$7 - \bigcirc < {}^{+}|\bigcirc - 7|$$

(c) '2.5' in each frame.

$$7 - \bigcirc < {}^{+}|\bigcirc - 7|$$

Sample 2. Write a '(9 + 7)' in each frame.

$$3 \times \boxed{} - 7 = \boxed{} + 25$$

Solution. $$\underbrace{\underbrace{3 \times \boxed{(9 + 7)} - 7}_{3 \times 16 - 7}}_{41} = \underbrace{\underbrace{\boxed{(9 + 7)} + 25}_{16 + 25}}_{41}$$

The new sentence is true.

12. (a) '$(3 \times 9 + 13)$' in each frame.

$$\boxed{} + 5 = 3 \times \boxed{} - 7$$

(b) '$(10 \times 11 - 104)$' in each frame.

$$\boxed{} + 5 = 3 \times \boxed{} - 7$$

(c) '$\left[\frac{8}{3} + \frac{2}{3} \times (16 - 11)\right]$' in each frame.

$$\boxed{} + 5 = 3 \times \boxed{} - 7$$

13. (a) '$(15 - 3)$' in each frame.

$$2 \times \boxed{} + 2 = 43 - \boxed{}$$

(b) '$(-8 + 23)$' in each frame.

$$2 \times \boxed{} + 2 = 43 - \boxed{}$$

(c) '$(7 - 20)$' in each frame.

$$2 \times \boxed{} + 2 = 43 - \boxed{}$$

B. The exercises below are like those in Part A, except that several types of frames are used in the same sentence. Follow the instructions for putting numerals in the frames, and tell whether each new sentence is true or false.

1. (a) Write a '4' in each '$\bigcirc$' and a '3' in each '$\square$'.

$$2 \times \bigcirc + \square = 7 - \square + \bigcirc$$

   (b) Write a '3' in each '$\bigcirc$' and a '4' in each '$\square$'.

$$2 \times \bigcirc + \square = 7 - \square + \bigcirc$$

   (c) Write a '−5' in each '$\bigcirc$' and a '6' in each '$\square$'.

$$2 \times \bigcirc + \square = 7 - \square + \bigcirc$$

2. (a) '−7' in each '$\square$' and '−3' in each '⬡'.

$$4 \times \square - 2 \times \text{⬡} = \square + 5 \times \text{⬡}$$

   (b) '5' in each '$\square$' and '2' in each '⬡'.

$$4 \times \square - 2 \times \text{⬡} = \square + 5 \times \text{⬡}$$

   (c) '0' in each '$\square$' and '0' in each '⬡'.

$$4 \times \square - 2 \times \text{⬡} = \square + 5 \times \text{⬡}$$

3. (a) '5' in each '$\square$', '3' in each '$\bigcirc$', and '4' in each '⬡'.

$$(\square + \bigcirc) + \text{⬡} = \square + (\bigcirc + \text{⬡})$$

   (b) '−4' in each '$\square$', '2' in each '$\bigcirc$', and '−4' in each '⬡'.

$$(\square + \bigcirc) + \text{⬡} = \square + (\bigcirc + \text{⬡})$$

   (c) '873' in each '$\square$', '−9384' in each '$\bigcirc$', and '−76.2' in each '⬡'.

$$(\square + \bigcirc) + \text{⬡} = \square + (\bigcirc + \text{⬡})$$

4. (a) '8' in each '○', '2' in each '⬡', and '20' in each '□'.

$$5 \times \bigcirc \times ⬡ = \square$$

(b) '−1' in each '○', '−2' in each '⬡', and '10' in each '□'.

$$5 \times \bigcirc \times ⬡ = \square$$

(c) '0' in each '○', '0' in each '⬡', and '0' in each '□'.

$$5 \times \bigcirc \times ⬡ = \square$$

5. (a) '5' in each '○' and '$\frac{16}{5}$' in each '□'.

$$\bigcirc \times \square = \square \times \bigcirc$$

(b) '2.4' in each '○' and '−2.4' in each '□'.

$$\bigcirc \times \square = \square \times \bigcirc$$

(c) '783' in each '○' and '9359' in each '□'.

$$\bigcirc \times \square = \square \times \bigcirc$$

6. (a) '4' in each '□' and '6' in each '⬡'.

$$-\square + 3 \times ⬡ = -(\square - 3 \times ⬡)$$

(b) '−3' in each '□' and '−5' in each '⬡'.

$$-\square + 3 \times ⬡ = -(\square - 3 \times ⬡)$$

(c) '$-\frac{2}{3}$' in each '□' and '$\frac{5}{7}$' in each '⬡'.

$$-\square + 3 \times ⬡ = -(\square - 3 \times ⬡)$$

SUBSTITUTION

Consider the sentence:

$$(*) \qquad 3 \times \square + 4 \times \bigcirc = 17 + \square\,.$$

As you have seen, this sentence can be used as a <u>pattern</u> for writing true-or-false sentences. One way to do this is to write numerals in the frames, observing the rule that we write copies of the same numeral in all frames of the same shape. For example, here is a sentence which follows the pattern of (*):

$$3 \times \boxed{2} + 4 \times (5) = 17 + \boxed{2}\,.$$

Another way of following the pattern is to write numerals <u>in</u> <u>place</u> <u>of</u> the frames:

$$3 \times 2 + 4 \times 5 = 17 + 2\,.$$

Of course, we replace all frames of a given shape by copies of the same numeral. In the example above, we replaced each '$\square$' in (*) by a '2' and each '$\bigcirc$' by a '5'. For short, we say that we <u>substituted</u> '2' for '$\square$' and '5' for '$\bigcirc$' in (*). Here are some more sentences which follow the pattern of (*). Tell what substitutions were made in (*) to obtain these sentences.

(1) $3 \times 51 + 4 \times 9 = 17 + 51$

(2) $3 \times -6 + 4 \times 8 = 17 + -6$

(3) $3 \times 7 + 4 \times 7 = 17 + 7$

(4) $3 \times (5 + 2) + 4 \times 7 = 17 + (5 + 2)$

(5) $3 \times (2 \times 6) + 4 \times 3 = 17 + 2 \times 6$

(6) $3 \times (4 + 2 \times 9) + 4 \times (17 - 3 \times 5) = 17 + (4 + 2 \times 9)$

Here are other examples of sentences with frames in them, and of sentences obtained by substituting for the frames.

$5 \times \square \times \triangle > 7 - \triangle$	$6 \times (\square - \triangle) = 8 \times \square \times (7 - \triangle)$
$5 \times 3 \times 4 > 7 - 4$	$6 \times (5 - 2) = 8 \times 5 \times (7 - 2)$
$5 \times 9 \times {}^-5 > 7 - {}^-5$	$6 \times (3 - 3) = 8 \times 3 \times (7 - 3)$

EXERCISES

A. For each of the following sentences, write the sentence you get by making the indicated substitutions.

1. Substitute '2' for '△' and '7' for '□' in:

$$9 \times \triangle + 3 \times \square = 15 \times \triangle \times \square .$$

2. Substitute '−9' for '□' and '5' for '△' in:

$$3 \times (\square + \triangle) - 2 \times (\triangle + \square) = \square + \triangle .$$

3. Substitute '(4 + 1)' for '□', '(3 + 5)' for '○', and '(9 + 2)' for '△' in:

$$\square \times (\bigcirc \times \triangle) = \square \times \bigcirc \times \triangle .$$

4. Substitute '8 + 3' for '□' in:

$$3 \times \square + 5 \times \square = 88.$$

5. Substitute '3 × 5' for '□' in:

$$6 + \square = 2 \times \square - 9$$

6. Substitute '15 - 3 × 2' for '△' and '−7' for '□' in:

$$2 \times \triangle - 5 \times \square = 12 \times (\triangle - \square).$$

B. Substitute, and tell whether the resulting sentence is true or false. Try to predict the answer before substituting and computing, but be sure to check your prediction.

1. '5' for '□', '6' for '△', and '9' for '○' in:

$$(\triangle + \bigcirc) \times \square = \triangle \times \square + \bigcirc \times \square .$$

2. '3 + 2 × 5' for '▭' in:

$$10 \times ▭ = ▭ + 117.$$

3. '15 × 2 - 5 × 2' for '⬭' in:

$$6 \times ⬭ + 5 = 135.$$

4. '5' for '△', '12' for '□', and '13' for '⬡' in:

$$\triangle \times \triangle + \square \times \square = ⬡ \times ⬡ .$$

5. '9' for '△' and '−7' for '□' in:

$$(2 \times \triangle - 7) \times (\square + 7) = 0.$$

6. '4 + 2 × 7' for '▭' and '6 × 3 - 2 × 5' for '⬭' in:

▭ × ⬭ = ⬭ × ▭ .

7. '3 × 5 - 7 × 2' for '△' and '7 × 2 - 3 × 5' for '⬡' in:

△ × △ = ⬡ × ⬡ .

8. '61' for '⬡', '37' for '□', and '93' for '△' in:

⬡ + □ + △ = ⬡ + (□ + △).

9. '3' for '△' and '2' for '□' in:

7 × △ + 2 × □ = 9 × △ × □ .

10. '5' for '□' in: (□ + $\frac{1}{2}$) × (□ + $\frac{1}{2}$) = □ × (□ + 1) + $\frac{1}{4}$.

C. Sometimes instead of substituting numerals for frames in a sentence, we substitute expressions which themselves contain frames. This gives us a new sentence which contains frames.

Sample. Write the sentence you get when you substitute

'3 × □' for '▭' and '4 + ⬡' for '⬭' in:

▭ + ⬭ = ⬭ + ▭ .

Solution. 3 × □ + (4 + ⬡) = 4 + ⬡ + 3 × □ .

[Why do we need to use only one pair of parentheses?]

1. Substitute '6 - 2 × ▭' for '⬭' in:

5 × ▭ + 3 × ⬭ = 17.

2. Substitute ' △ - 2 × □' for '▭' and '3 × □ + − △' for '⬭' in:

▭ × 3 - 6 = 11 + ⬭ × 4.

3. (a) Substitute '3 + □ ' for '▭' and '2 - △ ' for '⬭' in:

▭ + ⬭ = ⬭ + ▭ .

(b) In the sentence you obtained in (a), substitute '5' for '□' and '9' for '△'. Is the resulting sentence true? Could you have predicted your answer before doing any computations?

4. (a) Substitute '$3 + \triangle$' for '$\triangle$' and '$2 + \square$' for '$\bigcirc$' in:

$$\square \times (\triangle + \bigcirc) = \square \times \triangle + \square \times \bigcirc .$$

(b) Substitute '-3' for '$\square$' and '2' for '$\triangle$' in the sentence you obtained in (a). Is the resulting sentence true?

(c) In the sentence originally given in (a), substitute '-3' for '$\square$', '$3 + 2$' for '$\triangle$', and '$2 + -3$' for '$\bigcirc$'. Is the resulting sentence true?

5. (a) Substitute '$- \square$' for '$\triangle$' in:

$$\triangle = - - \triangle .$$

(b) Substitute '$^{-}2$' for '$\square$' in the sentence you obtained in (a). Is the resulting sentence true?

(c) Substitute '$\square - \triangle$' for '$\triangle$' in the sentence originally given in (a).

(d) Substitute '$^{-}3$' for '$\square$' and '$^{-}3$' for '$\triangle$' in the sentence you obtained in (c). Is the resulting sentence true?

<u>D</u>. For each of the following sentences, find a substitution which will make it true, and a substitution which will make it false.

<u>Sample</u>. $4 \times \square = 80$

<u>Solution</u>. (a) Substituting '20' for '$\square$' you get:

$$4 \times 20 = 80,$$

which is a true sentence.

(b) Substituting '11' for '$\square$' you get:

$$4 \times 11 = 80,$$

which is a false sentence.

1. $7 \times \square = 56$
2. $3 \times \square + 2 = 20$
3. $\square + \square = 100$
4. $2 \times \square + 8 \times \square = 60$
5. $3 \times \square + 17 \times \square = 20$
6. $3 \times \square + 17 \times \square = 80$
7. $5 \times \square + 11 \times \square = 16 \times \square$
8. $13 \times (\square + 2) = 26$
9. $7 \times (\square \times 5) = 35 \times \square$
10. $6 \times \square \times 4 \times \square = 24 \times \square$

E. In the preceding exercises you have been using sentences with frames as patterns for writing other sentences which follow the patterns. Now see if you can reverse the process by writing for each exercise below a sentence with frames which serves as a pattern for the sentences given in the exercise.

1.

$$3 + 9 = 9 + 3$$

$$^{-}8 + 0 = 0 + {}^{-}8$$

$$1 + 1 = 1 + 1$$

$$2 \times 3 + 6 \times 5 = 6 \times 5 + 2 \times 3$$

Pattern sentence: ____________________

2.

$$4 + 7 = 8 + 3$$

$$4 + 2 = 8 + 5$$

$$4 + \square = 8 + 7$$

$$4 + 3 \times 5 = 8 + (6 - \triangle)$$

Pattern sentence: ____________________

3.

$$5 + 9 < 5 - 9$$

$$6 \times 5 + {}^{-}3 < 6 \times 5 - {}^{-}3$$

$$5 + \square + 2 < 5 + \square - 2$$

$$\square + \triangle + (\bigcirc - \triangle) < \square + \triangle - (\bigcirc - \triangle)$$

Pattern sentence: ____________________

4.

$$2 + 3 \times 7 = (2 + 3) \times 7$$

$$5 + 9 \times 7 = (5 + 9) \times 7$$

$$1 + 0 \times 7 = (1 + 0) \times 7$$

$$0 + 1 \times 7 = (0 + 1) \times 7$$

$$5 + \triangle + (9 + \square) \times 7 = [5 + \triangle + (9 + \square)] \times 7$$

Pattern sentence: ____________________

5. All the instances of the commutative principle for multiplication.

Pattern sentence: ____________________

6.

$$6 \times (3 + 4) = 6 \times 3 + 6 \times 4$$

$${}^{-}8 \times ({}^{-}5 + {}^{-}6) = {}^{-}8 \times {}^{-}5 + {}^{-}8 \times {}^{-}6$$

$$(4 + 9) \times (7 + 3) = (4 + 9) \times 7 + (4 + 9) \times 3$$

$$15 \times (4 + 7 + 9) = 15 \times (4 + 7) + 15 \times 9$$

$${}^{-}1 \times (6 - \triangle + \square) = {}^{-}1 \times (6 - \triangle) + {}^{-}1 \times \square$$

Pattern sentence: ____________________

7.

$$(2 + 8) \times 3 = 2 \times 3 + 8 \times 3$$

$$(6 + 5 + 7) \times 8 = (6 + 5) \times 8 + 7 \times 8$$

$$[7 + (2 - 3)] \times \square = 7 \times \square + (2 - 3) \times \square$$

$$(1 \times 2 + 3) \times (4 + 5 + 6) = (1 \times 2) \times (4 + 5 + 6) + 3 \times (4 + 5 + 6)$$

Pattern sentence: ____________________

8. All the instances of the associative principle for addition.

Pattern sentence: ____________________

9. All the instances of the principle for multiplying by 1.

Pattern sentence: ____________________

10.

$$1 \times 57 = 57$$

$$1 \times (8 + 3) = 8 + 3$$

$$1 \times (\square \times \triangle) = \square \times \triangle$$

$$1 \times -(3 - 7) = -(3 - 7)$$

Pattern sentence: ____________________

11.

$$(7 + 4) \times (7 - 4) = 7 \times 7 - 4 \times 4$$

$$(5 + 2 + 1) \times (5 + 2 - 1) = (5 + 2) \times (5 + 2) - 1 \times 1$$

$$[1 + 3 + (7 \times 5)] \times [1 + 3 - (7 \times 5)] = (1 + 3) \times (1 + 3) - (7 \times 5) \times (7 \times 5)$$

$$(8 + 3 \times \square) \times (8 - 3 \times \square) = 8 \times 8 - (3 \times \square) \times (3 \times \square)$$

Pattern sentence: ____________________

2.02 Pronouns.--Suppose someone challenges you to say 'True' or 'False' about the following sentence:

He was a president of the United States.

You don't need to know anything about presidents, or even about the United States, in order to know that it would not be correct to answer one way or the other. The sentence is neither true nor false. [If you put a man's name in place of 'He' in the given sentence then the new sentence would be either true or false.]

Trying to answer 'True' or 'False' about the sentence:

He was a president of the United States

is just like trying to answer 'True' or 'False' about the sentence:

$$9 + \bigcirc = 15.$$

Both sentences have "holes" in them. [You have seen that a frame such as '$\bigcirc$' serves as a hole. So does the word 'He'.] Neither sentence is true, and neither sentence is false.

A sentence which is either true or false is called a statement. For example, '9 + 6 = 15' and 'Albert Einstein was a president of the United States' are statements. Sentences which can be turned into statements by filling holes with names are called open sentences. An open sentence is neither true nor false.

Since the holes in an open sentence hold places for nouns, they are called pronouns. The pronouns in the mathematical sentences we have been working with hold places for those nouns which are names of numbers, that is, they hold places for numerals. And so, we shall call such pronouns pronumerals.

In general, pronouns in an open sentence hold places for nouns, and in particular, pronumerals in an open sentence hold places for numerals.

Draw a line under each pronoun in the following sentences.

(1) He is Al's father.

(2) $\square$ is an even positive number.

(3) $\square + \bigcirc = 17 - 2 \times \square$.

(4) She is taller than Mary and she is her elder sister.

Individual pronumerals in an open sentence show you where to write names. But, taken together, they show you more than that. For example, in order to convert sentence (3) into a statement, you can write three names. However, you would not write something like this:

$$79 + 85 = 17 - 2 \times 94,$$

or even something like this:

$$79 + 85 = 17 - 2 \times 85.$$

When substituting for pronumerals, you follow the rule that pronumerals of the same shape are to be replaced by copies of the same numeral. Pronumerals of the same shape "link" places where similar replacements are to be made. In sentence (3), the two '$\square$'s are to be replaced by copies of the same numeral, and the '$\bigcirc$' can be replaced either by another copy of the same numeral or by a different numeral.

The rules for replacing pronouns in English open sentences cannot be stated so simply. [In fact, they probably just can't be stated!] In sentence (4) we can substitute and get:

(a) Elsie is taller than Mary and Elsie is Mary's elder sister,

or, we can substitute and get:

(b) Elsie is taller than Mary and Mary is Elsie's elder sister.

If you pointed to Elsie and said sentence (4), you would probably have meant (a); but if, in speaking, you emphasized the second 'she', you might very well have meant (b). If you meant (a), you were linking the 'she's. If you meant (b), you were linking the first 'she' and

the 'her', and you were *not* linking the two 'she's. [Notice that in (b) you did not even replace the linked pronouns by copies of the same word; you replaced 'she' by 'Elsie' and 'her' by 'Elsie's'.] So, the rule that two occurrences of the same pronoun should be replaced by copies of the same name is not strictly followed in English. And, *different* pronouns are sometimes linked.

You could overcome these difficulties in the English language if you introduced additional pronouns and adopted strict linking rules for them. For example, suppose '⬡' is a feminine pronoun and that always in a sentence all '⬡'s are linked. Then, we could state sentence (4) in such a way that it would be impossible to get (b); and in another way such that it would be impossible to get (a).

To get (a) but not (b), we would write:

> ⬡ is taller than Mary and ⬡ is her elder sister.

To get (b) but not (a), we would write:

> ⬡ is taller than Mary and she is ⬡ 's elder sister.

If in the first of these sentences you substitute 'Elsie' for '⬡' [and 'Mary's' for 'her'], you will get sentence (a). If in the second sentence you again substitute 'Elsie' for '⬡' [and 'Mary' for 'she'], you will get sentence (b).

Actually, there are pronouns in English which obey strict linking rules. Look at the sentence:

> He went because he felt it was polite to do so.

Are the two 'he's linked? You really can't tell. But, here is a restatement in ordinary English using pronouns which *clearly* are not linked.

> A *first* *boy* went because a *second* *boy* felt it was polite to do so.

The English language allows you to manufacture pronouns by prefixing to a noun words like 'first', 'second', 'third', 'former', 'latter', etc.

Do you think that 'he' and 'his' in:

He knew a man who roomed with his cousin in school

are linked? Use the pronouns 'first man', and 'second man', etc. to restate this sentence in two ways, each showing a different linkage.

STANDARD PRONUMERALS

Although we have been using frames as pronumerals, it is more common to use letters. In doing so, we shall use the lower-case letters of the alphabet:

a, b, c, ..., x, y, z

as well as the upper-case letters:

A, B, C, ..., X, Y, Z.

In using these you must be careful to observe the difference between 'a' and 'A', 'b' and 'B', etc. These are just as different as '□' and '△' or as '□' and '▭'.

We use letter-pronumerals just as we used frame-pronumerals. You can see a pattern in an open sentence which has letter-pronumerals just as easily as you could with an open sentence which had frame-pronumerals. For example, from the open sentence:

$$c + 7 \times a + b = c - (a + b)$$

we can generate (or make) statements by substituting numerals for the pronumerals 'a', 'b', and 'c'. Here are some of these statements:

$$15 + 7 \times 3 + 9 = 15 - (3 + 9),$$

$$19 + 7 \times 8 + 6 = 19 - (8 + 6),$$

$$-4 + 7 \times 1 + 5 = -4 - (1 + 5).$$

We can also generate <u>open</u> sentences from the given sentence by substituting expressions which contain pronumerals. For example:

$$(9 + x) + 7 \times (3 - y) + (k + m) = (9 + x) - [(3 - y) + k + m)],$$

$$(y - w) + 7 \times (2 + r) + (s - y) = (y - w) - [(2 + r) + (s - y)].$$

Now, just as an open sentence gives you a pattern for other sentences, an expression like '9 + a + 5 × b' gives you a pattern for other expressions. You can use it to generate numerals, or to generate other expressions which themselves give patterns for generating numerals. For example, from:

$$9 + a + 5 \times b$$

you can generate these <u>numerals</u>:

$$9 + 75 + 5 \times 15$$

$$9 + -3 + 5 \times (7 - 5).$$

Also, from '9 + a + 5 × b' you can generate these <u>expressions</u>:

$$9 + (8 + x) + 5 \times (7 - y)$$

$$9 + (3 - c) + 5 \times (2 + k)$$

$$9 + (A + b) + 5 \times (a - B)$$

$$9 + (9 + a + 5 \times b) + 5 \times (9 + a + 5 \times b).$$

As you can see, the expressions and sentences which can be generated from <u>pronumeral expressions</u> and from open sentences can look quite complicated. One small way in which they can be made to look less complicated is to follow the convention of omitting the multiplication signs. For example:

'3(5 + 9)' is an abbreviation for '3 × (5 + 9)',

'ab' is an abbreviation for 'a × b',

'q(3a + 4p)' is an abbreviation for 'q × (3 × a + 4 × p)',

'(−2a + 13)d' is an abbreviation for '(−2 × a + 13) × d',

'5x' is an abbreviation for '5 × x',

'x5' is an abbreviation for 'x × 5',

<u>but</u> '55' is <u>not</u> an abbreviation for '5 × 5',

and 'a - b' is <u>not</u> an abbreviation for 'a ×−b'.

Another abbreviation for '×' is obtained by using a '·' in place of a '×'. For example, '3·7' is an abbreviation for '3 × 7', and 'u·v' is an abbreviation for 'u × v'.

EXERCISES

A. *A value of a pronumeral* is a number whose name may be substituted for the pronumeral. A pronumeral expression also has values. These are the numbers whose names can be obtained by substituting numerals for the pronumerals in the expression. For example, the value of 'a + 2b' for the value 3 of 'a' and 5 of 'b' is $3 + 2 \cdot 5$, or 13.

Find the value of each of the following pronumeral expressions for the given values of the pronumerals.

Pronumeral	'a'	'A'	'b'	'B'	'c'	'd'	'x'	'y'	'z'
Value	3	7	5	-3	0	-4	$\frac{1}{2}$	$\frac{2}{3}$	$-\frac{5}{4}$

Sample. $(8a + 2b)(A + B)$

Solution. $(8 \cdot 3 + 2 \cdot 5)(7 + -3)$

$= (24 + 10)4$

$= 34 \cdot 4$

$= 136$.

[*Note*: You may be tempted to skip steps. Don't, until you are sure it won't lead to errors.]

1. $7a + 3d$
2. $8d - 5c$
3. $ab + 3y$
4. $2ax - 3ay$
5. $6xy + 4z$
6. $(8B - 3b)(8B - 3b)$
7. $aa + bb$
8. $a + a + b + b$
9. $xx + x + x$
10. $\dfrac{2a + 5b}{3d}$
11. $\dfrac{5B - 3y}{8z}$
12. $\dfrac{9a + 7a}{4b}$
13. $\dfrac{7d + A}{3b + A}$
14. $\dfrac{9xy}{3xz}$
15. $\dfrac{a(b - c)}{a(b + c)}$
16. $2a[d - 3(b + 2)]$
17. $(2d - c)[8a - 6x + 2A(3 - 5B)]$
18. $(5 + 2a) \div (3 - 6d)$
19. $(4x - 7y) \div (9y - 2z)$
20. $57 - 5.7 + bA$
21. $2 \cdot 4a - 4 \cdot 2A - 3 \cdot 5 \cdot 7y$
22. $3a + 2 \cdot 5(bB - 1)$
23. $7dd - 3d + 1 - (d - 1)(d - 1)$
24. $(A - a)(A - a)(A - a)$
25. $(3 - 2y)(6 - \frac{1}{2}y) - (y + 3)$

[More exercises are in *Parts A* and *B*, *Supplementary Exercises*.]

B. Complete these tables by filling in the blanks with numerals for the values of the pronumeral expressions [or: of the pronumeral] which correspond with the given values.

1.

x	1	2	-4	3.5	
3x + 4					25

2.

y	-2	-1	0	1	2
1 - 2y					

3.

z	-2	-1	0	1	
zz + 1					10

4.

k	6.2	-3.7		9.3
3(2k + 4)			0	

5.

t	5	30		17
8t + 12t			200	

6.

s	4	6	28	
3(5s) + 5(17s)				900

7.

m	6	-3	78	2/3
10m ÷ (2m)				

8.

d	2	-3	6.7	-4.3
d(d + 1) - dd				

9.

x	2	3	5		
y	7	-2		6	
8x + 5y			50	90	26

10.

p	2	3	-2	-3	
q	3	2	-3	-2	6
p - q					15

C. Use each of the following open sentences to generate a statement. Tell whether the statement is true or false. Try to generate some true statements and some false statements.

1. $5x + 4 = 14$
2. $5 - 8y = -3$
3. $A + 4 = 5 + 6A$
4. $7t - 1 = 1 + t$
5. $3z = z + 8$
6. $10 - 2s = s + 5$
7. $2B + 4 \neq 2(2 + B)$
8. $8 - 3c = 1\frac{1}{2}(5\frac{1}{3} - 2c)$
9. $x + 1 = x$
10. $2z - 1 = 1 + 2z$
11. $\frac{5r - 1}{5r + 1}$
12. $\frac{3m + 2}{8 - 3m} = \frac{6m + 5}{2 - 4m}$
13. $6j - j > 2$
14. $k - 3k < 8$
15. $10g - 8\frac{1}{2}g = 1.5g$
16. $12f + 7\frac{1}{3}f = 19\frac{1}{3}f$
17. $3xx - 12 = 0$
18. $uu = u + 20$
19. $(v - 7)(v + 7) = vv - 49$
20. $(w - 3)(2 + w) = ww - w - 6$

[More exercises are in <u>Part C</u>, <u>Supplementary Exercises</u>.]

D. Here are fifteen pronumeral expressions. By substituting in any of these expressions you can generate other expressions which have the same pattern. Each of the exercises is an expression which can be generated from one or more of the fifteen expressions. For each exercise, tell which of the expressions it can be generated from, and be prepared to give the substitutions.

(a) $x + y$	(b) ab	(c) $y + x$
(d) $xy + z$	(e) $a + bc$	(f) $x(y + z)$
(g) $ab + ac$	(h) $(m + n)p$	(i) $uw + vw$
(j) $P(QR)$	(k) $x + (y + z)$	(ℓ) $a + b + c$
(m) $x \cdot 1$	(n) $1 \cdot x$	(o) xyz

Sample 1. $(4 + 5)2 + 3 \cdot 2$

Solution. (a) ['$(4 + 5)2$' for 'x', '$3 \cdot 2$' for 'y']

(c) ['$(4 + 5)2$' for 'y', '$3 \cdot 2$' for 'x']

(d) ['$4 + 5$' for 'x', '2' for 'y', '$3 \cdot 2$' for 'z']

(e) ['$(4 + 5)2$' for 'a', '3' for 'b', '2' for 'c']

(i) ['$4 + 5$' for 'u', '2' for 'w', '3' for 'v']

Sample 2. $2x + 2y + 3z$

Solution. (a) ['$2x + 2y$' for 'x', '$3z$' for 'y']

(c) ['$2x + 2y$' for 'y', '$3z$' for 'x']

(e) ['$2x + 2y$' for 'a', '3' for 'b', 'z' for 'c']

(ℓ) ['$2x$' for 'a', '$2y$' for 'b', '$2z$' for 'c']

1. $9 + 5 \cdot 6$	2. $3 \cdot 5 + 2$	3. $3a + 3b$
4. $2 + 8 + 3 + 5$	5. $2 + 8 + (3 + 5)$	6. $2 + (8 + 3) + 5$
7. $(5 + 2 \cdot 3)1$	8. $9 \cdot 1 + 3 \cdot 1$	9. $x \cdot 1 + x \cdot 1$
10. $2ab$	11. $2a(3b)$	12. $(2a + 3b)c$
13. $(2a + 3b)5c$	14. $7 \cdot 5x$	15. $7(5x)$
16. $5x + 7x$	17. $x5 + x7$	18. $1 \cdot 1$
19. $x + y$	20. $(x + y)(u + v)$	21. $(x + y)(x + y)$
22. $7 + 2k$	23. $(7 + 2)k$	24. $7k + 2k$

E. Each exercise contains four expressions. Your job is to write a pronumeral expression from which you can generate the first three but not the fourth.

Sample. $\left\{\begin{array}{l} 8 \cdot 5 + 5 \\ 8 \cdot 3 + 3 \\ 8 \cdot 2 + 2 \end{array}\right.$

$7 \cdot 9 + 9$

Solution. Pronumeral expressions from which the first three can be generated are:

$$x,\ x + y,\ xy + z,$$

$$xy + y,\ 8x + y,\ 8x + x.$$

'$7 \cdot 9 + 9$' can also be generated from 'x', '$x + y$', '$xy + z$', and '$xy + y$', but not from '$8x + y$' or '$8x + x$'.

Answer. $8x + y$

[Another correct answer: $8x + x$. Of course, you could use other pronumerals instead of 'x' and 'y'.]

1. $\left\{\begin{array}{l} 9 + 15 \\ 9 + -41 \\ 9 + \frac{1}{2} \end{array}\right.$
   $7 + 15$

2. $\left\{\begin{array}{l} 3 - 2 + 3 \\ 9 - 2 + 9 \\ 3x - 2 + 3x \end{array}\right.$
   $3 - 5 + 3$

3. $\left\{\begin{array}{l} 5 \cdot 9 + 4 \cdot 8 \\ 5 \cdot 8 + 4 \cdot 8 \\ 5 \cdot -3 + 4 \cdot 12 \end{array}\right.$
   $5 \cdot 7 + 8 \cdot 7$

4. $\left\{\begin{array}{l} 85 \cdot 1 \\ -2 \cdot 1 \\ x \cdot 1 \end{array}\right.$
   $1 \cdot x$

5. $\left\{\begin{array}{l} 3x + 0 \\ 3y + 0 \\ 3z + 0 \end{array}\right.$
   $0 + 0$

6. $\left\{\begin{array}{l} 1 + 2 \cdot 5 \\ 1 + 2x \\ 1 + 2y \end{array}\right.$
   $3a$

7. $\left\{\begin{array}{l} 5 + (9 + 1) \\ a + (b + c) \\ 2x + (3y + z) \end{array}\right.$
   $9 + 5 + 1$

8. $\left\{\begin{array}{l} 5 \cdot 3 \cdot 9 \\ 6 \cdot 2x \\ 8x \cdot 3 \end{array}\right.$
   $9(2x)$

9. $\left\{\begin{array}{l} 3 + (5 + 2) + 7 \\ 6 + 8 + 5 \\ 7 + 4 + (5 + 9) \end{array}\right.$
   $6 + (2 + 8)$

10. $\left\{\begin{array}{l} 7(3+5) \\ 7(8+2) \\ 7(5+3) \end{array}\right.$
$(4+5)7$

11. $\left\{\begin{array}{l} 6 \cdot 9 + 6 \cdot 2 \\ 6 \cdot 3 + 6 \cdot 5 \\ 6 \cdot 4 + 6 \cdot 8 \end{array}\right.$
$2 \cdot 6 + 3 \cdot 6$

12. $\left\{\begin{array}{l} 4(5+x+3) \\ 4(4+8+y) \\ 4(5+9) \end{array}\right.$
$4 \cdot 3$

EXPLORATION EXERCISES

A. Remember Al Moore who lived in Alaska and wanted to learn mathematics by correspondence with his pen pal, Stan Brown? Well, it happened that Al really did it. He learned about real numbers and pronumerals and open sentences. Stan had told him about various principles of real numbers which helped Al get short cuts, and also helped him remember lots of computing facts. [For example, as soon as he had memorized the fact that 7×8 is 56, he also knew that 8×7 is 56.] Al now wanted to know more about these principles. So, he wrote Stan and said:

> Just what *is* the left distributive principle?

Stan replied:

> I thought you knew that. Here are some instances of it.

$$7(5+8) = 7 \cdot 5 + 7 \cdot 8$$

$$-9(7 + -3) = -9 \cdot 7 + -9 \cdot -3$$

$$-4(y+2) = -4 \cdot y + -4 \cdot 2$$

Al wrote back:

> Say, I don't want instances of it. I want to know what the principle *is*! After all, am I supposed to remember these examples every time I want to use the left distributive principle? And are these *all* of the instances? Remember, I don't have anyone up here I can ask. How can I tell an instance of the left distributive principle when I meet it?

So, Stan answered:

> Well, of course, I can't write down every instance. That would be impossible. But I can give you a rule for getting any instance or recognizing one when you see it. First, there is a numeral, followed by a left parenthesis, then a numeral, then a plus sign, then a numeral, then a right parenthesis, then an equality sign, then a numeral, then a multiplication dot, then a numeral, then a plus sign, then a numeral, then a multiplication dot, then, finally, a numeral. That's an instance of the left distributive principle.

Al sent an airmail reply:

> Aha! So, I suppose this is an instance of the left distributive principle:
>
> $$5(9 + 17) = 8 \cdot 6 + 41 \cdot 7.$$
>
> I followed your rule!

Stan was quick to answer:

> No, that's not it. I see I didn't make myself clear. I'll try again. Take the open sentence:
>
> $$x(y + z) = xy + xz.$$
>
> This gives you a pattern for generating instances. Just substitute a numeral for 'x', a numeral for 'y', and a numeral for 'z', and put in two multiplication dots on the right side to avoid confusion.

Al thought about this, and his next letter said:

> Well, I think I can follow that rule all right. But, didn't you tell me a long time ago that the left distributive principle said something about numbers? Your rule just tells me about numerals and sentences.

And, in my first lesson I learned that numerals aren't numbers. Tell me what the left distributive principle says about <u>numbers</u>.

Stan tries:

It says that for each real number you take, if you multiply it by the sum of a real number and a real number, you get the same number as you would if you took that first number and multiplied it by one of the other numbers and . . . . Wait, I'll start again.

For each first real number you take, if you multiply it by the sum of your second real number and your third real number, you get the same number as you would if you added the numbers you get by multiplying your first number by your second number and by multiplying your first number by your third number. Whew!!

Al complains:

'Whew' is right! That's complicated. It sounded as though this sentence:

$$5(3 + 4) = 5 \cdot 4 + 5 \cdot 3$$

is an instance of the principle. But, by that pattern business you wrote about in one of your other letters, this sentence wouldn't be an instance. Right?

Wearily, Stan replies:

Darn! I meant that you should add the third number <u>to</u> the second, and that you should add the product of the first number by the third <u>to</u> the product of the first number by the second. Maybe I didn't say it that way; I forget.

Al now tries to be helpful:

> I think maybe you ought to use that pattern-sentence along with the talk about choosing numbers to help keep things straight. How about this?
>
> For each first real number I pick, for each second real number, and for each third real number, it turns out that
>
> the 1st number × (the 2nd number + the 3rd number)
>
> equals
>
> the 1st number × the 2nd number + the 1st number × the 3rd number.
>
> Is this what you were trying to tell me?

Stan, with glee:

> Yeah, that's it! And you've given me an idea of how to say it in a shorter way. Remember when we talked about phrases like 'the first real number', 'the second real number', and 'the third real number' being like pronouns? Well, let's use pronumerals. Here goes.
>
> For each x, for each y, for each z,
>
> $$x(y + z) = xy + xz.$$
>
> This is it, isn't it?

Al closes this exchange with:

> I think we've got it now. And, I like the short way you said the same thing I did. But you didn't have to use 'x', 'y', and 'z' in that order did you? Couldn't I say that this is the left distributive principle, just as well?
>
> For each y, for each z, for each x,
>
> $$y(z + x) = yz + yx.$$
>
> Or, you could even use different letters.
>
> For each a, for each p, for each t,
>
> $$a(p + t) = ap + at.$$

Say aloud each principle for the real numbers, first by using pronounds like 'first number', 'second number', etc., and then write it in concise form by using letter pronumerals.

1. Commutative principle for multiplication

2. Commutative principle for addition

3. Associative principle for multiplication

4. Associative principle for addition

5. Distributive principle for multiplication over addition

6. Principle for adding 0

7. Principle for multiplying by 1

8. Principle of opposites

9. Principle for multiplying by 0

10. Principle for subtraction

B. Each of the following is a general statement about numbers. Translate it into a sentence beginning with 'For each x, ...'.

Sample. Whatever real number you pick, if you multiply it by 2, and then multiply 3 by the product, the result is the product of 6 by the number chosen.

Solution. For each x, $3(x2) = 6x$.

1. Whatever real number you pick, if you multiply 3 by it, then multiply 7 by it, and then add the second product to the first, the result is the product of 10 by the chosen number.

2. For each real number you pick, if you add 2 to it and multiply the sum by 7, you get the same result as you would by adding the product of 2 by 7 to the product of the number you picked by 7.

3. Pick any real number. Add 5 to it. Multiply 5 by this sum. Multiply by 4. Subtract 100. Multiply by $\frac{1}{20}$. The result is the number you started with.

4. Pick a real number. Add 5 to it, and 7 to it. Multiply the sums. The result is the product of the chosen number by itself, plus the product of 12 by the chosen number, plus 35.

* * *

In Unit 1 you learned procedures for adding and multiplying real numbers. That is, you learned ways of solving problems like:

$^{+}5 + {}^{-}6 = ?$ $\qquad\qquad$ $^{-}3 \times {}^{-}5 = ?$

But, you were not asked to state the rules you followed. Let's look into the problem of stating a rule for, say, adding two negative numbers. Of course, you already know <u>how</u> to add such numbers. Can you state a rule which describes exactly what you do with the numbers to get the sum? Here is one such careful description.

> For each first real number, for each second real number,
> if the first real number is negative and
> the second real number is negative
> then the sum of the real numbers is the negative
> number which corresponds with the sum of
> the number of arithmetic which corresponds
> with the first real number and the number of
> arithmetic which corresponds with the second
> real number.

The description can be made briefer by using letter pronumerals. Here is a first attempt.

> For each x, for each y,
> if x is negative and y is negative
> then x + y is the negative number which corresponds
> with the sum of the number of arithmetic which
> corresponds with x and the number of arithmetic
> which corresponds with y.

Can we make further improvements? The phrase

'the number of arithmetic which corresponds with x'

can be abbreviated to '$|x|$' since, as you recall from Unit 1, the absolute value of a real number is the number of arithmetic which corresponds

with it. So, a shorter description is:

For each x, for each y,
if x is negative and y is negative
then x + y is the negative number
which corresponds with $|x| + |y|$.

We can make a final improvement by recalling that in order to form a name for a negative real number we just put a raised minus sign in front of a name of the corresponding number of arithmetic. So, a concise rule for adding negative numbers is:

For each x, for each y,
if x is negative and y is negative
then $x + y = {}^-(|x| + |y|)$.

* * *

C. State in a concise way the rule for

1. adding positive numbers.

2. multiplying a negative number by a positive number.

3. multiplying a positive number by a negative number.

4. multiplying a positive number by a positive number.

5. multiplying a negative number by a negative number.

D. 1. What rule is the following?

For each x, for each y,
if x is positive and y is negative then

(a) if $|x| \geq |y|$ then $x + y = {}^+(|x| - |y|)$

and (b) if $|x| < |y|$ then $x + y = {}^-(|y| - |x|)$.

2. State the rule for adding a positive number to a negative number.

3. State rules for adding the real number 0 and multiplying by the real number 0. [Do you know these rules by some other names?]

4. What principle of real numbers makes it unnecessary for you to remember the rule of Exercise 2 of Part D if you remember the rule of Exercise 1 of Part D?

5. What principle of real numbers makes it unnecessary for you to remember the rule of Exercise 3 of Part C if you remember the rule of Exercise 2 of Part C?

2.03 Generalizations.--Here is a generalization statement about numbers:

$$\text{For each } x, \quad 1 + 2x = 3x.$$

Translated into ordinary English, this statement tells you that no matter what real number you pick, if you multiply 2 by this number and add the product to 1, the result is the product of 3 by the chosen number. Do you believe what this generalization statement tells you? Whatever your answer is, you should be able to give evidence for your belief. If you believe that the generalization is true, you might try to justify it on the basis of the principles for real numbers. If you believe that it is false, you might try to find a counter-example, that is, a number such that when you multiply 2 by this number and add the product to 1, the result is different from the product of 3 by this number.

Actually, the generalization is false. A counter-example is 7.

$$1 + 2 \cdot 7 = 15, \quad 3 \cdot 7 = 21, \quad \text{and} \quad 15 \neq 21.$$

[Notice that the generalization is false, despite the fact that $1 + 2 \cdot 1 = 3 \cdot 1$. A generalization is false even if it has only one false instance, no matter how many true instances it has.]

If you had first guessed that the generalization is true, this may have been because you thought that it is a consequence of the distributive principle for multiplication over addition. For example, you might have thought that:

$$(*) \qquad 1 + 2 \cdot 7 = (1 + 2)7$$

is an instance of the distributive principle. Why isn't it? [Or, you may have thought that (*) is an instance of one of the associative principles. Why isn't it?]

Now, consider another generalization about numbers:

$$\text{For each } x,\quad 3(x2) = 6x.$$

To make sure we understand what this says, let's look at a few instances. To write an instance, we substitute a numeral for 'x' in the open sentence which follows the 'For each x, '.

(1) $3(5 \cdot 2) = 6 \cdot 5$

(2) $3(8 \cdot 2) = 6 \cdot 8$

(3) $3(^{-}7 \cdot 2) = 6 \cdot {}^{-}7$

Statement (1) is true because $3(5 \cdot 2) = 3 \cdot 10 = 30$, and $6 \cdot 5 = 30$.

Statement (2) is true because $3(8 \cdot 2) = 3 \cdot 16 = 48$, and $6 \cdot 8 = 48$.

Statement (3) is true because $3(^{-}7 \cdot 2) = 3 \cdot {}^{-}14 = {}^{-}42$, and $6 \cdot {}^{-}7 = {}^{-}42$.

So, we have <u>verified</u> each of the three instances.

Do you think you could find a substitution for 'x' which would generate a false statement? If you think you couldn't, how can you be sure you're right? You certainly couldn't test and verify each instance as you did (1), (2), and (3). What you need is a <u>method</u> for testing any instance which you can be sure will verify each instance that you test. The method used in testing instances (1), (2), and (3) was to multiply the test number by 2, multiply 3 by this product, and compare this result with the product of 6 by the test number. But, it is not immediately clear that this computing method will result in a verification of each instance tested.

Consider another method of testing instance (1). Let's start with the expression on the left of '=' and try to transform it into the expression on the right of '='.

$3(5 \cdot 2) = 3(2 \cdot 5)$ [cpm]

$3(2 \cdot 5) = (3 \cdot 2)5$ [apm]

$(3 \cdot 2)5 = 6 \cdot 5.$ [$3 \cdot 2 = 6$]

Hence, $3(5 \cdot 2) = 6 \cdot 5.$

So, we say that '$3(5 \cdot 2) = 6 \cdot 5$' is a consequence of the commutative and associative principles for multiplication, and the computing fact that $3 \cdot 2 = 6$.

Let's use this method to test instance (2), this time putting a frame around the numeral for the test number.

$$3(\boxed{8} \cdot 2) = 3(2 \cdot \boxed{8}) \quad [\text{cpm}]$$

$$3(2 \cdot \boxed{8}) = (3 \cdot 2)\boxed{8} \quad [\text{apm}]$$

$$(3 \cdot 2)\boxed{8} = 6 \cdot \boxed{8}. \quad [3 \cdot 2 = 6]$$

Hence, $3(\boxed{8} \cdot 2) = 6 \cdot \boxed{8}$.

So, instance (2) is a consequence of the commutative and associative principles for multiplication, and the computing fact that $3 \cdot 2 = 6$.

Do you see how to test instance (3)? Just erase the numerals in the frames and write a '$^-7$' in each frame. Do you see that when you erase the numerals in the frames getting:

$$3(\square \cdot 2) = 3(2 \cdot \square) \quad [\text{cpm}]$$

$$3(2 \cdot \square) = (3 \cdot 2)\square \quad [\text{apm}]$$

$$(3 \cdot 2)\square = 6 \cdot \square. \quad [3 \cdot 2 = 6]$$

Hence, $3(\square \cdot 2) = 6 \cdot \square$.

you have a testing pattern which can be used to test any instance, and that such a test will always lead to a verification of the instance tested? In fact, the test-pattern shows you that each instance is a consequence of the commutative and associative principles for multiplication, and the fact that $3 \cdot 2 = 6$. So, we know that the generalization:

For each x, $3(x2) = 6x$

is a consequence of the commutative and associative principles for multiplication, and the computing fact that $3 \cdot 2 = 6$. The test-pattern is a derivation of the generalization from the cpm, the apm, and '$3 \cdot 2 = 6$'. For short, it is a proof of the generalization. The proof shows that if we accept the premisses:

For each x, for each y, $xy = yx$,
For each x, for each y, for each z, $xyz = x(yz)$,
$3 \cdot 2 = 6$,

then we must accept the conclusion:

For each x, $3(x2) = 6x$.

You should have just as much faith in the truth of the conclusion as you have in the truth of the premisses.

Now, let's consider still another generalization:

$$\text{For each } x,\quad x(2x + 3) = 2(xx) + 3x.$$

This generalization tells us, for example, that

$$7(2 \cdot 7 + 3) = 2(7 \cdot 7) + 3 \cdot 7.$$

Do you believe this instance? You could test it by computing. But, if we test it in such a way as to reveal a test-pattern, then we shall have proven not only the instance but the generalization itself.

$$\boxed{7}(2 \cdot \boxed{7} + 3) = \boxed{7}(2 \cdot \boxed{7}) + \boxed{7} \cdot 3 \quad [\ell\text{dpma}]$$

$$\boxed{7}(2 \cdot \boxed{7}) + \boxed{7} \cdot 3 = (2 \cdot \boxed{7})\boxed{7} + \boxed{7} \cdot 3 \quad [\text{cpm}]$$

$$(2 \cdot \boxed{7})\boxed{7} + \boxed{7} \cdot 3 = 2(\boxed{7} \cdot \boxed{7}) + \boxed{7} \cdot 3 \quad [\text{apm}]$$

$$2(\boxed{7} \cdot \boxed{7}) + \boxed{7} \cdot 3 = 2(\boxed{7} \cdot \boxed{7}) + 3 \cdot \boxed{7}. \quad [\text{cpm}]$$

Hence, $$\boxed{7}(2 \cdot \boxed{7} + 3) = 2(\boxed{7} \cdot \boxed{7}) + 3 \cdot \boxed{7}.$$

The test-pattern is easy to see if we erase the numerals from the frames, and even easier to see if we then replace the frames by a letter.

$$k(2k + 3) = k(2k) + k3 \quad [\ell\text{dpma}]$$

$$k(2k) + k3 = (2k)k + k3 \quad [\text{cpm}]$$

$$(2k)k + k3 = 2(kk) + k3 \quad [\text{apm}]$$

$$2(kk) + k3 = 2(kk) + 3k. \quad [\text{cpm}]$$

Hence, $$k(2k + 3) = 2(kk) + 3k.$$

This test-pattern can be used to verify any instance of the generalization:

$$\text{For each } x,\quad x(2x + 3) = 2(xx) + 3x.$$

So, it is a proof of this generalization. It shows that the generalization is a consequence of three principles for real numbers. What are they? [Notice that even when you have a test-pattern, you are still not able to test <u>every</u> instance of the generalization [Why not?]. However, the test-pattern gives you a sure-fire method of refuting anyone who claims to have a counter-example.]

EXERCISES

A. Each of the following is a generalization about real numbers. Some are true and some are false. Your job is to decide which, and in each case to give either a proof or a counter-example.

Sample 1. For each y, $7 + (3 + y) = y + 10$.

Solution. You may suspect that this is true, and be able to start writing a proof immediately. However, if you have difficulty in seeing how to construct a test-pattern, get to work on an instance. For example, try using the principles to verify:

$$7 + (3 + \boxed{5}) = \boxed{5} + 10.$$

We notice that the '$\boxed{5}$' is not in parentheses on the right side. This suggests transforming the left side '$7 + (3 + \boxed{5})$' by the associative principle for addition.

$$7 + (3 + \boxed{5}) = (7 + 3) + \boxed{5}.$$

Since $7 + 3$ is 10,

$$(7 + 3) + \boxed{5} = 10 + \boxed{5}.$$

Then next we can use the commutative principle for addition.

$$10 + \boxed{5} = \boxed{5} + 10.$$

We can now write a test-pattern.

$$7 + (3 + m) = (7 + 3) + m \quad [apa]$$

$$(7 + 3) + m = 10 + m \quad [7 + 3 = 10]$$

$$10 + m = m + 10. \quad [cpa]$$

Hence, $7 + (3 + m) = (7 + 3) + m$.

1. For each t, $3(5t) = 15t$.

2. For each q, $3 + 6q = 9q$.

3. For each r, $3 + 6r = 3(1 + 2r)$.

4. For each x, $x \cdot 1 + x = 2x$.

5. No matter what number you pick, if you add it to 9 and add this sum to 1, you get 10 plus the chosen number.

6. For each r, $8r + 7 = 7r + 8$.

7. For each x, $xxx = 3x$.

8. For each x, $3x \cdot 4 = 12x$.

9. For each x, $9 + 7x + 3 = 7x + 12$.

[More exercises are in Part D, Supplementary Exercises.]

Sample 2. For each k, $3k + (9k - 2) = 12k - 2$.

Solution. Here is a test-pattern.

$$\begin{array}{rcll} 3k + (9k - 2) &=& 3k + (9k + -2) & [\text{ps}] \\ 3k + (9k + -2) &=& 3k + 9k + -2 & [\text{apa}] \\ 3k + 9k + -2 &=& (3 + 9)k + -2 & [\text{dpma}] \\ (3 + 9)k + -2 &=& 12k + -2 & [3 + 9 = 12] \\ 12k + -2 &=& 12k - 2. & [\text{ps}] \end{array}$$

Hence, $3k + (9k - 2) = 12k - 2$.

We can save almost half the writing if we abbreviate the test-pattern as follows.

$$\begin{array}{rcll} 3k + (9k - 2) &=& 3k + (9k + -2) & [\text{ps}] \\ &=& 3k + 9k + -2 & [\text{apa}] \\ &=& (3 + 9)k + -2 & [\text{dpma}] \\ &=& 12k + -2 & [3 + 9 = 12] \\ &=& 12k - 2. & [\text{ps}] \end{array}$$

To save space horizontally, you may arrange your work as follows.

$$\begin{array}{ll} \ 3k + (9k - 2) & \\ & \text{ps} \\ = 3k + (9k + -2) & \\ & \text{apa} \\ = 3k + 9k + -2 & \\ & \text{dpma} \\ = (3 + 9)k + -2 & \\ & 3 + 9 = 12 \\ = 12k + -2 & \\ & \text{ps} \\ = 12k - 2. & \end{array}$$

10. For each m, $3m - 7 + 5m = 8m - 7$.

11. For each q, $(2q \cdot 5)q = 10(qq)$.

12. For each x, $6x(3x) = 18(xx)$.

13. For each r, $3r - r = 3$.

14. For each m, $2m(3 + 5m) = 10(mm) + 6m$.

15. For each x, $x - 1 = 1 - x$.

16. For each y, $3y + 7 + 5y - 3 = 8y + 4$.

17. For each x, $3x + 4(x + 7) = 7(x + 4)$.

18. For each n, $4n(3n) = 7nn$.

19. For each number you pick, if you add it to itself, you get the product of 2 by the chosen number.

☆20. For each A, $A(A + 2) + A(A + 3) = 2\ AA + 5A$.

[More exercises are in Part E, Supplementary Exercises.]

* * *

Consider the following generalization:

(*) For each x, $(x + 3)(x + 7) = (x + 7)(x + 3)$.

Do you think it's true? Let's look at one of its instances:

$$(5 + 3)(5 + 7) = (5 + 7)(5 + 3).$$

Since '(5 + 3)' and '(5 + 7)' are numerals for real numbers, do you see that from the open sentence:

$$xy = yx$$

we could generate this instance? And, do you see that each instance of (*) can be generated from 'xy = yx'? Hence, each instance of (*) follows from the commutative principle for multiplication:

For each x, for each y, $xy = yx$.

Therefore, each instance of (*) is true. So, (*) is true. In fact it is a consequence of the commutative principle for multiplication.

* * *

B. Each of the following generalizations is a consequence of one of the principles of real numbers. Tell which principle.

1. For each x, $(x + 4)(x + 3)(x + 5) = (x + 4)[(x + 3)(x + 5)]$.

2. For each y, $(y + 2)(y + 3) = (y + 2)y + (y + 2)3$.

3. For each k, $3k + 5 = 5 + 3k$.

4. For each A, $(2A + 7)[(8A + 1) + (5 + 2A)]$
$= (2A + 7)(8A + 1) + (2A + 7)(5 + 2A)$.

5. For each t, $(6t + 1)(7t) + (6t + 1)9 = (6t + 1)(7t + 9)$.

6. For each s, $3(s - 7) + 9 + 8(s - 15) = 3(s - 7) + [9 + 8(s - 15)]$.

7. For each x, $(xxx + xx + x + 1) \cdot 0 = 0$.

[More exercises are in Part F, Supplementary Exercises.]

C. Each of the following is the first part of a generalization. Your job is to complete the generalization [from the choices given] in such a way that you can prove it. [You should be prepared to give the proof.]

1. For each x, $3 + 5x + 5 =$ __________.

(a) $8x + 5$ (b) $3 + 10x$ (c) $5x + 8$ (d) $13x$

2. For each y, $2y + 3y + 2 =$ __________.

(a) $6y + 2$ (b) $2y + 5y$ (c) $(5 + 2)y$ (d) $5y + 2$

3. For each x, $x + 4 + x - 2 =$ __________.

(a) $2(x + 1)$ (b) $2x + 6$ (c) $4x - 2$ (d) $5x - 2$

4. For each z, $3z(9z) =$ __________.

(a) $27z$ (b) $12zz$ (c) $27zz$ (d) $12(zz)$

5. For each x, $3(x + 4) + 3x =$ __________.

(a) $3(xx + 4)$ (b) $6x + 12$ (c) $6x + 4$ (d) $10x$

(continued on next page)

6. For each k, $2(k + 5) + 7(k + 4) =$ __________.

(a) $9(k + 9)$ (b) $9(kk + 9)$ (c) $9k + 9$ (d) $9k + 38$

7. For each m, $(3m)(5m)(4m) =$ __________.

(a) $60(mmm)$ (b) $180m$ (c) $12mmm$ (d) $60m$

8. For each p, $(p + 3) + (p + 5) + (p + 6) =$ __________.

(a) $4p + 14$ (b) $3p + 14$ (c) $ppp + 14$ (d) $p + 14$

9. For each k, $5k(2 + 3k) =$ __________.

(a) $25kk$ (b) $10k + 15kk$ (c) $25kkk$ (d) $10k + 3k$

10. For each r, $(3r + 7) - (3r + 7) =$ __________.

(a) $6r$ (b) $-6r + 14$ (c) 0 (d) $6r - 7$

11. For each s, $5(3s + 1) \cdot [7(3s + 2)] =$ __________.

(a) $35(3s + 1)(3s + 2)$ (b) $12(6s + 3)$

(c) $35(6s + 3)$ (d) $(15s + 5)(3s + 2)$

12. For each x, $(x + 3)(x + 2) =$ __________.

(a) $xx + 6$ (b) $(x + 3)x + (x + 3)2$

(c) $2x + 5$ (d) $[(x + 3) + x] + [(x + 3) + 2]$

☆13. For each N, $(N + \frac{1}{2})(N + \frac{1}{2}) =$ __________.

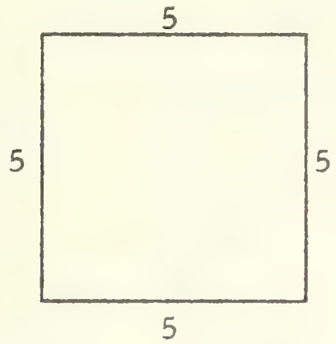
(a) $2N + 1$ (b) $N + \frac{1}{2}N + \frac{1}{4}$

(c) $N(N + 1) + \frac{1}{4}$ (d) $NN + \frac{1}{4}$

EXPLORATION EXERCISES

A. Here are some geometric figures which you probably studied in earlier grades. See how many of these figures you remember. Name each figure and compute the distance around it [that is, compute the perimeter].

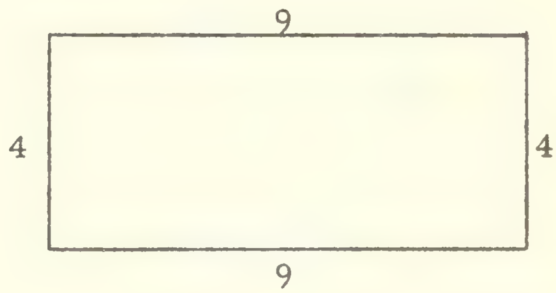
1.



Name ____________

Perimeter ________

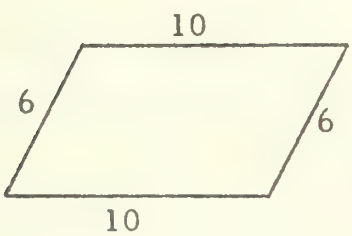
2.



Name ____________

Perimeter ________

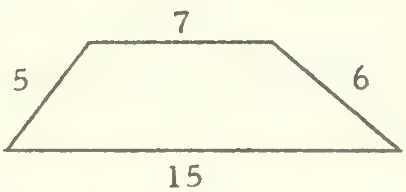
3.



Name ____________

Perimeter ________

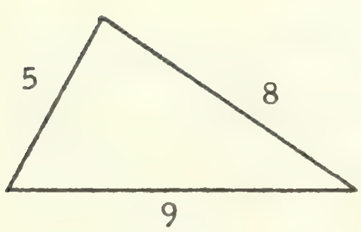
4.



Name ____________

Perimeter ________

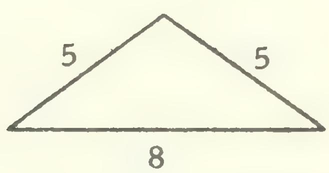
5.



Name ____________

Perimeter ________

6.



Name ____________

Perimeter ________

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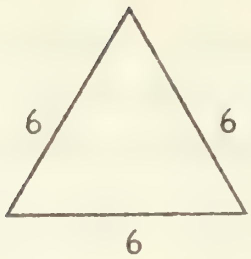
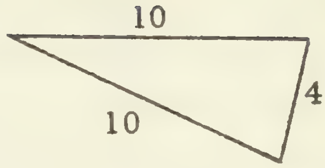
7.



Name ______________

Perimeter __________

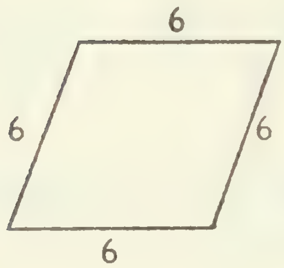
8.



Name ______________

Perimeter __________

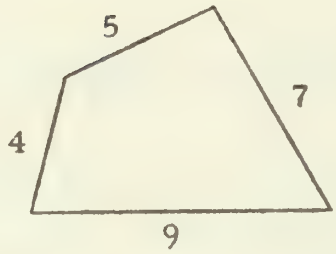
9.



Name ______________

Perimeter __________

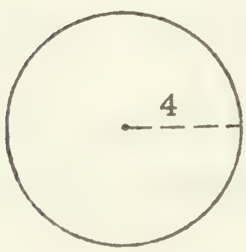
10.



Name ______________

Perimeter __________

11.

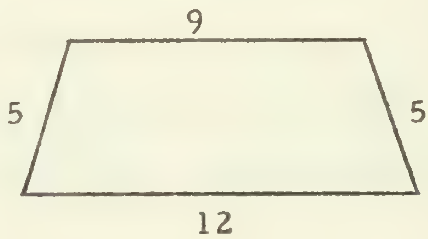


Name ______________

Perimeter __________

[Is there another word you use in Exercise 11 instead of 'perimeter'?]

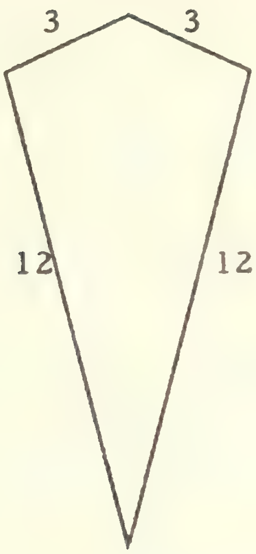
12.



Name ______________

Perimeter __________

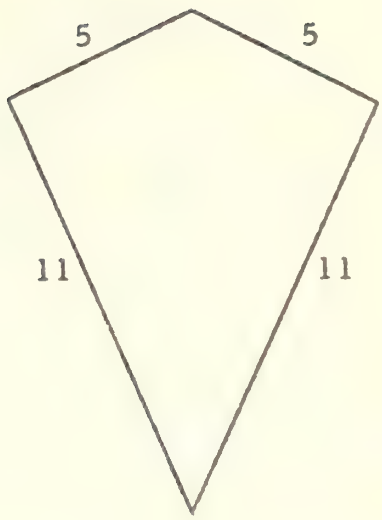
13.



Name ________________

Perimeter ____________

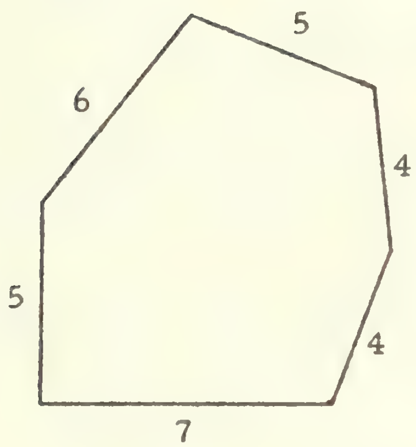
14.



Name ________________

Perimeter ____________

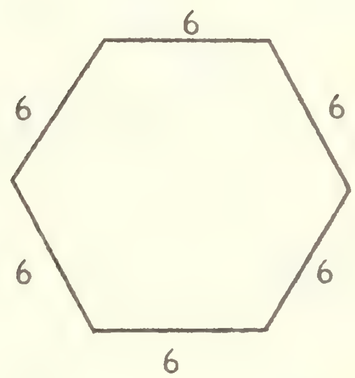
15.



Name ________________

Perimeter ____________

16.



Name ________________

Perimeter ____________

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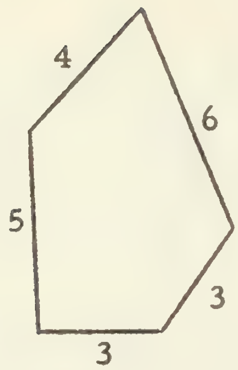
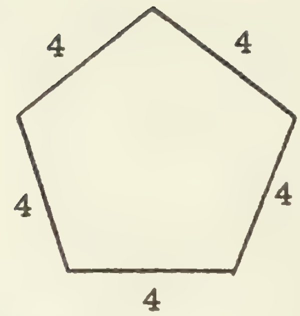
17.



Name ____________________

Perimeter ________________

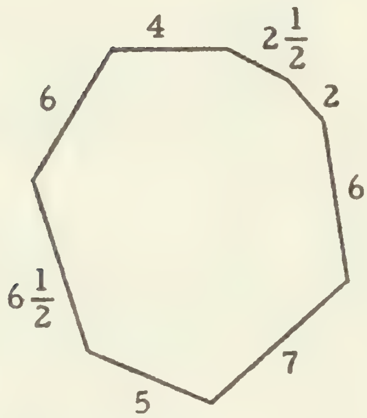
18.



Name ____________________

Perimeter ________________

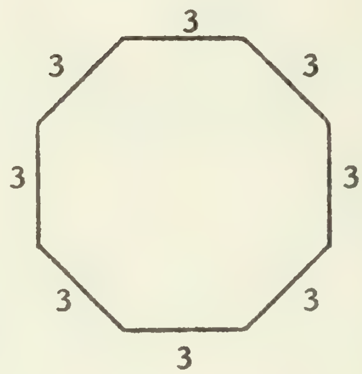
19.



Name ____________________

Perimeter ________________

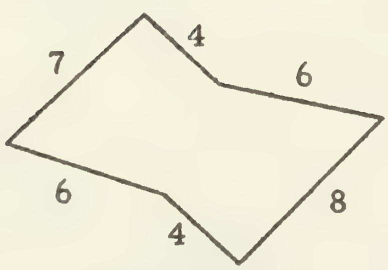
20.



Name ____________________

Perimeter ________________

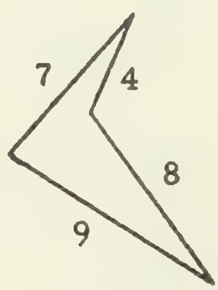
21.



Name ____________________

Perimeter ________________

22.



Name ____________________

Perimeter ________________

B. Use a ruler and compasses to make careful drawings of the figures described below. [Choose a convenient unit.]

1. A triangle with sides measuring 3, 5, and 7.
2. An equilateral triangle with one side measuring 2.
3. A rectangle with one side 2 units long and another side 5 units long.
4. A triangle whose perimeter is 12, and two of whose sides measure 3 and 4. [If you put two such triangles together with their longest sides matching, what kind of figure do you get?]
5. A kite with one side measuring 2 and another side measuring 5.
6. A square with one side 3 units long.
7. A circle with radius 2 units long.
8. A regular hexagon with side 2 units long. [Hint: Use the circle you drew for Exercise 7.]
9. An isosceles triangle with one side 3 units long and another side 1 unit long.
10. An isosceles triangle two of whose sides measure 3 and 4.
11. A circle whose circumference is 10π.
12. A rhombus [not a square], one of whose sides is 3 units long.
13. A parallelogram two of whose sides measure 3 and 5.
14. A square whose diagonal is 4 units long.
15. A parallelogram whose diagonals are 6 and 8 units long.
16. A square whose perimeter is 12.
17. A rectangle one of whose sides measures 3 more than twice the other, and whose perimeter is 18.

(continued on next page)

18. A kite with perimeter 20 and a side 4 units long.

19. An isosceles triangle whose perimeter is 30 and one of whose sides is twice as long as the other.

20. An equilateral triangle whose perimeter is 17.

C. Find the perimeter of each figure described below.

1. A rectangle with one side 4 units long and another side 7 units less than 3 times the first.

2. A square one of whose sides has the same length as a side of an equilateral triangle of perimeter 12.

3. A parallelogram whose sides have the same lengths as the sides of the rectangle in Exercise 1.

4. A quadrilateral whose sides are such that the average of their measures is 5.

5. An octagon whose sides are such that the average of their measures is $5\frac{1}{4}$.

6. A hexagon the measures of whose sides are consecutive whole numbers and whose longest side measures 19.

7. A parallelogram whose shorter side is 9 units long and the measure of whose longer side is twice the sum of the measure of its shorter side and 3.

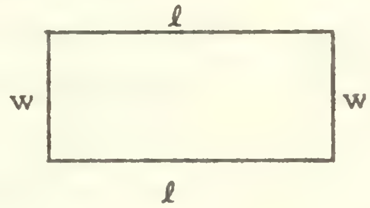
8. An isosceles triangle whose base [the side whose length differs from that of the others] is 5 units long, and 9 units less than twice one of the other sides.

9. A circle whose diameter is 3 units shorter than twice the side of a square whose perimeter is 60.

10. An isosceles trapezoid the longer of whose parallel sides measures 12, the shorter of whose parallel sides measures 15 less than twice the measure of the longer, and the length of each of whose nonparallel sides is half the length of the shorter parallel side.

2.04 <u>Simplification of expressions</u>.--One of the uses of pronumerals and pronumeral expressions is to write <u>formulas</u> which can be used in solving problems. For example, in solving problems about the perimeters of rectangles, you might use a formula like:

(*) $P = \ell + w + \ell + w.$



Then, to compute the perimeter of a rectangle whose sides measure 4 and 5, you find the value of the expression '$\ell + w + \ell + w$' for the values 4 and 5 of 'w' and 'ℓ'. The value of '$\ell + w + \ell + w$' in this case is 5 + 4 + 5 + 4. So, the perimeter is 5 + 4 + 5 + 4. We simplify '5 + 4 + 5 + 4' to '18'.

Now, every rectangle-perimeter problem which uses the formula (*) would involve a simplification like the one we went through in simplifying '5 + 4 + 5 + 4' to '18'. For example, another problem might require you to simplify '87.6 + 49.3 + 87.6 + 49.3'.

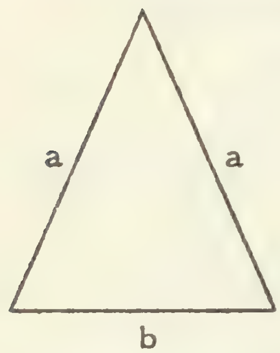
Let's look at a way of simplifying such numerical expressions, a way which can be generalized to all such expressions.

$$\begin{aligned} & [(87.6 + 49.3) + 87.6] + 49.3 \\ & && \text{[why?]}\ \underline{\qquad\qquad} \\ = \; & (87.6 + 49.3) + (87.6 + 49.3) \\ & && \text{[why?]}\ \underline{\qquad\qquad} \\ = \; & (87.6 + 49.3) \cdot 1 + (87.6 + 49.3) \cdot 1 \\ & && \text{[why?]}\ \underline{\qquad\qquad} \\ = \; & (87.6 + 49.3)(1 + 1) \\ & && \text{[why?]}\ \underline{\qquad\qquad} \\ = \; & (87.6 + 49.3) \cdot 2 \\ & && \text{[why?]}\ \underline{\qquad\qquad} \\ = \; & 2(87.6 + 49.3). \end{aligned}$$

Does this suggest to you a simpler formula than (*)?

[Note that although measures of pieces of straight lines like the measures of the sides of a rectangle are numbers of arithmetic, in simplifying expressions for perimeters we can act as though these measures were nonnegative real numbers. Thus, we can use the principles for real numbers to justify our simplifications.]

Consider a formula for the perimeter of an isosceles triangle. From what we mean by 'perimeter', one such formula is:



$$P = a + b + a.$$

We can get a simpler formula by simplifying the expression '$a + b + a$'. Can you guess what the simpler expression will be? Here is how we use the principles of real numbers to simplify this expression.

$$\begin{array}{ll}
(a + b) + a & \\
& \text{cpa} \\
= a + (a + b) & \\
& \text{apa} \\
= (a + a) + b & \\
& \text{pm1} \\
= (a \cdot 1 + a \cdot 1) + b & \\
& \ell\text{dpma} \\
= a(1 + 1) + b & \\
& 1 + 1 = 2 \\
= a2 + b & \\
& \text{cpm} \\
= 2a + b. &
\end{array}$$

So, a simple formula is:

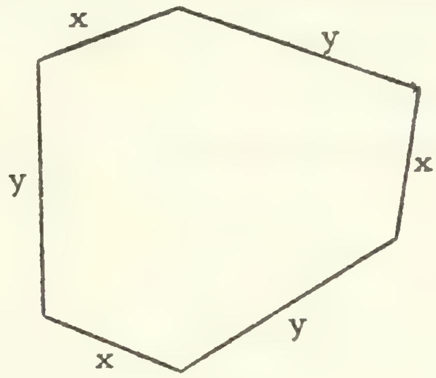
$$P = 2a + b.$$

Note that the simplification procedure from '$(a + b) + a$' to '$2a + b$' shows you that the generalization:

For each a, for each b, $a + b + a = 2a + b$

is a consequence of the principles [and the computing fact] listed on the right. This being so, you can use this generalization as you would any of the principles in justifying a step in another simplification problem.

For example, suppose you wanted a formula for finding the perimeter of a hexagon which looks like this.



From the meaning of 'perimeter', a formula is:

$$P = y + x + y + x + y + x.$$

By using the associative principle several times we can simplify:

$$y + x + y + x + y + x$$

to:

$$[y + x + y] + [x + y + x].$$

The generalization we obtained on page 2-46 justifies simplifying this to:

$$[2y + x] + [2x + y].$$

The commutative and associative principles for addition allow us to rephrase the expression as:

$$[2y + y] + [2x + x].$$

We should be able to simplify this expression to '$3y + 3x$'. To justify this, we can use the generalization:

For each k, $2k + k = 3k$.

[Give a proof of this last generalization.] Finally, the left distributive principle justifies going from:

$$3y + 3x$$

to:

$$3(y + x).$$

So, as you must have guessed from the beginning, a simpler formula is:

$$P = 3(y + x).$$

EQUIVALENT EXPRESSIONS

You have seen that the principles of real numbers can be used to simplify pronumeral expressions in just the same way as they are used in simplifying numerical expressions. In the work with perimeter formulas we simplified

'$\ell + w + \ell + w$' to '$2(\ell + w)$',

'$a + b + a$' to '$2a + b$',

and '$y + x + y + x + y + x$' to '$3(y + x)$'.

Since much of your work in mathematics will require skill in simplifying expressions, you will need to know not only how the principles are used in such simplifications, but also how to carry out the simplifications quickly and mechanically. You will learn to do this through practice, for by doing lots of simplifications you will discover short cuts. The important thing to remember about short cuts is that they are not magic, but that they are consequences of the principles, together with computing facts.

When you have a numerical expression like:

$$3 \times 7 + 5 \times -6$$

and simplify it to:

$$-9,$$

both the expression you start with and the final expression are numerals for the same number. And, we can state this fact by writing an equality sign between the two numerals, getting the true sentence:

$$3 \times 7 + 5 \times -6 = -9,$$

or, by saying that the numerals '$3 \times 7 + 5 \times -6$' and '-9' are <u>equivalent</u> <u>numerical</u> <u>expressions</u>.

[Note: We would not say that '$3 \times 7 + 5 \times -6$' equals '-9' because 'equal' has the same meaning as 'is the same as', and, clearly, the expressions '$3 \times 7 + 5 \times -6$' and '-9' are different. Although they are different, they stand for the same number, and this is what we mean when we say they are equivalent numerical expressions.]

Now, suppose we have a pronumeral expression like:

$$7x + 3x + 5,$$

and simplify it by means of our principles and computing facts to:

$$10x + 5.$$

If we write an equality sign between the two pronumeral expressions:

$$7x + 3x + 5 = 10x + 5,$$

we get an open sentence. Can you generate a false sentence from this open sentence? The answer is 'no', and we can express this fact by stating the generalization:

For each x, $7x + 3x + 5 = 10x + 5$.

Another way of saying that you can't generate a false sentence from the open sentence:

$$7x + 3x + 5 = 10x + 5$$

is to say that '$7x + 3x + 5$' and '$10x + 5$' are equivalent pronumeral expressions. If you pick a value of 'x' and substitute a numeral for this value of 'x' in both '$7x + 3x + 5$' and '$10x + 5$', you get a pair of equivalent numerical expressions.

> Equivalent numerical expressions are numerals for the same number.
>
> Equivalent pronumeral expressions are expressions such that for each substitution both expressions have the same value.

[Note: We would not say that '$7x + 3x + 5$' equals '$10x + 5$', because they are different pronumeral expressions. Instead we say that '$7x + 3x + 5$' is equivalent to '$10x + 5$' meaning that the expressions have the same value for each value of 'x'.]

EXERCISES

A. Each of the following exercises contains a pair of expressions. In some cases, the expressions are equivalent. In the others, they are not. Tell which pairs are equivalent and which pairs are not. If you claim that two expressions are not equivalent, you should be able to support your claim by giving values of the pronumerals in the expressions which lead to different values for the two expressions. If you claim that two expressions are equivalent, you should be able to show how you can transform one expression into the other by using principles and computing facts.

Sample 1. $5x + y + 3x + 4y$

$6xy + 7xy$

Solution. I claim that these are not equivalent. So, I just write:

Not equivalent.

If I am asked to justify this answer, I should be able to give a substitution for 'x' and a substitution for 'y' for which the two expressions have different values. Suppose I try '2' for 'x' and '3' for 'y'. The corresponding value of '$5x + y + 3x + 4y$' is $5 \cdot 2 + 3 + 3 \cdot 2 + 4 \cdot 3$, that is, is 31. The corresponding value of '$6xy + 7xy$' is $6 \cdot 2 \cdot 3 + 7 \cdot 2 \cdot 3$, or 78.

Sample 2. $7x + 3y + 2x + 9y + 15$

$3(3x + 4y + 5)$

Solution. I notice the second expression can be transformed into '$9x + 12y + 15$' by the left distributive principle and some computing facts. I can transform the first into '$9x + 12y + 15$' by the use of the associative, commutative, and distributive principles along

with some computing facts. So, I know that the expressions are equivalent. Therefore, I just write:

Equivalent.

If someone disputes me, I can ask him to give me values of 'x' and 'y' for which the expressions have different values. If someone asks me how I know that they are equivalent, I would prove:

For each x, for each y,

$$7x + 3y + 2x + 9y + 15 = 3(3x + 4y + 5).$$

[You supply the reasons.]

$$\begin{aligned}
&\{[(7x + 3y) + 2x] + 9y\} + 15 \\
&= \{[7x + (3y + 2x)] + 9y\} + 15 \quad \text{________} \\
&= \{[7x + (2x + 3y)] + 9y\} + 15 \quad \text{________} \\
&= \{[(7x + 2x) + 3y] + 9y\} + 15 \quad \text{________} \\
&= \{(7x + 2x) + (3y + 9y)\} + 15 \quad \text{________} \\
&= \{(7 + 2)x + (3 + 9)y\} + 15 \quad \text{________} \\
&= \{9x + 12y\} + 15 \quad \text{________} \\
&= \{(3 \cdot 3)x + (3 \cdot 4)y\} + 3 \cdot 5 \quad \text{________} \\
&= \{3(3x) + 3(4y)\} + 3 \cdot 5 \quad \text{________} \\
&= 3\{3x + 4y\} + 3 \cdot 5 \quad \text{________} \\
&= 3(3x + 4y + 5). \quad \text{________}
\end{aligned}$$

1. $3a + 2b + 7a$
   $10a + 2b$

2. $5x + 4 + 2y$
   $9x + 2y$

3. $8a(2b) + 3ab$
   $19ab$

4. $5p + 6 + 3p + 2$
   $8(p + 1)$

5. $11x + 2y$
   $6x + 2y + 5x$

6. $a + 3b + 5ab$
   $8ab$

7. $6x(2 + 3x) + 5xx$
   $12x + 23xx$

8. $7 + 4y$
   $11y$

9. $3(x + 7y)$
   $3x + 7y$

* * *

We have been talking about pairs of equivalent expressions. What does this have to do with the problem of simplifying an expression?

To simplify an expression is to transform it into an equivalent one which is simpler. What do we mean by 'simpler'? In the perimeter problems, we said that '$2(\ell + w)$' was simpler than '$\ell + w + \ell + w$'. Do you agree? Why?

Often, when you are trying to decide which of two equivalent pronumeral expressions is the simpler, you think about which one would be easier to find values of. [Usually, this is the one with fewer marks in it.] When we ask you to simplify a pronumeral expression, you can use this "evaluation test". [But, in some later exercises you will transform an expression into an equivalent one which is not simpler to evaluate but which is simpler for some other purpose.]

* * *

B. Simplify.

Sample 1. $3x + 5 + 4x + 3$

Solution. $3x + 5 + 4x + 3$

$= (3x + 4x) + (5 + 3)$

$= 7x + 8.$

[As answer, we write: $7x + 8$. And, this means that we are claiming that, for each x, $3x + 5 + 4x + 3 = 7x + 8$.]

1. $a + 3 + 4a$
2. $6b + 5 + 2b$
3. $b + 6 + b$
4. $9 + 2c + 7$
5. $11a + 7 + a$
6. $4x + 5 + {}^-2x$
7. $r + 2r + 3r$
8. $6 + 7p + 2$
9. $8 + {}^-3k - 5$

Sample 2. $2x + 5y + 3 + 7x + 2y + 7$

Solution. $2x + 5y + 3 + 7x + 2y + 7$

$= (2x + 7x) + (5y + 2y) + (3 + 7)$

$= 9x + 7y + 10.$

10. $3a + 2b + 7a$

11. $6x + 2y + 7 + 9x + 3y$

12. $6c + 5 + 2c + 8d$

13. $x + 3y + 8 + 2x + y + 9$

14. $7m + 2 + 3m + 5n$

15. $12 + 6p + 3q + {}^{-}5p + 7q + 2$

16. $7s + 2s + 5s + 6t$

17. ${}^{-}3x + {}^{-}9y + {}^{-}7x + {}^{-}5 + {}^{-}6y$

18. $\frac{2}{3}x + \frac{3}{4} + \frac{1}{3}x + \frac{7}{8}$

19. $\frac{2}{5}y + \frac{1}{4}x + \frac{1}{3}y + \frac{1}{5}x$

20. $2.1k + 3.8m + 2.7k$

21. $0.8p + 7.9 + 3.1p + {}^{-}6.3$

Sample 3. $7ab + 3c + {}^{-}6ab + 9c$

Solution. $7ab + 3c + {}^{-}6ab + 9c$

$= (7ab + {}^{-}6ab) + (3c + 9c)$

$= ab + 12c.$

Sample 4. $3xx + 2x + 5xx + 5 + 9x$

Solution. $3xx + 2x + 5xx + 5 + 9x$

$= (3xx + 5xx) + (2x + 9x) + 5$

$= 8xx + 11x + 5$

$= x(8x + 11) + 5.$

[Either '8xx + 11x + 5' or 'x(8x + 11) + 5' is acceptable as an answer, although the latter is simpler according to our evaluation test. However, there are many places in mathematics where '8xx + 11x + 5' is more useful.]

22. $6ab + 5c + 3ab + 2c$

23. ${}^{-}10rs + 3t + 5rs + 6rs + {}^{-}2t$

24. $3x + 2xx + 9x + 7xx + 4$

25. $3mm + 2m + 5 + 6mm + 5m + 7$

26. $7pq + 3rs + 2pq + 12rs$

27. $6y + 2yy + 9 + {}^{-}1yy + {}^{-}7y + {}^{-}9$

28. $7(2a + 3b) + 6(a + 3b)$

29. $2(5a + b) + 3(b + 2a)$

30. $6(2x + y) + 3(4x + 5y)$

31. $3(5x + 1) + 9(1 + 2x)$

32. $4(1 + 3x) + 5(x + 2xx)$

33. $7(y + 3yy) + y(3 + 7y)$

34. $1/2(8 + 4x) + 1/3(3x + 1)$

35. $2/5(10y + 15x) + 1/7(21x + 7)$

36. $1/3(3xx + 6x) + 7(x + 2xx)$

37. $3/7(5x + 2y) + 2/7(3x + 9y)$

(continued on next page)

<u>Sample</u> <u>5</u>. $(3x)(2y)z$

<u>Solution</u>. $(3x)(2y)z$

$= (3 \cdot 2)xyz$

$= 6xyz.$

<u>Sample</u> <u>6</u>. $(2ac)(3ab)(5abc)$

<u>Solution</u>. $(2ac)(3ab)(5abc)$

$= (2 \cdot 3 \cdot 5)aaabbcc$

$= 30aaabbcc.$

38. $(6m)(2n)(3p)$ 39. $(2m)(3n)(5m)$ 40. $p(3qp)(7pqq)$

41. $(6xx)(2xy)(^-3xyy)$ 42. $(^-2x)(^-4xy)(^-1yx)$ 43. $5(3x)(^-2x)(^-3xy)$

44. $(3ab)(^-2ab)(^-1ab)$ 45. $(4xy)(2yz)(3xz)$ 46. $(^-1ab)(^-1bc)(^-1cd)$

47. $(\frac{1}{2}x)(\frac{3}{4}y)(\frac{7}{8}z)$ 48. $(\frac{2}{3}m)(\frac{1}{5}m)(\frac{1}{2}m)$ 49. $(\frac{3}{7}k)(\frac{7}{9}m)(\frac{3}{5}n)$

50. $(\frac{^-1}{3}a)(\frac{^-2}{3}b)(\frac{^-3}{4}c)$ 51. $(\frac{^-6}{7}t)(\frac{1}{3}r)(\frac{5}{8}s)$ 52. $\frac{2}{5}(\frac{3}{4}e)(\frac{10}{11}f)(\frac{1}{2}g)$

53. $(7.2x)(3.8y)(2.1z)$ 54. $(0.1x)(0.2y)(0.3y)$ 55. $^-6.1(^-3.8a)(^-5.1b)$

56. $(3x)(2y) + (7x)(5y)$ 57. $(9a)(2b) + (^-3a)(^-2b)$

58. $(4p)(^-3q) + 3(4p)(^-1q)$ 59. $(7m)(2n)(^-3p) + (^-5m)(^-6n)(3p)$

<u>C</u>. Simplify.

1. $3x + 7 + 2x$ 2. $5m + 2m + 6$ 3. $8k + k + 5k$

4. $7r + 5 + 2r + 4$ 5. $3t + 9 + 5t - 6$

6. $10a + 3b + 5a + 7$ 7. $12j + 5k + 2 + 9k + 4j$

8. $2d + 3c + 7d + 2c$ 9. $9x + 3y + 7 + 2y + 12x$

10. $5m + 2k + 3m + 1$ 11. $7d + 5r + 3d + 5r$

12. $4x + 3y + (-7)x + 2y$ 13. $(-1)k + (-5)k + 7k + (-2)k$

14. $7rs + 2t + 3rs + 5u$ 15. $9ab + 3c + 7d + 12ab$

16. $6xyz + 2x + 3yz + 5x$ 17. $3mn + 2n + 5nm + 4n$

18. $3(2x + 1) + 5(1 + 3x)$ 19. $2(5x + 7) + 5(3x + 2)$

20. $6(5 + 4x) + 3(2 + 7x)$

21. $4(61x + 72) + 8(61x + 72)$

22. $4(3 + 8y) + 7(4y + 1)$

23. $12(2 + 5y) + 7(1 + 3y)$

24. $5(1 + 2a + 3b) + 7(2 + 4a + 6b)$

25. $3(3x + 2y + 5z) + 9(2x + 7y + 9z)$

26. $7x + 2 + (-1)(3x + 1)$

27. $(-2)(4 + 3y) + (-3)(1 + 5y)$

28. $3a(2b)(6c)$

29. $4x(3y)(7z)$

30. $a(5b)(7c)$

31. $2x(3xx)(5x)$

32. $2y(4xy)(3xy)$

33. $(-6)x(-8)y$

34. $[(-2)xy][(-8)xyx]$

35. $[(-1)ab][(-3)aab]$

36. $[(-2)xy][5yx]$

37. $2ab(a + 3b) + 5ab(2a + 7b) + 8ab(4a + 9b)$

38. $5xy(2x + 5y) + 7xy(5x + 3y) + 4xy(x + y)$

39. $4a(3a + 2b + 5c) + 5b(4a + 2b + 3c) + 7c(a + b + 2c)$

40. $7(2x + 3yx + 5y) + 4(8x + 2xy + 7y) + (-2)(3x + 5xy + y)$

41. $3x(5xx + 3x + 5) + 2x(4xx + 5x + 7) + x(xx + x + 1)$

42. $8a + 2b + 3x + 5c + 7x + 2b + 5a + 6c + 7a + (-3)x$

43. $4x + 2y + (-7)z + 6x + 3z + (-2)y + (-6)x + 3y + (-4)z$

44. $2[(7x + 3) + (5x + 1)]$

45. $2[(3 + 5x) + (9 + 7x)]$

46. $5[(2y + 1) + (1 + y)] + 7[(8y - 1) + (3y - 2)]$

47. $4x[(3y + 2x) + (8x + 4y)] + 5y[(2x + 8y) + (3x + 5y)]$

48. $5a[(7a + b) + (7b + a)] + 6b[(2a + 3b) + (5b + 6a)]$

49. $12a[(8a + 3b) + (4a + b)] + 4b[(8a + 3b) + (4a + b)]$

50. $x(x + 7) + 7(x + 7)$

51. $a(a + 2) + 7(a + 2)$

52. $\frac{1}{2}x(3 + y) + \frac{3}{4}y(x + 5)$

53. $\frac{3}{4}x(x + y + 1) + \frac{1}{5}y(5y + 2x + 3)$

54. $(\frac{1}{5}a)(\frac{3}{5}b) + (\frac{2}{7}b)(\frac{3}{8}a)$

55. $\frac{2}{3}(\frac{3}{4}xy)(\frac{1}{3}x) + \frac{1}{5}(\frac{5}{7}xy)(\frac{14}{15}y)$

[More exercises are in <u>Part G</u>, <u>Supplementary Exercises</u>.]

D. Complete each of the following into a true sentence by writing the simplest expression you can in the blank.

Sample. For each x, the sum of $(6x + 1)$ and $(3x + 4)$ is ______________.

Solution. The expression '$(6x + 1) + (3x + 4)$' would make the sentence true. But, an expression simpler than '$(6x + 1) + (3x + 4)$' and equivalent to it is '$9x + 5$'. So, write '$9x + 5$' in the blank.

1. For each x, the sum of $(9x + 3)$ and $(5 + 7x)$ is ____________.

2. For each x, for each y, the sum of $(2x + 7y)$ and $(5x + 9y)$ is ____________.

3. For each x, the product of $(7x)$ by $(3x + x)$ is ____________.

4. For each x, for each y, for each z, the sum of $(x + 3y + z)$, $(3x + 2y + 5z)$, and $(2x + y + 3z)$ is ____________.

5. For each A, for each B, the sum of $(3A)$ and $(4A + 7B)$ is ____________.

6. For each k, the sum of $(3k + 2)$ and the product of 6 by $(3 + 5k)$ is ____________.

7. For each p, the product of the sum of $(3p + 5)$ and $(5p + 6)$ by 7 is ____________.

8. For each x, for each y, the sum of $(3x)$ and $(5y)$ is __________.

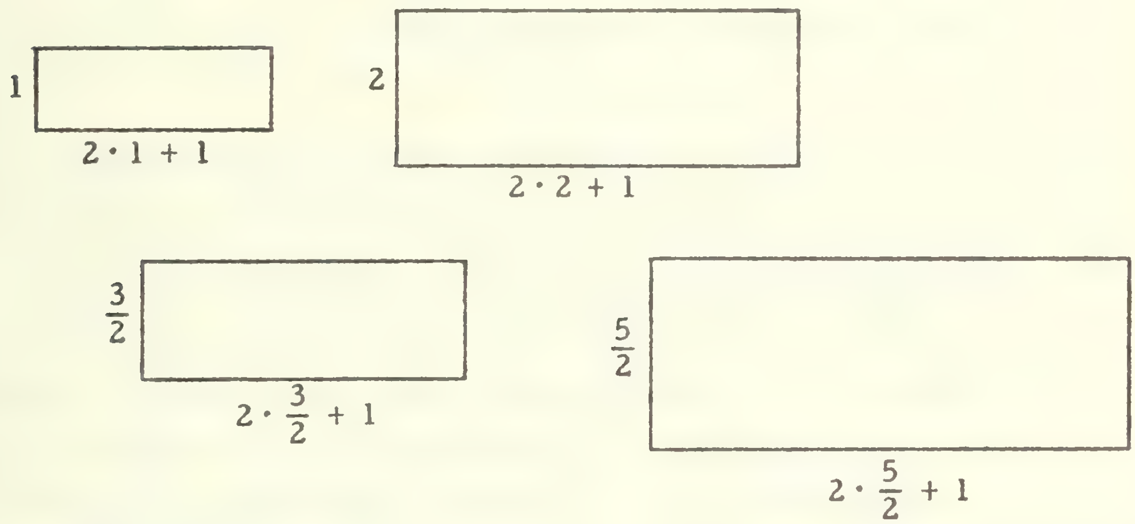
9. For each k, for each j, the product of the sum of $(2k + j)$ and $(3j + 5k)$ by the sum of $(5k + 7j)$ and $(k + 9j)$ is __________.

10. For each x, for each y, the sum of $(\frac{1}{2}x + \frac{1}{3}y)$ and $(\frac{1}{4}x + \frac{1}{5}y)$ is ____________.

E. Write the simplest formula you can for the perimeter of each figure described.

Sample 1. A rectangle whose length measures 1 more than twice the measure of the width.

Solution. How many rectangles fit this description? Here are pictures of just a few of them.



Our job is to find a formula for computing the perimeters of all such rectangles. For the rectangles shown above, the perimeters are given by the expressions:

$$2[(2 \cdot 1 + 1) + 1],$$

$$2[(2 \cdot \frac{3}{2} + 1) + \frac{3}{2}],$$

$$2[(2 \cdot 2 + 1) + 2],$$

and: $$2[(2 \cdot \frac{5}{2} + 1) + \frac{5}{2}].$$

Do you see a pattern for these numerical expressions? Each of them can be obtained by substituting for 'x' in the pronumeral expression:

$$2[(2x + 1) + x].$$

So, a formula for the perimeter of such rectangles is the sentence:

$$P = 2[(2x + 1) + x].$$

We can simplify the expression on the right of the

equality sign.

$$\begin{aligned} &2[(2x + 1) + x] \\ &= 2[x + (2x + 1)] \\ &= 2[(x + 2x) + 1] \\ &= 2[3x + 1] \\ &= 6x + 2. \end{aligned}$$

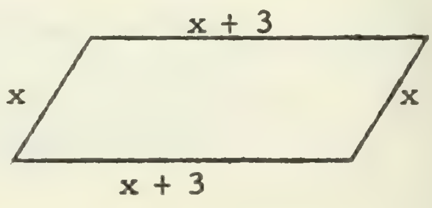
Hence, a simple formula is:

$$P = 6x + 2.$$

Note that the values of 'x' in this formula are the measures of the widths of rectangles which fit the given description. Let's check the formula with an example. Suppose we take the largest of the rectangles pictured on page 2-57, the one whose width measures $\frac{5}{2}$. According to the <u>formula</u>, the perimeter is $6 \cdot \frac{5}{2} + 2$, that is, the perimeter is 17. According to the <u>description</u>, the length measures 6; so, the perimeter is $2(6 + \frac{5}{2})$, which is 12 + 5, or 17.

<u>Sample 2</u>. A parallelogram whose longer side measures 3 more than the shorter side.

<u>Solution</u>. Instead of drawing several examples of such parallelograms, we draw only one, and label its sides with pronumeral expressions instead of numerals. If 'x' is a pronumeral whose values are the measures of the shorter sides of such parallelograms then, for such values of 'x', the values of 'x + 3' are the measures of the longer sides. Expressions for the perimeters of such parallelograms can be generated from the pronumeral expression:



$$x + (x + 3) + x + (x + 3).$$

Hence, a formula for the perimeter of such parallelograms is '$P = x + (x + 3) + x + (x + 3)$'. A simpler formula is:

$$P = 4x + 6.$$

1. A rectangle whose longer side measures 7 more than 5 times the measure of the shorter side.

2. A parallelogram whose shorter side is $\frac{1}{3}$ as long as the longer side.

3. A parallelogram whose longer and shorter sides differ in measure by 9.

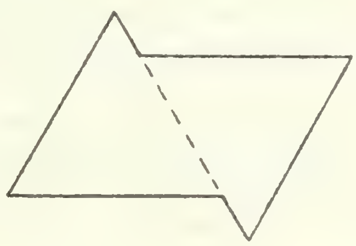
4. A hexagon the measures of whose sides are consecutive whole numbers. [Example: A hexagon with sides 2", 3", 4", 5", 6", and 7" long.]

5. An isosceles trapezoid for which the longer of its parallel sides is three times as long as the shorter of its parallel sides, and the sum of the measures of its nonparallel sides is the sum of the measures of its parallel sides.

6. A kite whose shorter side measures 3 more than $\frac{1}{5}$ the measure of its longer side.

7. A rectangle made by putting two squares next to each other.

8. A hexagon made by putting two equilateral triangles with equal perimeters next to each other so that the length of the overlap is $\frac{3}{4}$ the length of a side.



9. A pentagon, three of whose sides have the same measure, and, of the remaining two sides, the shorter side is $\frac{1}{2}$ as long as the longer.

☆10. A rectangle for which the average of the measures of its four sides is twice the measure of its shorter side.

[More exercises are in Part H, Supplementary Exercises.]

2.05 Theorems and basic principles.--In proving generalizations you probably noticed that you could have shortened your work if you had had principles like:

$$\text{For each } x, \quad 1 \cdot x = x,$$

and:

$$\text{For each } x, \quad \text{for each } y, \quad \text{for each } u, \quad \text{for each } v,$$

$$(xy)(uv) = (xu)(yv).$$

In fact, you may have already stated such principles and derived them from the ones mentioned in Unit 1.

Statements which can be derived from the basic principles are called theorems. For example, when you simplify the expression

'$3x + 2y + 7x + 1$' to '$10x + 2y + 1$',

you are claiming that the generalization:

$$\text{For each } x, \quad \text{for each } y, \quad 3x + 2y + 7x + 1 = 10x + 2y + 1$$

is a theorem. And, when you give a test-pattern for the generalization, you are showing that it is a theorem. So, you have already proven many theorems.

A theorem can be used in justifying a step in a proof in the same way that you have used the basic principles. Those that are most useful for this purpose are worth remembering and even giving names to. An example of such a theorem is the left distributive principle for multiplication over addition. Although this is one of the principles mentioned in Unit 1, it can be derived from the basic ones. Let's prove that for each x, for each y, for each z, $x(y + z) = xy + xz$.

$$x(y + z) = (y + z)x \quad [\text{cpm}]$$

$$(y + z)x = yx + zx \quad [\text{dpma}]$$

$$yx + zx = xy + xz. \quad [\text{cpm}]$$

$$\text{Hence,} \qquad x(y + z) = xy + xz.$$

[Notice that the commutative principle for multiplication was applied twice in the last step.] So, the left distributive principle for multiplication over addition is [as was mentioned in Unit 1] a consequence of the commutative principle for multiplication and the distributive principle for multiplication over addition.

For definiteness and ease of reference we state the basic principles which we shall use in this unit. [The last basic principle will be discussed later, but we state it here for completeness.] We also introduce an abbreviation for 'for each'. It is '$\forall$'.

Commutative principles

$$\forall_x \forall_y \quad x + y = y + x.$$

$$\forall_x \forall_y \quad xy = yx.$$

Associative principles

$$\forall_x \forall_y \forall_z \quad x + y + z = x + (y + z).$$

$$\forall_x \forall_y \forall_z \quad xyz = x(yz).$$

Distributive principle

$$\forall_x \forall_y \forall_z \quad (x + y)z = xz + yz.$$

Principles for 0 and 1

$$\forall_x \quad x + 0 = x.$$

$$\forall_x \quad x \cdot 1 = x.$$

Principle of Opposites

$$\forall_x \quad x + -x = 0.$$

Principle for Subtraction

$$\forall_x \forall_y \quad x - y = x + -y.$$

Principle of Quotients

$$\forall_x \forall_{y \neq 0} \quad (x \div y) \cdot y = x.$$

[Notice that we have not included the principle for multiplying by 0. This is because we shall later derive it from the principles just stated.]

EXERCISES

Prove the following theorems.

1. $\forall_x \; 1 \cdot x = x.$ ["The 1 times theorem"]

2. $\forall_x \forall_a \forall_b \forall_c \; ax + bx + cx = (a + b + c)x.$ ["Extended distributive theorem"]

3. $\forall_x \forall_y \forall_a \forall_b \; (ax)(by) = (ab)(xy).$ ["Product rearrangement theorem"]

4. $\forall_x \forall_y \forall_a \forall_b \; (a + x) + (b + y) = (a + b) \div (x + y).$ ["Sum rearrangement theorem"]

Up to now in Unit 2 you have learned about pronumerals and how they can be used in stating generalizations about real numbers. You have seen how to derive theorems from the basic principles, and how to use the basic principles and the theorems in finding short cuts for simplifying expressions. These short cuts apply to problems involving addition and multiplication. Now we want to work toward short cuts dealing with opposition, subtraction, and division. We shall prove theorems like:

$$\forall_x \forall_y \; x - y = -(y - x),$$

and use them to simplify expressions like:

$$3b - (a - b).$$

So, your two main purposes for the rest of this unit are to prove theorems which will help you develop more short cuts in simplifying expressions, and to gain skill in applying these short cuts. The skills you acquire in simplifying expressions will be used throughout the rest of your work in mathematics.

2.06 Oppositing and subtracting.--We saw in Unit 1 that in order to define the operation subtraction it was convenient to have the operation oppositing [and we introduced the minus sign as a name for this operation]. Oppositing is such that

(*) For each x, $x + -x = 0$.

We call (*) the principle of opposites. [Read 'x + −x' as 'x plus the opposite of x' and not as 'x plus negative x'.]

EXERCISES

A. Give the opposite of each number listed.

1. $^{+}3$ 2. $^{-}7$ 3. -4 4. $+8.2$ 5. $- -3$ 6. 0

B. True or false?

1. For each x, $+x$ is a positive number.
2. For each x, $-x$ is a negative number.
3. For each x, if x is negative then $-x$ is positive.
4. For each x, if x is positive then $-x$ is negative.
5. For each x, if x is 0 then $-x$ is 0.
6. For each x, if $-x$ is positive then x is negative.
7. For each x, if $-x$ is negative then x is positive.
8. For each x, if $+x$ is positive then x is positive.
9. For each x, if $+x$ is negative then x is negative.

C. Fill the blank with an oppositing sign to make the sentence true unless the sentence is already true.

1. $^{+}534 +$ _____ $^{+}534 = 0$
2. $- {}^{+}534 +$ _____ $^{+}534 = 0$
3. $-{}^{-}721 +$ _____ $-{}^{-}721 = 0$
4. $^{-}721 +$ _____ $^{-}721 = 0$
5. $^{+}5 \cdot -{}^{+}2 =$ _____ $({}^{+}5 \cdot {}^{+}2)$
6. $^{+}5 \cdot -{}^{-}2 =$ _____ $({}^{+}5 \cdot {}^{-}2)$
7. $-{}^{+}2 \cdot {}^{+}7 =$ _____ $({}^{+}2 \cdot {}^{+}7)$
8. $-{}^{-}2 \cdot {}^{+}7 =$ _____ $({}^{-}2 \cdot {}^{+}7)$
9. $-{}^{+}3 \cdot -{}^{+}4 =$ _____ $({}^{+}3 \cdot {}^{+}4)$
10. $-{}^{-}3 \cdot -{}^{+}4 =$ _____ $({}^{-}3 \cdot {}^{+}4)$
11. $-{}^{+}3 \cdot -0 =$ _____ $({}^{+}3 \cdot 0)$
12. $-{}^{-}3 \cdot 0 =$ _____ $({}^{-}3 \cdot 0)$

ADDITION PRINCIPLES

> Suppose Rita and Rhoda each picks a real number. Then, Aaron picks a real number and tells both Rita and Rhoda to add it to hers. If Rita and Rhoda pick the same number [and both add correctly], do they get the same sum?

> Suppose Rita and Rhoda each picks a real number. Then, Aaron picks a real number and tells both Rita and Rhoda to add it to hers. If they both get the same sum [and both add correctly], did they pick the same number?

These situations illustrate two important properties of addition. Let's state the principle which expresses the property illustrated in the first situation. For each number Rita picks, for each number Rhoda picks, and for each number Aaron picks, if Rita's number is the same as Rhoda's number then Rita's number plus Aaron's number is the same as Rhoda's number plus Aaron's number. For short:

$$\forall_x \forall_y \forall_z \text{ if } x = y \text{ then } x + z = y + z.$$

For example, this tells us that, since $8 = 4 \times 2$, $8 + 7 = 4 \times 2 + 7$. We accept this reasoning because $8 + 7 = 8 + 7$ and, supposing that $8 = 4 \times 2$, '4×2' is another name for 8. So, substituting '4×2' for the second '8', we see that $8 + 7 = 4 \times 2 + 7$.

Here is a test-pattern for this generalization.

Suppose that	$x = y$.
Since	$x + z = x + z$, $[\forall_a \; a = a.]$
it follows that	$x + z = y + z$.
Hence,	if $x = y$ then $x + z = y + z$.

The generalization:

$$\forall_x \forall_y \forall_z \text{ if } x = y \text{ then } x + z = y + z$$

is called <u>the uniqueness principle for addition</u>. ["If you add a number to a number you get a <u>unique</u> sum".]

[Notice that in the proof of the uniqueness principle we did not use any of the basic principles or theorems. We did cite a principle of logic--"a thing is equal to itself". So, the uniqueness principle for addition is itself a theorem of logic.]

Let's turn now to the second situation. Rita, Rhoda, and Aaron each picked a number. Rita and Rhoda added Aaron's number to their numbers and got the same sum. When they told Aaron this, he said, "You both chose the same number." How did he know?

Rita figured it out this way:

> "When I told him my sum, he knew that he could get my number from it by adding the opposite of his number. And, he knew he could get your number, Rhoda, by adding the opposite of his number to your sum. And, since we had the same sum, he knew he would come out with the same number both times [uniqueness principle]."

Here is a statement of the principle Aaron used:

$$\forall_x \forall_y \forall_z \text{ if } x + z = y + z \text{ then } x = y.$$

It is called the cancellation principle for addition.

Rita's explanation can be translated into a proof.

Suppose that $x + z = y + z$.

Then $x + z + -z = y + z + -z$, [uniqueness principle]

$x + (z + -z) = y + (z + -z)$, [apa, apa]

$x + 0 = y + 0$, [po, po]

and $x = y$. [pa0, pa0]

Hence, if $x + z = y + z$ then $x = y$.

Principles and theorems of logic are not usually cited in proofs. We cited the uniqueness principle in this last proof for the sake of clarity. Thus, the cancellation principle is a consequence just of the apa, the po, and the pa0.

Notice in both these proofs that the last sentence is a conditional sentence, that is, a sentence of the form: if... then___. Notice also that the first line in each proof is a supposition of the "if-part" and the next-to-last line is the "then-part". So, the proof consists in using the if-part together with principles or theorems to derive the then-part.

EXERCISES

Prove these theorems.

1. $\forall_x \forall_y \forall_z$ if $x = y$ then $z + x = z + y$. ["Left uniqueness principle for addition"]

2. $\forall_x \forall_y \forall_z$ if $z + x = z + y$ then $x = y$. ["Left cancellation principle for addition"]

3. $\forall_x \forall_y$ if $x = y$ then $-x = -y$. ["Uniqueness principle for opposition"]

4. $\forall_x \forall_y \forall_z$ if $x = y$ then $xz = yz$. ["Uniqueness principle for multiplication"]

5. $\forall_x \forall_y \forall_z$ if $x = y$ then $zx = zy$. ["Left uniqueness principle for multiplication"]

☆6. $\forall_u \forall_v \forall_x \forall_y$ if $u = v$ and $x = y$ then $u + x = v + y$.

☆7. $\forall_u \forall_v \forall_x \forall_y$ if $u = v$ and $u + x = v + y$ then $x = y$.

*

Let us now prove the principle for multiplying by 0. What do our basic principles tell us about 0?

$x + 0 = x$ [pa0]

Can we get an expression containing an '$x \cdot 0$' out of this?

$x(x + 0) = xx$ [left uniqueness principle for multiplication]

$xx + x \cdot 0 = xx$ [ℓdpma]

Now we have an '$x \cdot 0$' on one side. Can we get a '0' on the other?

$xx + x \cdot 0 = xx + 0$ [pa0]

$x \cdot 0 = 0$ [left cancellation principle for addition]

So, the principle for multiplying by 0 is a consequence of the principle for adding 0, the left distributive principle for multiplication over addition, and the left cancellation principle for addition. Earlier, we claimed that we could derive the principle for multiplying by 0 from the basic principles stated on page 2-61. This can be done because the ℓdpma and the left cancellation principle can be derived from our basic principles.

*

☆8. Derive the pm0 directly from the basic principles.

THE PRINCIPLE OF OPPOSITES

The principle of opposites:

$$\forall_x \; x + -x = 0$$

tells us that for each real number there is a real number [the opposite of the first] which when added to the first gives the sum 0. In Unit I we mentioned that we would be able to prove that there couldn't be <u>two</u> numbers such that the sum of each with the given number is 0. For example, take the number ${}^-485$. The principle of opposites tells us that ${}^-485 + -{}^-485 = 0$. We are now able to predict that there isn't another number [that is, a number different from the opposite of ${}^-485$] which when added to ${}^-485$ gives the sum 0. In other words, we can now show that no matter what number you pick, if you add it to ${}^-485$ and get 0, that number must be the opposite of ${}^-485$. Let's state this generalization:

$$\forall_y \text{ if } {}^-485 + y = 0 \text{ then } y = -{}^-485.$$

and build a test-pattern.

Suppose that	${}^-485 + y = 0.$	
Now,	${}^-485 + -{}^-485 = 0.$	[po]
So,	${}^-485 \div y = {}^-485 + -{}^-485.$	
Therefore,	$y = -{}^-485.$	[left cancellation prin.]
Hence,	if ${}^-485 + y = 0$ then $y = -{}^-485.$	

Could you write a test-pattern for the generalization:

$$\forall_y \text{ if } 9832 + y = 0 \text{ then } y = -9832?$$

Do so right here.

Now, consider the generalization:

$$(*) \qquad \forall_x \forall_y \text{ if } x + y = 0 \text{ then } y = -x.$$

We leave to you the job of writing a proof of (*).

Do you see how generalization (*) provides a way of showing that a second number is the opposite of a first number? Suppose a first boy picks a number, and a second boy picks a number. How can the boys use (*) to find out if the second number is the opposite of the first? Just find out if the first number plus the second number is 0. If it is 0 then the second number is the opposite of the first.

In Unit 1 you used this idea when you wanted to show, for example, that the opposite of $({}^{+}4 + {}^{-}7)$ is ${}^{-}4 + {}^{+}7$. What we used there was the theorem:

$$\forall_x \forall_y \text{ if } x + y = 0 \text{ then } -x = y.$$

We shall call this <u>the 0-sum theorem</u>. [Do you see that it is equivalent to (*)?] According to the 0-sum theorem, in order to show that

$$-\underbrace{({}^{+}4 + {}^{-}7)}_{x} = \underbrace{{}^{-}4 + {}^{+}7}_{y},$$

it is sufficient to show that

$$\underbrace{({}^{+}4 + {}^{-}7)}_{x} + \underbrace{({}^{-}4 + {}^{+}7)}_{y} = 0.$$

Let's do so.

$$\begin{aligned} & ({}^{+}4 + {}^{-}7) + ({}^{-}4 + {}^{+}7) && \\ &= ({}^{+}4 + {}^{-}4) + ({}^{+}7 + {}^{-}7) && \text{[Why?]} \\ &= \quad 0 \quad + \quad 0 && \text{[Why?]} \\ &= \quad 0. && \text{[Why?]} \end{aligned}$$

So, since $({}^{+}4 + {}^{-}7) + ({}^{-}4 + {}^{+}7) = 0$, it follows from (*) that

$$-({}^{+}4 + {}^{-}7) = {}^{-}4 + {}^{+}7.$$

EXERCISES

A. Let's use the 0-sum theorem to prove some other theorems about opposites.

Sample. The opposite of the opposite of a number is that number.

Solution. $\forall_a \; - -a = a.$

[The 0-sum theorem tells us that to prove this, we should prove that, for each a,

$$\underbrace{-a}_{\substack{\uparrow \\ x}} + \underbrace{a}_{\substack{\uparrow \\ y}} = 0.$$

Then, the 0-sum theorem would enable us to conclude that

$$- \underbrace{-a}_{\substack{\uparrow \\ x}} = \underbrace{a}_{\substack{\uparrow \\ y}}.$$

Let's write a test-pattern .]

$$\begin{aligned} &-a + a \\ &= a + -a \quad \text{cpa} \\ &= 0. \quad \text{po} \end{aligned}$$

Hence, $-a + a = 0.$

So, $- -a = a.$ [0-sum theorem]

This shows that the generalization '$\forall_a \; - -a = a$' is a consequence of the commutative principle for addition, the principle of opposites, and the 0-sum theorem.

1. $\forall_a \forall_b \; -(a + b) = -a + -b.$ ["Distributive theorem for opposition over addition"]

2. $\forall_c \forall_d \; -(c + -d) = d + -c.$

3. $\forall_p \forall_q \; -(pq) = p \cdot -q.$

4. $\forall_x \forall_y \; -(xy) = -xy.$ [Note that '$-xy$' is an abbreviation for '$(-x)y$' and for '$-x \cdot y$'.]

5. $\forall_x \forall_y$ if $x = -y$ then $-x = y.$

B. Prove these theorems.

Sample 1. $\forall_m \; - - -m = -m.$

Solution.
$$\left.\begin{aligned} & - - -m \\ & = -m. \end{aligned}\right\} \forall_a \; - -a = a.$$

Sample 2. $\forall_u \forall_v \; -u \cdot -v = uv.$

Solution.
$$\begin{aligned} & -u \cdot -v && \} \; \forall_p \forall_q \; -(pq) = p \cdot -q. \\ & = -(-u \cdot v) && \} \; \forall_x \forall_y \; -(xy) = -xy. \\ & = -[-(uv)] && \} \; \forall_a \; - -a = a \\ & = uv. \end{aligned}$$

1. $\forall_x \forall_y \; -xy = x(-y).$

2. $\forall_x \forall_y \forall_z \; -x(y + z) = -(xy) + -(xz).$

3. $\forall_x \forall_y \forall_z \; -x(-y + -z) = xy + xz.$

4. $\forall_x \; x \cdot -1 = -x.$

C. Prove the "-1 times theorem":

$$\forall_x \; -x = -1 \cdot x.$$

Then, use it to prove as many of the theorems in Parts A and B as you can. For example, here is a proof of the theorem of Exercise 1 of Part A.

$$\begin{aligned} & -(a + b) && \} \; -1 \text{ times theorem} \\ & = -1 \cdot (a + b) && \} \; \ell\text{dpma} \\ & = -1 \cdot a + -1 \cdot b && \} \; -1 \text{ times theorem} \\ & = -a + -b. \end{aligned}$$

SUBTRACTION

In Unit 1 you learned that the inverse of adding a real number is adding the opposite of that number. We can express this idea by the generalization:

$$\forall_x \forall_y \ (x + y) + -y = x.$$

Write a test-pattern for this theorem.

The principle for subtraction:

$$\forall_x \forall_y \ x - y = x + -y$$

tells us that adding the opposite of a real number is the same operation as subtracting the number. So, we see that the inverse of adding a real number is subtracting the real number. That is, that

$$\forall_x \forall_y \ (x + y) - y = x.$$

Prove this last theorem by writing a test-pattern.

EXERCISES

A. Each of the following exercises contains several sentences-to-be-completed. After you complete the sentences in an exercise, state the theorem which is illustrated by the completed sentences and be prepared to prove it.

Sample.

(a) $7 - 4 \cdot 5 = 7 + \_\_\_\_\_ \cdot 5$

(b) ${}^{-}9 - 8 \cdot 13 = {}^{-}9 + \_\_\_\_\_ \cdot 13$

(c) $5 - {}^{-}2 \cdot 5 = 5 + \_\_\_\_\_ \cdot 5$

(d) $-8 - -3 \cdot 7 = -8 + \_\_\_\_\_ \cdot 7$

Theorem: ____

(continued on next page)

Solution.

(a) $7 - 4 \cdot 5 = 7 + \underline{-4} \cdot 5$

(b) $^-9 - 8 \cdot 13 = {}^-9 + \underline{-8} \cdot 13$

(c) $5 - {}^-2 \cdot 5 = 5 + \underline{-{}^-2} \cdot 5$

(d) $^-8 - -3 \cdot 7 = {}^-8 + \underline{--3} \cdot 7$

Theorem: $\forall_x \forall_y \forall_z \; x - yz = x + -yz$

[The sentences illustrate the theorem that subtracting the product of a first number by a second number is the same as adding the product of the opposite of the first number by the second number.]

Here is a proof of this theorem.

$$\begin{aligned} & x - yz \\ &= x + -(yz) \quad \Big\} \text{ ps} \\ &= x + -yz. \quad \Big\} \; \forall_x \forall_y \; -(xy) = -xy. \end{aligned}$$

1. (a) $^+9 - {}^-5 +$ _____ $= {}^+9$

   (b) $^-3 - {}^+7 +$ _____ $= {}^-3$

   (c) $^-8 - {}^+2 + {}^+2 =$ _____

   (d) _____ $- {}^-7 + {}^-7 = {}^-43$

   Theorem: ____________________

2. (a) $-(70 -$ _____ $) = 90 - 70$

   (b) $-(81 -$ _____ $) = 35 - 81$

   (c) $-(10 -$ _____ $) = -3 - 10$

   (d) $-($_____ $- {}^-19) = {}^-19 - {}^-3$

   Theorem: ____________________

3. (a) $30 + (90 - 60) = 30 + 90 -$ _____

   (b) $72 + (53 - 84) = 72 + 53 -$ _____

   (c) $^{-}9 + (^{-}3 - {}^{+}7) = {}^{-}9 +$ _____ $- {}^{+}7$

   (d) $15 + (29 - {}^{-}31) = 15 + 29 -$ _____

   Theorem: ____

4. (a) $10 - (5 +$ _____ $) = 10 - 5 - 2$

   (b) $7 - ($ _____ $+ 4) = 7 - 9 - 4$

   (c) $^{-}5 - ($ _____ $+ {}^{-}3) = {}^{-}5 - {}^{-}6 - {}^{-}3$

   (d) $5 - (5 +$ _____ $) = 5 - 5 - 5$

   Theorem: ____

5. (a) $9 - (3 - 5) = 9 - 3 +$ _____

   (b) $7 - (10 - 7) = 7 - 10 +$ _____

   (c) $11 - ($ _____ $- 5) = 11 - 12 + 5$

   (d) $^{-}3 - (^{-}9 - {}^{-}5) = {}^{-}3 - {}^{-}9 +$ _____

   Theorem: ____

6. (a) $5 + (12 - 9) = 5 -$ _____ $+ 12$

   (b) $61 + (37 - 58) = 61 -$ _____ $+ 37$

   (c) $^{-}2 + (^{-}7 - 5) = {}^{-}2 - 5 +$ _____

   (d) $3 + (^{-}4 - {}^{-}6) =$ _____ $- {}^{-}6 + {}^{-}4$

   Theorem: ____

(continued on next page)

7. (a) $5(6 - 3) = 5 \cdot 6 - 5 \cdot \underline{\qquad}$

(b) ${}^{-}5(7 - 18) = -5 \cdot 7 - {}^{-}5 \cdot \underline{\qquad}$

(c) $4 \cdot 3 - 4 \cdot 17 = 4(3 - \underline{\qquad})$

(d) $6(9 - {}^{-}2) = 6 \cdot 9 - \underline{\qquad} \cdot {}^{-}2$

Theorem: ______________________________

8. (a) $(9 - 3)8 = 9 \cdot 8 - 3 \cdot \underline{\qquad}$

(b) $(4 - 14)51 = 4 \cdot 51 - \underline{\qquad} \cdot 51$

(c) $6 \cdot 19 - 8 \cdot 19 = (6 - \underline{\qquad})19$

(d) $-3 \cdot 48 - 3 \cdot 48 = (\underline{\qquad} - 3)48$

Theorem: ______________________________

9. (a) $5 - (-90 - \underline{\qquad}) = 5 + 90 + 60$

(b) $7 - (\underline{\qquad} - 13) = 7 + 81 + 13$

(c) $6 - (\underline{\qquad} - {}^{-}1) = 6 + {}^{-}3 + {}^{-}1$

(d) $4 - (-15 - {}^{-}9) = 4 + 15 + \underline{\qquad}$

Theorem: ______________________________

10. (a) $15 - (17 - 13 - 19) = 15 - 17 + \underline{\qquad} + 19$

(b) $21 - (-8 - 10 - -3) = 21 - \underline{\qquad} + 10 + -3$

(c) $-5 - (-7 - 18 - 25) = \underline{\qquad} - -7 + 18 + 25$

(d) $60 - (60 - 30 - 30) = 60 - 60 + 30 + \underline{\qquad}$

Theorem: ______________________________

B. State the theorems referred to in the following exercises, and be prepared to prove them.

1. If I subtract a number from 0, I get the opposite of the number.

   Theorem: ______________________________

2. If I subtract 0 from a number, I get the number.

   Theorem: ______________________________

3. Rita and Rhoda each picks a real number. Aaron picks a number and tells each girl to add his number to hers. Then Rhoda subtracts her sum from Rita's sum, and finds that she gets the same answer as she would have if she had subtracted her original number from Rita's original number.

   Theorem: ______________________________

4. Just like Exercise 3 except that the girls subtract Aaron's number from each of theirs.

   Theorem: ______________________________

5. Albert and Beulah and Charles and Dora each picks a real number. Beulah subtracts hers from Albert's and Dora subtracts hers from Charles'. They add the differences. Then, Charles adds his number to Albert's and Dora adds her number to Beulah's. The girls subtract their sum from the boys' sum, and get the same result as when they added the differences.

   Theorem: ______________________________

C. Complete each of the following from the choices given to make a true generalization. [There is only one correct choice.]

1. $\forall_x \forall_y \forall_z \; x - (-y - z) =$ ________.

(a) $x - y - z$ (b) $x + y - z$ (c) $x - y + z$ (d) $x + y + z$

2. $\forall_x \forall_y \forall_z \; -(x - y - z) =$ ________.

(a) $x + y + z$ (b) $-x + y - z$ (c) $-x + y + z$ (d) $x - y + z$

3. $\forall_x \forall_y \forall_z \; x - yz =$ ________.

(a) $x - y - z$ (b) $x + y - z$ (c) $x + -yz$ (d) $x + -y \cdot -z$

4. $\forall_a \forall_b \forall_c \forall_d \; -(a - b)(c - d) =$ ________.

(a) $(a + b)(c + d)$ (b) $(b - a)(d - c)$

(c) $(b - a)(c + d)$ (d) $(b - a)(c - d)$

5. $\forall_a \forall_b \forall_c \forall_d \; -(a - b) - (c - d) =$ ________.

(a) $-a - b - c - d$ (b) $b - a + d - c$

(c) $-a + b - c - d$ (d) $b + a + d - c$

6. $\forall_u \forall_v \forall_x \forall_y \; -u(x + y) - v(x - y) =$ ________.

(a) $-ux + uy - vx - vy$ (b) $-ux - uy - vx + vy$

(c) $-ux - uy + vx - vy$ (d) $-ux + -uy + -vx + -vy$

7. $\forall_x \forall_y \; 8x - 2y - 5x + 7y =$ ________.

(a) $6xy - 2xy$ (b) $3x - 9y$ (c) $8x$ (d) $3x + 5y$

8. $\forall_a \forall_b \forall_c \; 3a - 2(a - b + c) =$ ________.

(a) $a - b + c$ (b) $a + 2b - 2c$

(c) $3a - 2 - a - b + c$ (d) $3a - 2 - a + b - c$

9. $\forall_x \forall_y \forall_z \; 3x - x - x - 2y - y - z =$ ________.

(a) $x - 3y - z$ (b) $3x - 2 - z$

(c) $-5x - 3y - z$ (d) $3 - x - 2 - z$

10. $\forall_a \forall_b \; -(3a - 5b)(4a - 2b)(5a - 7b) =$ ________.

(a) $(5b - 3a)(2b - 4a)(7b - 5a)$ (b) $(3a - 5b)(2b - 4a)(7b - 5a)$

(c) $-(5b - 3a)(2b - 4a)(7b - 5a)$ (d) $(5b + 3a)(4a + 2b)(5a + 7b)$

D. Simplify.

Sample 1. $3x - 2y - 7x$

Solution. $3x - 2y - 7x$

$= 3x + -2y + -7x$

$= 3x + -7x + -2y$

$= 3x - 7x - 2y$

$= (3 - 7)x - 2y$

$= -4x - 2y.$

[Read '$-4x - 2y$' as 'the opposite of 4, times x, minus, 2 times y'.]

[Note: In doing simplification exercises you do not need to write all the steps. As in earlier simplification exercises, you should look for short cuts, and be able to justify them by referring to principles or to theorems which you can prove from the principles.]

1. $5a + 2b - 8a$
2. $3x - 7y + 5x - y$
3. $9k - 3r - 6k + r$
4. $7p - 2q - 6p - p + 7$
5. $8x - y - x + 9y$
6. $5a - a - a - 7a$
7. $2s + 5t - 3s + t$
8. $5x + 3y + -x + -2y$
9. $x + 3y - -x - -2y$
10. $8m - 3n - 6n + -5m$
11. $2xx + 3x - 7xx$
12. $5aa + 6b - 3a + 7b$
13. $7a - 3b - 6c + 5a$
14. $-x - 3y + 4x - 7y$
15. $-2k + 3r - 7rk$
16. $9t - 3st + 7t - 6s$
17. $\pi x + 7x - 9y$
18. $3.4a - 2.7b + 6.3a$
19. $(a + b)x + cx$
20. $a + 2c(a + 2c) + 2ac$
21. $b + 2d + (b + 2)d$
22. $(x - y)u + (x - y)v$

[More exercises are in Part I, Supplementary Exercises.]

Sample 2. $5(3x - 4y) + 6x$

Solution. $5(3x - 4y) + 6x$

$= 5(3x) - 5(4y) + 6x$

$= (5 \cdot 3)x - (5 \cdot 4)y + 6x$

$= 15x - 20y + 6x$

$= 15x + {}^-20y + 6x$

$= 15x + 6x - 20y$

$= 21x - 20y.$

23. $9(2a - 5b) + 6a$

24. $7(2m - 3n) - 9n$

25. $8(x - 3y) + 2x - 5y$

26. $3(-2x - 7y) - y - z$

27. $2(4a - 5b) + 3(7a - 3b) + 5(2b - a)$

28. $9(2M - 3N) + 11(N - M) + 7(3 - 2M + 5N)$

29. $-6(5x + 3y) + 4x - 2y$

30. $-3(2a + 13b) - 7a + b$

31. $-4a(a - 3b) + 3b(a + b)$

32. $-7x(-5x - 4y) + xx + 3yx$

[More exercises are in Part I, Supplementary Exercises.]

Sample 3. $3r + 5s - (2r - 7s)$

Solution. $3r + 5s - (2r - 7s)$

$= 3r + 5s - 2r + 7s$

$= r + 12s.$

33. $5x + 2y - (x - y)$

34. $7a + 5b - (-a - b)$

35. $3m - 2n - (4m - 3n)$

36. $8k - 3j - (7k - 2j)$

37. $11x - (3 - 2x + 5y)$

38. $15r - (5r - 2s - 7) + 3s$

39. $a - (3b + 7a) - 2a$

40. $x - (6x - 3y) - (9y - 2x)$

[More exercises are in Part I, Supplementary Exercises.]

Sample 4. $5x - 6(x - 3y)$

Solution. $5x - 6(x - 3y)$

$$\begin{aligned}
&= 5x + -6(x + -3y) \\
&= 5x + [-6x + -6 \cdot -3y] \\
&= 5x + [-6x + (-6 \cdot -3)y] \\
&= 5x + [-6x + (6 \cdot 3)y] \\
&= 5x + [-6x + 18y] \\
&= 5x + -6x + 18y \\
&= 5x - 6x + 18y \\
&= (5 - 6)x + 18y \\
&= -1x + 18y \\
&= -x + 18y.
\end{aligned}$$

41. $3y - 7(2x - 5y)$

42. $8m - 6(-3n - 5m)$

43. $7k - 2(5k - 3j)$

44. $18r - 9(5s - 2r)$

45. $4x - 3y - 8(-3x - 2y)$

46. $5(2x + 7y) - 10(-x - y)$

47. $9x - (x - y) - 3(5x - 11y) - 7(-2x - 4y) - 7x$

48. $-4x - z(x - 2y) - 5(-x - 3 - 4y) - z(1 - 2x - 3y)$

49. $6A - 3(A - B + 1) - 7(2A + 3 - 5B) - 6(2B - 3A)$

50. $-z(11x - 2y) - 3z(5x - 4y)$

51. $-2a(6a - 2b) - 5b(a - 3b)$

52. $-x - 3y - 4(-x + 2y) - y + x$

53. $5x + 4y - 6(x - 3y - 1)$

54. $-5(2 - 3a + 5b) - 7(1 - a + 2b) - (-a - b - 3)$

55. $-6(5 - x - 3y) - 3[5 - 3x - 4y - (2x - 3y)]$

56. $-2[3(a - b) - 7(2a - 3b)] - 5[-6(-a - 3b) - (-a - b)]$

[More exercises are in Part I, Supplementary Exercises.]

<u>Sample</u> <u>5</u>. $3a(-2b)(-4c)(-d)$

<u>Solution</u>. $3a(-2b)(-4c)(-d)$

$= -[3a(2b)(4c)d]$

$= -[(3 \cdot 2 \cdot 4)\ abcd]$

$= -24abcd.$

<u>Sample</u> <u>6</u>. $-5x(-2y)(-3z)(-2x)$

<u>Solution</u>. $-5x(-2y)(-3z)(-2x)$

$= (5 \cdot 2 \cdot 3 \cdot 2)xxyz$

$= 60xxyz.$

57. $7x(2y)(-5z)$

58. $6a(-2b)(-5c)$

59. $-2m(-3p)(11q)$

60. $2A \cdot 7B \cdot -3B$

61. $-3X \cdot -2Y \cdot -4Z$

62. $-X \cdot -Y \cdot -Z$

63. $3ab \cdot 2ab \cdot 5ab$

64. $-2x \cdot -3y \cdot -xy \cdot -7$

65. $-3q \cdot -p \cdot -rp$

66. $-2mk \cdot -3km \cdot -m \cdot -k$

67. $3ab(-2a + 7b) - 7aab + 5abb - acb + bba$

68. $-2x(3 - 2y + 7x) - 5x(4x - 3y - 2) - 2xx + 3xy$

69. $-3a(5a - 2b) - 6b(-7a - 3b) - 3b \cdot 2a - 6a \cdot -b$

70. $4x[3 - 2(3x - 7y)] - 5y[-4 - 3(7 - 2x - 5y)]$

71. $1.8(-2.5y) + 6(3x + 2y) - 3(3x + 2y)$

72. $-5g(3g - 2h) + \frac{1}{2}g(2g + 4y) - 5g \cdot h + 10 \cdot -g$

73. $-\triangle(3\triangle - 4\square) - 10\triangle(-\triangle - \square) - 3\triangle(4\square) + 3\triangle(-6\triangle)$

74. $-\frac{3}{5}s(4t)(-1\frac{2}{3}r) + 3s(-2r)(5t) - 7st(r - 5t)$

75. $6p(-3q) - 2(3p - 5q) + 2(p - q) - 7q(-p)$

76. $5cd(9c - 4d) - 4c(9d) + 5d \cdot 12c + 6cd(3d - 4c)$

[More exercises are in <u>Part</u> <u>I</u>, <u>Supplementary</u> <u>Exercises</u>.]

2.07 Division.--In Unit 1 you learned that dividing by a number is the inverse of multiplying by that number. For example,

$$({}^{+}3 \times {}^{-}4) \div {}^{-}4 = {}^{+}3$$

$$\text{and} \quad ({}^{-}12 \times {}^{+}5) \div {}^{+}5 = {}^{-}12.$$

If you wish to do a division problem, such as:

$${}^{+}16 \div {}^{-}2 = ?,$$

you can do it by solving the problem:

$$? \times {}^{-}2 = {}^{+}16.$$

You also learned that there are several numerals which name the quotient of ${}^{+}16$ by ${}^{-}2$. Probably the simplest looking is '${}^{-}8$'. Others are '${}^{+}16 \div {}^{-}2$' itself, and the fractions '$\frac{{}^{+}16}{{}^{-}2}$' and '${}^{+}16/{}^{-}2$'.

DOES MULTIPLYING BY 0 HAVE AN INVERSE?

Since division is the inverse of multiplication, and subtraction is the inverse of addition, division and subtraction have much in common. But, they differ in one important respect.

It is always possible to subtract a second real number from a first real number, but you can divide a first real number by a second real number only if the second is not 0.

Let's look into this problem of "dividing by 0". Do you recall from Unit 1 the description of an operation and an inverse operation as sets of pairs of numbers? We can construct a list of pairs which belong, for example, to dividing by 2, by first constructing a list of pairs which belong to multiplying by 2. You get a pair which belongs to dividing by 2 by interchanging the components of a pair which belongs to multiplying by 2.

multiplying by 2

(5, 10)	(17, 34)	(-2, -4)
(0, 0)	(9, 18)	(1, 2)
(-1, -2)	$(\frac{3}{5}, \frac{6}{5})$	$(\frac{3}{2}, 3)$
(10, 20)	(18, 36)	(2.5, 5)
(4.5, 9)	$(\frac{1}{2}, 1)$	(-4, -8)
	. . .	

dividing by 2

(10, 5)	(34, 17)	(-4, -2)
(0, 0)	(18, 9)	(2, 1)
(-2, -1)	$(\frac{6}{5}, \frac{3}{5})$	$(3, \frac{3}{2})$
(20, 10)	(36, 18)	(5, 2.5)
(9, 4.5)	$(1, \frac{1}{2})$	(-8, -4)
	. . .	

To do a problem which involves dividing by 2, for example:

$$18 \div 2 = ?$$

we look for a pair which belongs to dividing by 2 and which has first number 18. This is the pair (18, 9). So, the solution of the problem '18 ÷ 2 = ?' is 9.

Now, let's get a list of pairs which belong to "dividing by 0". This set of pairs ought to be the inverse of multiplying by 0.

multiplying by 0

(4, 0)	(-3, 0)	(0, 0)	$(\frac{3}{5}, 0)$
(-2, 0)	(10, 0)	(17, 0)	$(\pi, 0)$
(93, 0)	. . .	(378, 0)	

So, some of the pairs in "dividing by 0" would be

"dividing by 0"

(0, 4)	(0, -3)	(0, 0)	$(0, \frac{3}{5})$
(0, -2)	(0, 10)	(0, 17)	$(0, \pi)$
(0, 93)	. . .	(0, 378)	

Now, let's try to use this list to solve a "division by 0" problem:

$$93 \div 0 = ?.$$

We want to find a pair belonging to "division by 0" whose first number is 93. A glance at the list shows us no such pair. Perhaps we need a bigger list! Do you think a bigger list would help? The multiplication principle of 0 tells us that there can be <u>no</u> pair in the multiplying by 0 list with second number 93. So, there can be no pair in the "dividing by 0" list with first number 93. This means that there is no solution to the problem:

$$93 \div 0 = ?.$$

So, this means that marks such as '$93 \div 0$' and '$\frac{93}{0}$' and '93/0' are not numerals. <u>There is no number whose name is '$93 \div 0$' or '$\frac{93}{0}$', or '93/0'</u>.

What about the problem:

$$0 \div 0 = ?.$$

Once again, we go to the list for "dividing by 0" to seek a pair, this time one with first number 0. The principle for multiplying by 0 tells us that every pair in the "dividing by 0" list has 0 for first number [Explain.]. So, if '$0 \div 0$' <u>were</u> a numeral, it would have to be a numeral for each number! If this were the case, we would have, for example,

$$0 \div 0 = 17 \text{ and } 0 \div 0 = 10.$$

And, this would mean that

$$17 = 10,$$

which is certainly <u>not</u> the case. The only way out of this unpleasant situation is to decide, as in the case of '$93 \div 0$', that '$0 \div 0$' is not a numeral.

We conclude, then, that "dividing by 0" is not an operation; multiplying by 0 <u>does</u> <u>not</u> have an inverse. A short way of saying this is:

YOU CAN'T DIVIDE BY ZERO!

[NOBODY CAN!]

EXERCISES

<u>A</u>. You have seen that you can't divide <u>by</u> 0. Here are some exercises which ask you <u>to</u> <u>divide</u> <u>0</u> <u>by</u> numbers other than 0. Fill in the blanks to make true sentences.

1. $0 \div 8 =$ ________ because ________ $\cdot 8 = 0$.
2. $0 \div -3 =$ ________ because ________ $\cdot -3 = 0$.
3. $\frac{0}{5} =$ ________ because ________ $\cdot 5 = 0$.
4. For each $x \neq 0$, $\frac{0}{x} =$ ______ because, for each x, ______ $\cdot x = 0$.

<u>B</u>. Fill in the blanks to make true sentences.

1. For each x, $x \div 1 =$ ________ because, for each x, ______ $\cdot 1 = x$.
2. For each x, $x \div -1 =$ ________ because, for each x, ______ $\cdot -1 = x$.
3. For each $x \neq 0$, $x/x =$ ________ because, for each x, ______ $\cdot x = x$.

[Give names to the first generalization in each of Exercises 1 and 2].

DOES MULTIPLYING BY A NONZERO NUMBER HAVE AN INVERSE?

Tell how many solutions there are for the problem:

$$(1) \quad ? \cdot 0 = 93.$$

Tell how many solutions there are for the problem:

$$(2) \quad ? \cdot 0 = 0.$$

We have seen that the first problem has no solutions, and that the second problem has many. It was these answers which showed us that multiplying by 0 does not have an inverse.

Now, tell how many solutions there are for the problem:

$$(3) \quad ? \cdot {}^-2 = {}^+16.$$

Clearly, $^-8$ is one solution. Is there another solution? How about these problems:

$$(4) \quad ? \cdot {}^-2 = {}^+12 \qquad (5) \quad ? \cdot {}^-2 = {}^-41 \qquad (6) \quad ? \cdot {}^-2 = 0.$$

How many solutions does each of these have?

When we say that multiplying by $^-2$ has an inverse operation, we mean precisely that for each first number, you can find just one number whose product by $^-2$ is that first number. That is, we mean that each problem like (3), (4), (5), and (6) has exactly one solution. In other words,

for each x, there is just one z such that $z \cdot {}^-2 = x$.

What we have said about multiplying by $^-2$ could be said about multiplying by any nonzero number. So, multiplying by each nonzero real number has an inverse just if

$$(*) \quad \forall_x \forall_{y \neq 0} \text{ there is just one } z \text{ such that } z \cdot y = x.$$

Since we believe it to be the case that multiplication by each nonzero real number does have an inverse, we want (*) to be a theorem. One way to make sure that it is a theorem would be to take (*) itself as one of our basic principles. But, as you will see, it is sufficient to take the principle of quotients as a basic principle. Using the principle of quotients and the other basic principles we can derive (*).

QUOTIENTS

<u>The principle of quotients</u>:

$$\forall_x \forall_{y \neq 0}\ (x \div y) \cdot y = x$$

tells us, for example, that, since $5 \neq 0$, $6 \div 5$ is a real number whose product by 5 is 6. In fact, it tells us that, for each x, and for each nonzero y,

there is at least one z [the quotient of x by y] such that $z \cdot y = x$.

Hence, in order to establish (*), it is sufficient to show that there is no number other than $x \div y$ whose product by y is x--that is, it is sufficient to prove:

$$\forall_x \forall_{y \neq 0} \forall_z \text{ if } z \cdot y = x \text{ then } z = x \div y.$$

We shall call this generalization <u>the division theorem</u>, and you will prove it later in an exercise.

Since '$\frac{x}{y}$' is an abbreviation for '$(x \div y)$', the principle of quotients can be written:

$$\forall_x \forall_{y \neq 0}\ \frac{x}{y} \cdot y = x.$$

The principle of quotients tells you, for example, that

$$(6 \div 2) \cdot 2 = 6, \quad \frac{36}{4} \cdot 4 = 36, \quad \frac{1}{9} \times 9 = 1, \quad \frac{3 \cdot 7}{5 \cdot 7}(5 \cdot 7) = 3 \cdot 7.$$

Similarly, the division theorem can be written:

$$\forall_x \forall_{y \neq 0} \forall_z \text{ if } z \cdot y = x \text{ then } z = \frac{x}{y}.$$

It tells us, for example, that

$$\text{if } 3 \cdot 2 = 6 \text{ then } 3 = \frac{6}{2}.$$

So, since $3 \cdot 2 = 6$, we know from the division theorem that $3 = \frac{6}{2}$;

$$\frac{6}{2} = 3 \ \underline{\text{because}} \ 3 \cdot 2 = 6.$$

Similarly,

$$\frac{36}{4} = 9 \ \underline{\text{because}} \ 9 \cdot 4 = 36,$$

$$(5 \times 6) \div 6 = 5 \ \underline{\text{because}} \ 5 \cdot 6 = 5 \times 6,$$

$$\frac{3 \cdot 7}{5 \cdot 7} = \frac{3}{5} \ \underline{\text{because}} \ \frac{3}{5}(5 \cdot 7) = (\frac{3}{5} \cdot 5)7 = 3 \cdot 7.$$

[In the last example we also used the principle of quotients [Where?]. What other principle did we use?]

Another use of the division theorem is to prove such generalizations as:

$$\forall_x \frac{5x - 5}{5} = x - 1.$$

Here is a test-pattern.

$$\begin{aligned} (x - 1)5 &= x5 - 1 \cdot 5 && \text{[dpms]} \\ &= 5x - 5 \cdot 1 && \text{[cpm]} \\ &= 5x - 5. && \text{[pm1]} \end{aligned}$$

Hence, $\quad x - 1 = \dfrac{5x - 5}{5}. \quad$ [division theorem]

Notice that while the division sign '÷', like '+', '×', and the subtraction sign '-' has associated with it a pair of grouping symbols [which, according to the conventions you learned in Unit 1, can often be omitted], the fraction-bar '——' does not require grouping symbols. For example, although

'3 × (5 ÷ 6)' <u>may</u> <u>not</u> be abbreviated to '3 × 5 ÷ 6'

[since the latter is an abbreviation for '(3 × 5) ÷ 6'],

'3 × (5 ÷ 6)' <u>may</u> be abbreviated to '$3 \times \frac{5}{6}$'.

Furthermore, the use of the fraction-bar instead of '÷' often permits one to omit other grouping symbols. For example,

'3 × [(5 + 4) ÷ (6 - 3)]' may be abbreviated to '$3 \times \frac{5 + 4}{6 - 3}$',

thus getting rid of two pairs of parentheses, as well as the pair of brackets. In general, the outermost grouping symbols in both the numerator and the denominator of a fraction may be omitted.

EXERCISES

<u>A</u>. Abbreviate each of the following expressions by using fraction-bars instead of division signs and as few grouping symbols as necessary.

1. (7 ÷ 3) + (5 ÷ 3)
2. 6 × (4 ÷ 3)
3. (6 × 4) ÷ 3
4. 9 × 5 ÷ (2 × 4)
5. 9 × 5 ÷ 2 × 4
6. 6 ÷ 2 × (9 ÷ 3)
7. 6 ÷ 2 × 9 ÷ 3
8. 6 ÷ (2 × 9) ÷ 3
9. 10 ÷ 5 ÷ (4 ÷ 3)

B. You can use the division theorem [and the basic principles] to simplify expressions containing fractions. Simplify each of the following, and show how the division theorem justifies the simplification.

Sample 1. $\frac{2}{3} + \frac{5}{7}$

Solution. The division theorem tells us that, for each x, and for each nonzero y,

$$\text{if } (\frac{2}{3} + \frac{5}{7})y = x \quad \text{then } \frac{2}{3} + \frac{5}{7} = \frac{x}{y}.$$

So, to simplify '$\frac{2}{3} + \frac{5}{7}$' requires finding simple numerals for a number x and a nonzero number y such that $(\frac{2}{3} + \frac{5}{7})y = x$. Now, for each y, $(\frac{2}{3} + \frac{5}{7})y = \frac{2}{3}y + \frac{5}{7}y$. If we use the value $3 \cdot 7$ for 'y', the corresponding value of '$\frac{2}{3}y + \frac{5}{7}y$' is $\frac{2}{3}(3 \cdot 7) + \frac{5}{7}(3 \cdot 7)$. The principle of quotients together with the associative and commutative principles for multiplication tell us that this value is $2 \cdot 7 + 5 \cdot 3$. So, we know that

$$(\frac{2}{3} + \frac{5}{7})(3 \cdot 7) = 2 \cdot 7 + 5 \cdot 3,$$

and the division theorem tells us that $\frac{2}{3} + \frac{5}{7} = \frac{2 \cdot 7 + 5 \cdot 3}{3 \cdot 7}$. So, '$\frac{2}{3} + \frac{5}{7}$' simplifies to '$\frac{29}{21}$'.

[Note: Do you see that this justifies, in this instance, the rule for "adding fractions by finding a common denominator"?]

Sample 2. $\frac{a}{10} - \frac{b}{5}$

Solution. $(\frac{a}{10} - \frac{b}{5})10 = \frac{a}{10} \cdot 10 - \frac{b}{5} \cdot 10 = a - 2b.$

So, by the division theorem, '$\frac{a}{10} - \frac{b}{5}$' simplifies to '$\frac{a - 2b}{10}$'.

[We say that '$\frac{a - 2b}{10}$' is simpler because it is a single fraction, whereas '$\frac{a}{10} - \frac{b}{5}$' contains two fractions.]

1. $\frac{7}{8} + \frac{2}{3}$
2. $\frac{3}{5} - \frac{2}{15}$
3. $\frac{8}{3} + 0.7$
4. $\frac{x}{3} + \frac{x}{5}$
5. $\frac{y}{6} - \frac{y}{2}$
6. $\frac{3x}{2} - \frac{5y}{7}$

<u>Sample 3.</u> $\frac{3}{x} + \frac{5}{y}$

<u>Solution.</u> $(\frac{3}{x} + \frac{5}{y})(xy)$

$$\left.\begin{aligned} &= \frac{3}{x}(xy) + \frac{5}{y}(xy) \\ &= 3y + 5x. \end{aligned}\right\} \text{[providing that neither 'x' nor 'y' has the value 0]}$$

So, by the division theorem, $\forall_{x \neq 0} \forall_{y \neq 0}\ \frac{3}{x} + \frac{5}{y} = \frac{3y + 5x}{xy}$.

We write our answer as: $\frac{3y + 5x}{xy}$, $[x \neq 0,\ y \neq 0]$.

7. $\frac{5}{a} + \frac{2}{b}$
8. $\frac{3}{2x} - \frac{5}{7y}$
9. $\frac{8}{3k} + \frac{7}{4m}$
10. $\frac{3}{7x} + \frac{2}{5y}$
11. $\frac{2}{7x} + \frac{3}{5y}$
12. $\frac{a}{b} + \frac{c}{d}$

PROVING THE DIVISION THEOREM

You have seen how the division theorem can be used in simplifying expressions. It is now time to show that the division theorem is actually a theorem. That is, to derive it from the principle of quotients and the other basic principles. As preparation for doing so, let's consider an analogous situation concerning subtraction.

For subtraction, the analogue of the principle of quotients is:

(1) $\forall_x \forall_y\ (x - y) + y = x,$

and the analogue of the division theorem is:

(2) $\forall_x \forall_y \forall_z$ if $z + y = x$ then $z = x - y$.

Now, we can use (1) to derive (2) as follows.

Suppose that $z + y = x$.
Then $z + y = (x - y) + y$, [(1)]
and $z = x - y$. [cancellation principle]

Hence, if $z + y = x$ then $z = x - y$.

However, besides (1) we had to use the cancellation principle for addition. Now, it turns out that we can use (1) to derive the cancellation principle [without using either the principle of opposites or the principle for subtraction]. Turn the page to see how.

Suppose that $x + z = y + z$.

Then $x + z + (0 - z) = y + z + (0 - z)$,

$x + [z + (0 - z)] = y + [z + (0 - z)]$, [apa]

$x + [(0 - z) + z] = y + [(0 - z) + z]$, [cpa]

$x + 0 = y + 0$, [(1)]

and $x = y$. [pa0]

Hence, if $x + z = y + z$ then $x = y$.

So, we could use (1) as a basic principle in place of the principle of opposites and the principle for subtraction. Analogously, you can use the principle of quotients [together with our other basic principles] to prove the cancellation principle for multiplication:

$$\forall_x \forall_y \forall_{z \neq 0} \text{ if } xz = yz \text{ then } x = y.$$

Then, use this and the principle of quotients to prove the division theorem:

$$\forall_x \forall_{y \neq 0} \forall_z \text{ if } z \cdot y = x \text{ then } z = x \div y.$$

[The principle of quotients and the division theorem together tell us that multiplication by each nonzero real number has an inverse. That is, (*) on page 2-85 is a consequence of the principle of quotients and our other basic principles.]

EXERCISES

Prove each of the following theorems.

Sample. "Cancellation principle for multiplication"

$$\forall_x \forall_y \forall_{z \neq 0} \text{ if } xz = yz \text{ then } x = y.$$

Solution. Suppose that $xz = yz$.

Then $xz(1 \div z) = yz(1 \div z)$,

$x[z(1 \div z)] = y[z(1 \div z)]$, [apm]

$x[(1 \div z)z] = y[(1 \div z)z]$, [cpm]

$x \cdot 1 = y \cdot 1$, [pq; $z \neq 0$]

and $x = y$. [pm1]

Hence, [for $z \neq 0$,] if $xz = yz$ then $x = y$.

[Note that in applying the principle of quotients we had to

introduce the restriction '$z \neq 0$'. So, the test-pattern proves:
$\forall_x \forall_y \forall_{z \neq 0}$ if $xz = yz$ then $x = y$, not: $\forall_x \forall_y \forall_z$ if $xz = yz$ then $x = y$.]

1. $\forall_x \forall_{y \neq 0} \forall_z$ if $zy = x$ then $z = \frac{x}{y}$. ["Division theorem"]

2. $\forall_x\ \frac{x}{1} = x$. 3. $\forall_{x \neq 0}\ \frac{x}{x} = 1$. 4. $\forall_x\ \frac{x}{-1} = -x$.

5. $\forall_{x \neq 0}\ \frac{0}{x} = 0$. 6. $\forall_x \forall_{y \neq 0}$ if $\frac{x}{y} = 0$ then $x = 0$.

THE 0-PRODUCT THEOREM

It is easy to derive the generalization:

$$\forall_x \forall_y \text{ if } x = 0 \text{ then } xy = 0$$

from the commutative principle for multiplication and the principle for multiplying by 0 [Do so.]. So, this generalization is a theorem. Do you think that you could prove:

$$\forall_x \forall_y \text{ if } xy = 0 \text{ then } x = 0?$$

Since $1 \cdot 0 = 0$ but $1 \neq 0$, this second generalization is not a theorem. How can you "correct" this generalization to get a theorem?

The generalization:

(1) $\quad \forall_x \forall_{y \neq 0}$ if $xy = 0$ then $x = 0$

is a theorem. Here is a proof.

Suppose that $xy = 0$.

Then $xy = 0y$. [pm0; cpm]

So, $x = 0$. [cancellation principle for multiplication; $y \neq 0$]

Hence [for $y \neq 0$], if $xy = 0$ then $x = 0$.

Notice that (1) is logically equivalent to:

(1′) $\quad \forall_x \forall_{y \neq 0}$ if $x \neq 0$ then $xy \neq 0$.

And, (1′) is logically equivalent to:

(2) $\quad \forall_x \forall_y$ if $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.

Also, (2) is logically equivalent to:

(2′) $\quad \forall_x \forall_y$ if $xy = 0$ then $x = 0$ or $y = 0$.

We shall call (2′) <u>the 0-product theorem</u>. Sometimes when we refer to this theorem, you will want to think of the equivalent generalization (2).

SIMPLIFYING EXPRESSIONS CONTAINING FRACTIONS

In doing Exercise 12 on page 2-89 you actually proved the theorem:

$$\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{y} + \frac{u}{v} = \frac{xv + uy}{yv}.$$

This theorem justifies the rule you learned in an earlier grade for "adding fractions". So, the rule you learned follows from the basic principles. This is the case for all the rules you learned about fractions. You will use these same rules in simplifying pronumeral expressions such as:

$$\frac{15xy}{3x} + 4x(\frac{3y}{2x} + \frac{7y}{4x}) + \frac{6ab}{2xy} \times \frac{5xy}{4ab} \div \frac{3x}{2}.$$

But, before you do so, you will want to show that these rules also follow from the basic principles. This amounts to stating generalizations which justify the rules and showing that these sentences are theorems.

<u>Adding or subtracting fractions</u>

For this you have two rules based on the generalizations:

$$\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{y} + \frac{u}{v} = \frac{xv + uy}{yv},$$

and:

$$\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{y} - \frac{u}{v} = \frac{xv - uy}{yv}.$$

Here is a test-pattern for the first generalization.

$$\begin{aligned}
&(\frac{x}{y} + \frac{u}{v})(yv) && \\
&= \frac{x}{y}(yv) + \frac{u}{v}(yv) && \text{Why?} \\
&= \frac{x}{y}(yv) + \frac{u}{v}(vy) && \text{Why?} \\
&= \frac{x}{y}yv + \frac{u}{v}vy && \text{Why?} \\
&= xv + uy. && \text{pq; } [y \neq 0,\ v \neq 0]
\end{aligned}$$

So, $(\frac{x}{y} + \frac{u}{v})(yv) = xv + uy.$

Hence, $\frac{x}{y} + \frac{u}{v} = \frac{xv + uy}{yv}.$ [division theorem; $yv \neq 0$]

Since, by the 0-product theorem, the restrictions '$y \neq 0$' and '$v \neq 0$' imply the restriction '$yv \neq 0$', we may disregard the latter. So, the above test-pattern is a proof of the first generalization.

Now, write a test-pattern for the theorem which justifies the rule for subtracting fractions.

* * *

Multiplying fractions

$$\frac{3}{5} \times \frac{7}{8} = ?$$

Since $(\frac{3}{5} \times \frac{7}{8}) \times (5 \times 8) = 3 \times 7$, it follows from the division theorem that
$$\frac{3}{5} \times \frac{7}{8} = \frac{3 \times 7}{5 \times 8}.$$

The rule about multiplying numerator-numbers and multiplying denominator-numbers is justified by this generalization:

$$\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0} \ \frac{x}{y} \cdot \frac{u}{v} = \frac{xu}{yv}.$$

Prove this theorem.

EXERCISES

Simplify.

Sample 1. $\frac{2a}{3b} \cdot \frac{5c}{7d}$

Solution.
$$\left.\begin{aligned} &\frac{2a}{3b} \cdot \frac{5c}{7d} \\ &= \frac{(2a)(5c)}{(3b)(7d)} \end{aligned}\right\} [3b \neq 0;\ \ 7d \neq 0]$$
$$= \frac{10ac}{21bd}.$$

Answer. $\frac{10ac}{21bd}$, $[b \neq 0,\ \ d \neq 0]$

[Notice that since $3 \neq 0$, the 0-product theorem tells us we can replace the restriction '$3b \neq 0$' by '$b \neq 0$'.]

1. $\frac{3x}{7y} \cdot \frac{2k}{5m}$
2. $\frac{9r}{2s} \cdot \frac{11r}{10s}$
3. $\frac{a}{b} \cdot \frac{cd}{5b}$
4. $\frac{-5p}{7j} \cdot \frac{-4m}{3k}$
5. $\frac{1}{x} \cdot \frac{2}{y} \cdot \frac{3}{z}$
6. $\frac{7k}{2} \cdot \frac{3r}{-5} \cdot \frac{3e}{-8}$

(continued on next page)

Sample 2. $\frac{w}{z}(\frac{x}{y} + \frac{u}{v})$

Solution.
$$\left.\begin{aligned} &\frac{w}{z}(\frac{x}{y} + \frac{u}{v}) \\ &= \frac{w}{z}(\frac{xv + uy}{yv}) \\ &= \frac{w(xv + uy)}{z(yv)}. \end{aligned}\right\} \begin{aligned} &[y \neq 0,\ v \neq 0] \\ &[z \neq 0,\ yv \neq 0] \end{aligned}$$

Answer. $\frac{w(xv + uy)}{zyv}$, $[y \neq 0,\ v \neq 0,\ z \neq 0]$

7. $\frac{3}{a}(\frac{2}{b} + \frac{5}{c})$ 8. $\frac{r}{6}(\frac{t}{s} - \frac{u}{2})$ 9. $\frac{k}{m}(\frac{2x}{y} + \frac{3z}{u})$

* * *

Reducing fractions

$$\frac{6}{10} = \frac{3 \times 2}{5 \times 2} = \frac{3}{5}. \qquad \frac{6}{18} = \frac{1 \times 6}{3 \times 6} = \frac{1}{3}.$$

$$\frac{2.5}{15} = \frac{2.5 \times 10}{15 \times 10} = \frac{25}{150} = \frac{1 \times 25}{6 \times 25} = \frac{1}{6}.$$

The generalization which justifies these illustrations is:

$$\forall_x \forall_{y \neq 0} \forall_{z \neq 0}\ \frac{xz}{yz} = \frac{x}{y}.$$

Here is the beginning of a test-pattern for this generalization; you should complete it.

$$\left.\begin{aligned} &\frac{x}{y}(yz) \\ &= \end{aligned}\right\} \text{apm}$$

$$\vdots$$

Here is another test-pattern for the same generalization.

$$\begin{aligned} &\frac{xz}{yz} \\ &= \frac{x}{y} \cdot \frac{z}{z} \\ &= \frac{x}{y} \cdot 1 \\ &= \frac{x}{y}. \end{aligned} \quad \begin{aligned} &\text{multiplication theorem for fractions; } [y \neq 0,\ z \neq 0] \\ &\text{Why?} \\ &\text{Why?} \end{aligned}$$

Perhaps you are more familiar with the rule for reducing fractions in which you divide numerator-number and denominator-number by the same number. Here are two examples.

$$\frac{6}{10} = \frac{6 \div 2}{10 \div 2} = \frac{3}{5}. \qquad \frac{6}{18} = \frac{6 \div 6}{18 \div 6} = \frac{1}{3}.$$

And, you probably shorten the work by using cancel marks like this.

$$\frac{\overset{3}{\cancel{6}}}{\underset{5}{\cancel{10}}} = \frac{3}{5} \qquad \frac{\overset{1}{\cancel{6}}}{\underset{3}{\cancel{18}}} = \frac{1}{3}$$

These procedures for reducing fractions are justified by

$$\forall_x \forall_{y \neq 0} \forall_{z \neq 0} \; \frac{x}{y} = \frac{x \div z}{y \div z}.$$

This generalization can be proved by applying the theorem stated on page 2-94.

$$\begin{aligned} &\frac{x \div z}{y \div z} \\ &= \frac{(x \div z)z}{(y \div z)z} \quad \text{Why?} \\ &= \frac{x}{y}. \quad \text{Why?} \end{aligned}$$

Sometimes the process of reducing fractions is used in a disguised form in carrying out a simplification. For example:

$$\frac{7}{8} \times \frac{6}{11} = ? \qquad \frac{7}{\underset{4}{\cancel{8}}} \times \frac{\overset{3}{\cancel{6}}}{11} = \frac{21}{44}.$$

But, this is really a short cut for the following procedure:

$$\frac{7}{8} \times \frac{6}{11} = \frac{7 \times 6}{8 \times 11} = \frac{7 \times 6}{11 \times 8} = \frac{7}{11} \times \frac{\overset{3}{\cancel{6}}}{\underset{4}{\cancel{8}}} = \frac{7 \times \overset{3}{\cancel{6}}}{11 \times \underset{4}{\cancel{8}}} = \frac{7 \times \overset{3}{\cancel{6}}}{\underset{4}{\cancel{8}} \times 11} = \frac{7}{\underset{4}{\cancel{8}}} \times \frac{\overset{3}{\cancel{6}}}{11}.$$

So, the rule for cancelling before multiplying is justified by the multiplication theorem for fractions, the theorem for reducing fractions by dividing, and the commutative principle for multiplication.

EXERCISES

A. Reduce these fractions.

1. $\frac{15}{35}$ 2. $\frac{28}{44}$ 3. $\frac{18}{81}$ 4. $\frac{16}{64}$ 5. $\frac{19}{95}$

Sample 1. $\frac{12x}{15y}$

Solution. In reducing this fraction we can apply the theorem for reducing fractions by dividing.

$$\frac{12x}{15y} = \frac{12x \div 3}{15y \div 3} = \frac{4x}{5y}, \quad [y \neq 0].$$

[Notice that in simplifying '$12x \div 3$' to '$4x$' we assume that '$(12x) \div 3$' and '$(12 \div 3)x$' are equivalent. In other words, we assume that the generalization:

$$\forall_x \forall_y \forall_{z \neq 0} \ \frac{xy}{z} = \frac{x}{z}y$$

is a theorem. Prove this theorem.]

Sample 2. $\frac{25a}{35b}$ Solution. $\frac{\overset{5}{\cancel{25}}a}{\underset{7}{\cancel{35}}b} = \frac{5a}{7b}, \quad [b \neq 0]$

6. $\frac{26x}{34y}$ 7. $\frac{45x}{18x}$ 8. $\frac{21abc}{15abc}$ 9. $\frac{50xyz}{12xy}$ 10. $\frac{44x(a+b)}{52y(a+b)}$

* * *

There are several theorems which follow easily from the theorem:

$$(*) \qquad \forall_x \forall_y \forall_{z \neq 0} \ \frac{xy}{z} = \frac{x}{z}y$$

which you were asked to prove in Sample 1. These theorems justify some of the short cuts you have learned. Of course, you don't need these theorems to justify the short cuts because you can use earlier theorems for this purpose. For example, consider the problem:

$$\frac{3}{5} \times 7 = ?$$

In grade school you may have learned to do this problem this way:

$$\frac{3}{5} \times 7 = \frac{3}{5} \times \frac{7}{1} = \frac{3 \times 7}{5 \times 1} = \frac{3 \times 7}{5}.$$

This method is correct and is justified by the theorem for dividing by 1 [Exercise 2 on page 2-91], the multiplication theorem for fractions, and the principle for multiplying by 1. But, a short cut is:

$$\frac{3}{5} \times 7 = \frac{3 \times 7}{5},$$

and this is justified by (*). [In fact, the long way of doing the problem may suggest a proof of (*).]

Consider the problem of simplifying the expression:

$$\frac{4x + 6y}{2}.$$

Here is one procedure for simplifying:

(1) $$\frac{4x + 6y}{2} = (4x + 6y)\frac{1}{2} = 2x + 3y.$$

Another way to do this is as follows:

(2) $$\frac{4x + 6y}{2} = \frac{(2x + 3y)2}{2} = 2x + 3y.$$

Still a third way is:

(3) $$\frac{4x + 6y}{2} = \frac{2(2x) + 2(3y)}{2} = 2x + 3y.$$

Each of these procedures is justified in part by a special theorem.

In (1) we used the generalization:

$$\forall_x \forall_{y \neq 0} \quad \frac{x}{y} = x \cdot \frac{1}{y}.$$

[Dividing by a number is the same as multiplying by its reciprocal.]

In (2) we used the generalization:

$$\forall_x \forall_{y \neq 0} \quad \frac{xy}{y} = x.$$

[The inverse of multiplying by a nonzero number is dividing by that number.]

In (3) we used the generalization:

$$\forall_{x \neq 0} \forall_y \forall_z \quad \frac{xy + xz}{x} = y + z.$$

Each of these theorems is an easy consequence of (*), and an even easier consequence of the division theorem. <u>Prove these three theorems</u>.

Here is another easy consequence of (*):

$$\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0} \forall_{z \neq 0} \; \frac{xu}{yv} = \frac{(x \div z)u}{(y \div z)v}.$$

Prove it.

* * *

B. Simplify.

1. $\frac{2}{7} \times \frac{6}{11}$
2. $\frac{3}{8} \times \frac{14}{9}$
3. $\frac{a}{5} \times \frac{b}{6}$
4. $a \times \frac{b}{c}$
5. $10z \cdot \frac{1}{z}$
6. $xy \cdot \frac{z}{y}$
7. $10a \times \frac{3b}{4c}$
8. $\frac{3x}{2(a+b)} \cdot 8x(a+b)$
9. $\frac{2m}{5n} \times 15np$
10. $(ab) \div a$
11. $(16axy) \div (4xy)$
12. $(30x) \div 5$
13. $(3x) \div 2$
14. $\frac{5y}{2}$
15. $\frac{64xyz}{4y}$
16. $\frac{8 + 10}{2}$
17. $\frac{3x + 6y}{3}$
18. $\frac{(3x) \cdot (6y)}{3}$
19. $\frac{15ab + 25ac}{5a}$
20. $\frac{9x(y+z) - 3u(y+z)}{3(y+z)}$
21. $\frac{6xy - 8yz}{2y}$
22. $\frac{2x}{3} \cdot \frac{5y}{7}$
23. $\frac{y}{x} \cdot \frac{3x}{8}$
24. $\frac{2k}{5m} \cdot \frac{7n}{8k}$
25. $\frac{3st}{5rq} \cdot \frac{70r}{30s}$
26. $\frac{2x}{9(a+b)} \cdot \frac{12(a+b)}{16x}$
27. $\frac{1}{ab} \cdot \frac{1}{bc}$
28. $\frac{3}{a} \cdot \frac{5}{b}$
29. $\frac{3}{a} + \frac{5}{b}$
30. $\frac{3x}{2y} + \frac{7y}{5x}$
31. $\frac{1}{p} - \frac{1}{q}$
32. $\frac{12a}{3c} - \frac{5b}{15d}$
33. $\frac{a+x}{x} + \frac{b+y}{y}$

* * *

Least common denominator

Consider the problem:

$$\frac{3}{7} + \frac{2}{7} = ?$$

You wouldn't want to use the addition theorem for fractions:

$$\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{y} + \frac{u}{v} = \frac{xv + uy}{yv}.$$

to solve this problem. You would probably solve it this way:

$$\frac{3}{7} + \frac{2}{7} = \frac{3 + 2}{7} = \frac{5}{7}.$$

That is, you would use the generalization:

$$\forall_x \forall_y \forall_{z \neq 0}\ \frac{x}{z} + \frac{y}{z} = \frac{x + y}{z}.$$

[Use the division theorem to give a very quick proof of this theorem. Do you see that this theorem tells you that division distributes over addition? So, we can call this the distributive theorem for division over addition. Do you think that there is a left distributive theorem for division over addition? If there were, it would follow that $\frac{7}{9} + \frac{7}{5} = \frac{7}{9 + 5}$!]

Similarly, you wouldn't want to use the addition theorem for fractions to solve the problem:

$$\frac{2}{15} + \frac{7}{20} = ?$$

Instead you might proceed as follows:

$$\frac{2}{15} + \frac{7}{20} = \frac{2}{3 \cdot 5} + \frac{7}{4 \cdot 5} = \frac{2 \cdot 4}{(3 \cdot 5)4} + \frac{7 \cdot 3}{(4 \cdot 5)3} = \frac{2 \cdot 4 + 7 \cdot 3}{3 \cdot 4 \cdot 5}.$$

So, $\frac{2}{15} + \frac{7}{20} = \frac{29}{60}$. [Notice the way in which the least common denominator, '$3 \cdot 4 \cdot 5$', was found.] Explain the step:

$$\frac{2}{3 \cdot 5} + \frac{7}{4 \cdot 5} = \frac{2 \cdot 4}{(3 \cdot 5)4} + \frac{7 \cdot 3}{(4 \cdot 5)3},$$

and then explain the last step.

This procedure is justified by the generalization:

$$\forall_x \forall_{y \neq 0} \forall_{z \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{yz} + \frac{u}{vz} = \frac{xv + uy}{yvz}.$$

Prove it.

EXERCISES

A. Simplify.

1. $\frac{7}{18} + \frac{3}{14}$ 2. $\frac{5}{6} - \frac{3}{34}$ 3. $\frac{11}{24} + \frac{17}{81}$ 4. $\frac{1}{2} + \frac{2}{3} - \frac{4}{5}$

5. $\frac{3}{xy} + \frac{5}{yz}$ 6. $\frac{5}{2x} - \frac{3}{2y}$ 7. $\frac{9}{5a} + \frac{3}{15b}$ 8. $\frac{9}{5a} \cdot \frac{3}{15b}$

9. $\frac{7}{2rs} + \frac{3}{8st}$ 10. $5 + \frac{3}{16}$ 11. $a + \frac{b}{c}$ 12. $6.5 + 5.02$

B. Prove these theorems.

1. $\forall_x \forall_y \forall_{z \neq 0}\ \frac{x}{z} - \frac{y}{z} = \frac{x - y}{z}$.

2. $\forall_x \forall_{y \neq 0} \forall_{z \neq 0} \forall_u \forall_{v \neq 0}\ \frac{x}{yz} - \frac{u}{vz} = \frac{xv - uy}{yvz}$.

3. $\forall_x \forall_y \forall_{z \neq 0}\ x + \frac{y}{z} = \frac{xz + y}{z}$.

* * *

Dividing fractions

Consider the problem:

$$3 \div \frac{5}{7} = ?$$

The grade school rule tells you to "invert" the '$\frac{5}{7}$' to get '$\frac{7}{5}$', and then multiply 3 by $\frac{7}{5}$. Let's justify this procedure.

To solve the problem '$3 \div \frac{5}{7} = ?$' is to find the number whose product by $\frac{5}{7}$ is 3. That is,

$$? \times \frac{5}{7} = 3.$$

But, $\frac{7}{5} \times \frac{5}{7} = 1$ [Why?], and $3 \times 1 = 3$. So,

$$3 \times (\frac{7}{5} \times \frac{5}{7}) = 3.$$

Hence,

$$(3 \times \frac{7}{5}) \times \frac{5}{7} = 3.$$

So,

$$3 \div \frac{5}{7} = 3 \times \frac{7}{5}.$$

Justify the invert-and-multiply rule in general by proving that

$$\forall_x \forall_{y \neq 0} \forall_{z \neq 0} \; x \div \frac{y}{z} = x \cdot \frac{z}{y}.$$

EXERCISES

A. Prove these theorems.

1. $\forall_x \forall_{y \neq 0} \forall_{u \neq 0} \forall_{v \neq 0} \; \frac{x}{y} \div \frac{u}{v} = \frac{xv}{yu}.$

2. $\forall_{x \neq 0} \forall_{y \neq 0} \; \dfrac{1}{\frac{x}{y}} = \frac{y}{x}.$

3. $\forall_x \forall_{y \neq 0} \forall_{z \neq 0} \; \frac{x}{y} \div z = \frac{x}{yz}.$

B. Simplify.

1. $\frac{3}{5} \div \frac{7}{10}$
2. $1 \div \frac{3}{11}$
3. $\frac{3}{4} \div 6$
4. $0.3 \div 0.01$
5. $\frac{a}{2b} \div \frac{3}{5b}$
6. $\frac{5a}{3xy} \div \frac{2a}{9x}$
7. $\frac{13xyz}{2abc} \div \frac{26xz}{8ab}$
8. $\dfrac{\frac{2}{9}}{\frac{7}{9}}$
9. $\dfrac{\frac{8}{13}}{\frac{2}{15}}$
10. $\dfrac{\frac{x}{y}}{\frac{5x}{3y}}$
11. $\dfrac{\frac{9}{8ab}}{\frac{3c}{4b}}$

Sample. $\dfrac{\frac{2}{5} + \frac{3}{4}}{\frac{1}{10} + \frac{1}{4}}$

Solution. We could simplify this by simplifying the numerator and then the denominator, and then use the theorem for dividing fractions. Perhaps an easier way is to "clear the numerator and denominator of fractions" by using the theorem about multiplying numerator-number and denominator-number by the same nonzero number.

$$\frac{\frac{2}{5} + \frac{3}{4}}{\frac{1}{10} + \frac{1}{4}} = \frac{(\frac{2}{5} + \frac{3}{4})20}{(\frac{1}{10} + \frac{1}{4})20} = \frac{8 + 15}{2 + 5} = \frac{23}{7}.$$

[Why did we multiply the numerator- and denominator-numbers by 20?]

12. $\dfrac{\frac{2}{3} + \frac{1}{2}}{\frac{3}{2} + \frac{1}{6}}$ 13. $\dfrac{\frac{2}{9} + \frac{1}{4}}{\frac{1}{3} - \frac{1}{5}}$ 14. $\dfrac{\frac{3}{5} + 0.3}{0.7 - \frac{1}{2}}$ 15. $\dfrac{5 + \frac{3}{7}}{3 + \frac{5}{2}}$

16. $\dfrac{2 + \frac{1}{x}}{3 - \frac{2}{x}}$ 17. $\dfrac{\frac{1}{a} + \frac{1}{b}}{\frac{1}{a} - \frac{1}{b}}$ 18. $\dfrac{\frac{2}{5x} + \frac{1}{3x}}{\frac{1}{2x} + \frac{5}{6x}}$ 19. $\dfrac{1 + \frac{8}{a + b}}{1 - \frac{5}{a + b}}$

DIVISION AND OPPOSITION

In earlier sections you proved theorems about opposites of sums, products, and differences.

$$\forall_x \forall_y \quad -(x + y) = -x + -y.$$

$$\forall_x \forall_y \quad -(xy) = -xy = x \cdot -y.$$

$$\forall_x \forall_y \quad -(x - y) = -x - -y.$$

It is now natural to ask about the opposite of a quotient. From the theorems on the opposites of sums and differences, one might suspect that the opposite of a quotient is the quotient of opposites. For example, that

$$-\frac{18}{3} = \frac{-18}{-3} \text{ and } -\frac{-9}{-18} = \frac{9}{18}.$$

But, a bit of computing shows that these statements are false. Moreover, division is more closely related to multiplication than it is to addition or subtraction, so we should expect to get a better clue from the theorem on the opposite of a product. This suggests, for example, that

$$-\frac{18}{3} = \frac{-18}{3} \text{ and that } -\frac{18}{3} = \frac{18}{-3}.$$

Computation shows that these statements are true. So, we should investigate the generalization:

$$(*) \qquad \forall_x \forall_{y \neq 0} \quad -\frac{x}{y} = \frac{-x}{y} = \frac{x}{-y}.$$

Let's try to prove that

(a) $$\forall_x \forall_{y \neq 0} \; -\frac{x}{y} = \frac{-x}{y}.$$

One way to attack this would be through the 0-sum theorem. That is, we try to show that, for each x and each nonzero y,

$$\frac{x}{y} + \frac{-x}{y} = 0.$$

This is easy to do.

$$\frac{x}{y} + \frac{-x}{y} = \frac{x + -x}{y} = \frac{0}{y} = 0.$$

Another way to prove (a) is to use the division theorem. That is, we try to show that, for each x and each nonzero y,

$$(-\frac{x}{y})y = -x.$$

Carry out the proof yourself.

Still a third way of proving (a) is to use the -1 times theorem. Hint: $-\frac{x}{y} = -1 \cdot \frac{x}{y} = \ldots$. Finish this third proof of (a).

To complete the proof of (*), we need to prove:

(b) $$\forall_x \forall_{y \neq 0} \; -\frac{x}{y} = \frac{x}{-y}.$$

Prove this in at least two ways.

EXERCISES

A. The theorem about the product of opposites is:

$$\forall_x \forall_y \; (-x)(-y) = xy.$$

There is an analogous theorem for quotients. State it and prove it.

B. Simplify by finding equivalent expressions with fewer minus signs.

Sample 1. $-\frac{-3}{-7}$

Solution. $-\frac{-3}{-7} = -\frac{3}{7}$.

(continued on next page)

Sample 2. $-\dfrac{-x(a - b)}{-y(c - d)}$

Solution. $-\dfrac{-x(a - b)}{-y(c - d)}$

$$= -\frac{-[x(a - b)]}{-[y(c - d)]}$$

$$= -\frac{x(a - b)}{y(c - d)} \qquad [y(c - d) \neq 0]$$

$$= \frac{-[x(a - b)]}{y(c - d)}$$

$$= \frac{x \cdot -(a - b)}{y(c - d)}$$

$$= \frac{x(b - a)}{y(c - d)}.$$

Answer. $\dfrac{x(b - a)}{y(c - d)}$, $[y \neq 0,\ c \neq d]$

[An equally good answer is: $\dfrac{x(a - b)}{y(d - c)}$, $[y \neq 0,\ c \neq d]$.]

1 1. $-\dfrac{-5}{8}$ 2. $-\dfrac{9}{-7}$ 3. $-\dfrac{7 - x}{2}$ 4. $-\dfrac{8 \div y}{8 - y}$

5. $-\dfrac{-3 - y}{5}$ 6. $\dfrac{-(x - 3)}{-(3 - y)}$ 7. $-\dfrac{-(x - 3)}{-(3 - y)}$ 8. $\dfrac{4 - x - y}{m - 2}$

9. $\dfrac{(a - b)(c - d)}{-(a + b)}$ 10. $-\dfrac{(x - 1)(2 - x)(3 - x)}{-(x + 4) \cdot -(x + 5)}$

*

[Part J of the Supplementary Exercises provides computational practice in simplifying fractions which do not contain pronumerals.]

*

C. Simplify.

1. $\dfrac{40x}{5}$ 2. $\dfrac{54y}{9}$ 3. $\dfrac{72a}{-8}$ 4. $\dfrac{-16a}{4}$

5. $\dfrac{-30p}{-6}$ 6. $\dfrac{18xy}{2}$ 7. $\dfrac{36abc}{-9}$ 8. $\dfrac{-17rs}{-1}$

9. $\dfrac{6xy}{1/2}$ 10. $\dfrac{-7x}{-2}$ 11. $\dfrac{-204xyz}{2}$ 12. $\dfrac{-27xy}{-3}$

13. $\dfrac{5(a + b)}{5}$ 14. $\dfrac{-6(x - y)}{6}$

15. $\frac{15xy}{3x}$ 16. $\frac{20aab}{4a}$ 17. $\frac{9xxy}{3xy}$ 18. $\frac{9xy}{3xu}$

19. $\frac{5xyz}{5xyz}$ 20. $\frac{-18aab}{-2b}$ 21. $\frac{17a}{-a}$ 22. $\frac{24xxyz}{6xyyz}$

23. $\frac{-6xy}{3xyy}$ 24. $\frac{9aa}{-9aa}$ 25. $\frac{-15mnn}{-5mmn}$ 26. $\frac{-12abc}{-2axy}$

27. $\frac{5a}{6b} \times \frac{3x}{7b}$ 28. $\frac{5k}{3m} \times \frac{9mk}{2p}$ 29. $\frac{2xy}{3ab} \times \frac{12abc}{8xyz}$

30. $\frac{-3cd}{2rs} \times \frac{12rr}{7cc}$ 31. $\frac{3xx}{-2y} \times \frac{-8yy}{9x}$ 32. $\frac{5tp}{-3sr} \cdot \frac{9ssrp}{-20tt}$

33. $\frac{4a}{3b} \div \frac{6x}{5y}$ 34. $\frac{9ax}{4by} \div \frac{2aa}{3by}$ 35. $\frac{7apc}{2rs} \div \frac{3aap}{4rrc}$

36. $\frac{a - 5}{2} \times \frac{8}{a - 5}$ 37. $\frac{x + 7}{3(x - 2)} \times \frac{5(x - 2)}{6(x + 7)}$

38. $\frac{9(x + 1)}{3(x + 5)} \times \frac{15(x + 5)}{2(x + 1)}$ 39. $\frac{4(x + 5)(x + 3)}{9(x + 1)} \cdot \frac{3(x + 1)(x + 2)}{2(x + 3)(x + 4)}$

40. $\frac{5(a + 3)}{6(a - 4)} \div \frac{7(a + 3)}{6}$ 41. $\frac{2(9 + x)}{7(11 - x)} \div \frac{10(x + 9)}{21(11 - x)}$

42. $\frac{18d + 6e}{2}$ 43. $\frac{5m - 15}{5}$ 44. $\frac{3x - 9}{-3}$

45. $\frac{4.6x + 6.9}{2.3}$ 46. $\frac{2(9c - 3d)}{3}$ 47. $\frac{2}{3}(9c - 3d)$

48. $\frac{1}{7}(-7p - 14r)$ 49. $\frac{3}{5}(15x - 25y)$ 50. $(5 - 100a)\frac{4}{5}$

51. $\frac{2u + 2v}{4}$ 52. $\frac{12ab + 3ac}{-3}$ 53. $(9f - 12g) \div \frac{1}{3}$

54. $\frac{5xy - 7}{1/2}$ 55. $\frac{18x - 27y}{9/2}$ 56. $\frac{\frac{1}{2}a + \frac{3}{2}b}{2}$

57. $\frac{8xy + 4xz}{4x}$ 58. $\frac{3ab - 7ac}{a}$ 59. $\frac{8xyyz + 12xyz}{4xyz}$

60. $\frac{2ap - 7apq}{-ap}$ 61. $\frac{27xyz + 3z}{3z}$ 62. $\frac{18abc - 3a}{-3a}$

63. $\frac{3(x + 4)}{x + 4}$ 64. $\frac{(x - 2)(x + 7)}{x - 2}$ 65. $\frac{(y - 3)(y + 9)}{-(y - 3)}$

66. $\frac{(m - 5)(m + 7)}{5 - m}$ 67. $(a - 1) \cdot \frac{7a}{(a - 1)}$ 68. $2(x + 3) \cdot \frac{x}{x + 3}$

Sample 1. $28\left[\frac{x + 18}{4} - \frac{3}{7}(x - 3)\right]$

Solution. $28\left[\frac{x + 18}{4} - \frac{3}{7}(x - 3)\right]$

$= 28 \cdot \frac{x + 18}{4} - 28\left[\frac{3}{7}(x - 3)\right]$

$= 7(x + 18) - 4 \cdot 3(x - 3)$

$= 7x + 126 - 12(x - 3)$

$= 7x + 126 - (12x - 36)$

$= 7x + 126 - 12x + 36 = 162 - 5x.$

69. $12\left(\frac{a}{3} + \frac{a}{4}\right)$

70. $8\left(\frac{3n}{4} - \frac{5n}{2}\right)$

71. $8\left(\frac{3}{4}n - \frac{5}{2}n\right)$

72. $18\left(\frac{5x}{6} + \frac{5x}{9}\right)$

73. $24\left(\frac{2x}{3} - \frac{3y}{8}\right)$

74. $14\left(\frac{x}{7} - \frac{y}{2}\right)$

75. $6\left(\frac{a + 2}{3} + \frac{a + 3}{2}\right)$

76. $10\left(\frac{2k}{5} - \frac{3k}{2}\right)$

77. $15\left(\frac{3x}{5} - \frac{5x}{3}\right)$

78. $8\left(\frac{a + 5}{2} - \frac{a + 1}{4}\right)$

79. $12\left(\frac{4y + 1}{2} - \frac{2y + 3}{3} - \frac{5y - 4}{4}\right)$

80. $6\left(\frac{x + 11}{6} - \frac{10 - x}{3}\right)$

81. $24\left[\frac{y - 7}{8} - \frac{9 - y}{3} + \frac{y}{6}\right]$

82. $12\left[\frac{1}{3}(x + 5) - 4 - \frac{x - 8}{4} + \frac{1}{2}\right]$

Sample 2. $2(a - 5)\left(\frac{a}{a - 5} - \frac{3}{2}\right)$

Solution. $2(a - 5)\left(\frac{a}{a - 5} - \frac{3}{2}\right)$

$$\left.\begin{aligned} &= 2(a - 5)\frac{a}{a - 5} - 2(a - 5)\frac{3}{2} \\ &= 2a - (a - 5)3 \end{aligned}\right\}[a \neq 5]$$

$= 2a - 3a + 15$

$= -a + 15.\ [a \neq 5]$

Answer. $-a + 15$, $[a \neq 5]$, or: $15 - a$, $[a \neq 5]$

83. $k\left(\frac{3}{k} - 1\right)$

84. $2a\left(\frac{5}{2a} - 3\right)$

85. $7(a - 4)\left(\frac{5}{a - 4} + 1\right)$

86. $2a\left(\frac{5}{a} - \frac{1}{2}\right)$

87. $6x\left(\frac{4}{x} + \frac{3}{2x}\right)$

88. $7s\left(\frac{5}{7} - \frac{3}{s}\right)$

89. $2(b + 5)\left(\frac{b}{b + 5} - \frac{1}{2}\right)$

90. $12n\left(\frac{2n + 1}{2n} - \frac{3n - 2}{3n} - \frac{7}{12}\right)$

91. $2(x + 2)\left(\frac{3}{2} - \frac{x}{x + 2}\right)$

92. $8(b + 3)\left(\frac{b - 2}{b + 3} - \frac{3}{8}\right)$

93. $5y(y + 4)\left(\frac{6}{5y} - \frac{2}{y + 4}\right)$

94. $3x(x - 3)\left(\frac{x}{x - 3} - \frac{7}{x}\right)$

95. $m(m + 3)\left(\frac{2}{m + 3} - \frac{1}{m}\right)$

96. $9r(3r - 4)\left(-\frac{1}{3r} + \frac{2r}{3r - 4}\right)$

*

97. $\frac{x}{5} - \frac{x}{3}$

98. $\frac{2y}{7} + \frac{3y}{2}$

99. $\frac{9x}{2} - \frac{1}{3} + \frac{x}{3}$

100. $\frac{3a}{4} + \frac{a}{8}$

101. $\frac{4b}{5} - \frac{1}{10}$

102. $\frac{x}{3} + \frac{x}{5} + x - 1$

103. $\frac{8}{x} - \frac{3}{2x}$

104. $\frac{7}{y} + \frac{5}{3y}$

105. $\frac{9}{k} - \frac{1}{5k} + 3$

106. $\frac{2}{3z} - \frac{1}{2z} + 7 - \frac{3}{8z}$

107. $1 - \frac{2}{7r} + \frac{1}{2} - \frac{3}{r}$

108. $\frac{7t + 1}{10} - \frac{t - 9}{4}$

109. $\frac{1}{10}(7t + 1) - \frac{1}{4}(t - 9)$

110. $\frac{a + 4}{2} - \frac{a + 3}{5}$

111. $\frac{4a + 1}{2} - \frac{2a + 3}{3} - \frac{5a - 1}{4}$

112. $\frac{r + 1}{2} - \frac{r - 5}{3}$

113. $\frac{6s - 3}{5} - \frac{3s - 2}{2}$

114. $\frac{z - 7}{5} + 2 - \frac{z + 8}{10}$

115. $\frac{c - 2}{6} - \frac{c - 4}{4} - \frac{2}{3}$

116. $3 + \frac{n - 1}{3} - \frac{n + 14}{9}$

117. $\frac{y + 9}{9} + \frac{1}{3} - \frac{y - 7}{2} + 1$

Sample 3. $\frac{5}{a-2} - \frac{6}{a}$

Solution. $$\left.\begin{aligned}&\frac{5}{a-2} - \frac{6}{a}\\ &= \frac{5a - 6(a-2)}{(a-2)a}\end{aligned}\right\}[a \neq 2,\ a \neq 0]$$

$$= \frac{5a - 6a + 12}{(a-2)a}$$

$$= \frac{-a + 12}{(a-2)a}.$$

Answer. $\frac{-a+12}{(a-2)a}$, $[a \neq 2,\ a \neq 0]$

118. $\frac{3}{a+2} + \frac{5}{a+3}$

119. $\frac{4}{x-5} - \frac{7}{x-1}$

120. $\frac{9}{x-3} + \frac{2}{x-4}$

121. $\frac{2}{y-7} - \frac{3}{y-5}$

☆122. $\frac{5}{x} + \frac{3}{x-8} + 1$

☆123. $\frac{11}{2y+1} - \frac{5}{y} + \frac{4}{2y+1}$

☆124. $\frac{6}{a-3} + \frac{2}{a+5} - \frac{7}{a-3}$

☆125. $\frac{3}{4} - \frac{y}{y+2} + \frac{y}{3}$

☆126. $\frac{x+1}{x-8} + \frac{3}{5} + \frac{2}{x-8}$

☆127. $\frac{x}{x+3} - \frac{2x}{x+7} - \frac{3x}{x+3}$

128. $\dfrac{\frac{3}{2} - \frac{1}{2x}}{\frac{1}{3x} - \frac{1}{6}}$

129. $\dfrac{3 - \frac{1}{2t}}{7 + \frac{1}{3t}}$

130. $\dfrac{1}{\frac{1}{r} + \frac{1}{s}}$

131. $\dfrac{a + \frac{1}{3b}}{b + \frac{1}{4a}}$

132. $\dfrac{x - y}{\frac{2}{x} + \frac{3}{y}}$

133. $\dfrac{\frac{1}{a}}{\frac{1}{b} + \frac{1}{c}}$

134. $\dfrac{1}{1 - \dfrac{1}{1 - \dfrac{1}{1+1}}}$

135. $\dfrac{1}{2 + \dfrac{1}{2 + \dfrac{1}{2+1}}}$

[More exercises are in Part K, Supplementary Exercises.]

2.08 Comparing real numbers.--In Unit 1 you learned a procedure for deciding which of two real numbers is the larger. If you have a first number and a second number and if you can get the first number by adding a positive number to the second number, which is the larger number?

Suppose Rita picks a first number, Rhoda picks a second number, and Rhoda subtracts her number from Rita's. If the difference is a positive number, who picked the larger number? If the difference is not a positive number, who picked the larger?

We can summarize this subtraction procedure for telling which is the larger number as follows:

For each x, for each y,

(a) if $x - y$ is a positive number then $x > y$, and

(b) if $x - y$ is not a positive number then $x \not> y$.

EXERCISES

A. Each of the following exercises involves a generalization. In some cases the generalization is true, in others it is false. For each exercise, state the generalization in a concise way, and tell whether you think it is true or false.

Sample 1. If I add 7 to a first number, I get a second number which is larger than the first number.

Solution. If you need to, you can try a few examples to get the "feel" of the generalization.

$$1 + 7 = 8 \quad \text{and} \quad 8 > 1$$

$$^{-}5 + 7 = 2 \quad \text{and} \quad 2 > {}^{-}5$$

$$^{-}483 + 7 = {}^{-}476 \text{ and } {}^{-}476 > {}^{-}483$$

The generalization is:

$$\forall_x \quad x + 7 > x.$$

I think it's true.

<u>Sample 2</u>. If I add a second number to a first number, I get a third number which is larger than the first number.

The generalization is:

$$\forall_x \forall_y \; x + y > x.$$

This is false, and here's why. Add $^-5$ to 8; you get 3 which is smaller than the first number, 8.

1. If I subtract 3 from a first number, I get a second number which is smaller than the first number.
2. If I multiply a first number by 2, I get a second number which is larger than the first number.
3. If I divide a first number by 2, I get a second number which is smaller than the first number.
4. If I multiply a number by itself, I get the same or a larger number.
5. If I subtract a first number from a second number and get a positive difference, I would get a negative difference if I reversed the order of subtracting.
6. If I subtract a first number from a second number and get a nonpositive difference, I would get a positive difference if I reversed the order of subtracting.
7. If a first number is greater than a second number, and the second number is greater than the third number, then the first number is greater than the third number.
8. If a first number is less than a second number and a third number is less than this second number, then the first number is less than the third number.
9. I divide a first number by a second number, and I also divide the first number by a third number. If the second number is less than the third number, the first quotient is greater than the second quotient.

B. True or false?

1. $\forall_x \ |x| \geq x$. [An instance of this generalization is: $|^-5| \geq {}^-5$. To make sense out of this statement, we must assume that '$|^-5|$' is being used as an abbreviation for '$^+|^-5|$'. See Unit 1, page 1-110.]

2. $\forall_x \forall_y \ xy \leq |x| \cdot |y|$.

3. $\forall_x \ |-x| = |x|$.

4. $\forall_x \ -|xx| = x \cdot -x$.

5. $\forall_x \ |x(-x)| = xx$.

6. $\forall_x \forall_y \ |x + y| = |x| + |y|$.

7. $\forall_x \forall_y \ |x + y| \leq |x| + |y|$.

8. $\forall_x \forall_y \ |x - y| \leq |x| - |y|$.

9. $\forall_x \forall_y \forall_u \forall_v$ if $x > y$ and $u > v$ then $x + u > y + v$.

10. $\forall_x \forall_y \forall_z$ if $x > y$ then $x + z > y + z$.

11. $\forall_x \forall_y \forall_z$ if $x + z > y + z$ then $x > y$.

12. $\forall_x \forall_y \forall_z$ if $x > y$ then $xz > yz$.

13. $\forall_x \forall_y \forall_z$ if $xz > yz$ then $x > y$.

14. $\forall_{x \neq 0 \ \ y \neq 0}$ if $x > y$ then $\frac{1}{x} < \frac{1}{y}$.

15. $\forall_x \forall_y \forall_{z > 0}$ if $y < x$ then $yz < xz$.

16. $\forall_x \forall_y \forall_{z < 0}$ if $x > y$ then $xz > yz$.

17. $\forall_x \forall_{y > 0} \ |xy| \not> -xy$.

18. $\forall_{x < 0} \forall_{y > 0} \ |xy| \not< -xy$.

19. $\forall_x \forall_y$ if $x \geq y$ then $x - y \geq 0$.

20. $\forall_{x > 0}$ if $\frac{1}{x} > 2$ then $x < \frac{1}{2}$.

MISCELLANEOUS EXERCISES

A. For each open sentence, tell what value of the pronumeral can be used to generate a true sentence.

1. $4\square + 1 = 9$
2. $8\Diamond - 2 = 38$
3. $2\bigtriangledown + \bigtriangledown = 12$
4. $9\triangle - 2\triangle = 28$
5. $3x - 17 = 1$
6. $2y + 12 = 18$
7. $5 - 3A = -7$
8. $20 + 3B = 50$
9. $2m + 5m = 35$
10. $p - 4p = 17$
11. $\frac{1}{2}y = 12$
12. $\frac{1}{4}x + 7 = 22$
13. $\frac{1}{3}z - 5 = 75$
14. $7 + \frac{1}{3}y = 15$
15. $\frac{x - 4}{3} = 12$
16. $\frac{5 - y}{9} = 2$
17. $\frac{2A + 1}{4} = 8$
18. $\frac{12 - 7k}{2} = 5\frac{1}{2}$
19. $x = x + 9$
20. $|x - 3| = 5$
21. $3(x + 2) = 51$
22. $5(x + 7) + 8 = 108$
23. $8x + 2 - 5x = 27$
24. $4y + 6 - 17y = 6$
25. $3(x + 2) + 2(x - 3) + 9x = 42$

B. For each of the following expressions, write three expressions which are equivalent to it.

1. $3x + 2y + 9$
2. $x + 5x - 7$
3. $5 + 9 - 6$
4. $18 - 2x + 25$
5. $6(r - 3s)$
6. $4xx + 2x$
7. $4x(2y - 3z)$
8. $3ab + 5ac$
9. $\frac{1}{2}x - \frac{2}{3}y$
10. $\frac{11 - 6x}{9}$
11. $27xy$
12. $33xy \div (3x)$

*

For each of the following pronumeral expressions, write <u>one</u> expression which contains the same pronumerals but which is <u>not</u> equivalent to it, <u>and</u> <u>prove</u> that they are not equivalent.

13. $2x + 1$	14. $3yz$
15. $6k - 2$	16. $7s \times 3t$
17. $3xx - 2x$	18. $7 + 0x$
19. $\frac{3y + x}{x}$	20. $\frac{14xxyyy}{2xyy}$

<u>C</u>. Rearrange these numerals into columns with all numerals for the same number in the same column.

$5 \times 2 + 4$	$9 \times 1 - 6 \times 2$	$23 - 2 \cdot 3$	$7 \div -.5$	
$\frac{2}{3} \cdot 21$	$\frac{8}{1 + 1} + 5$	$-3 \cdot -4$	$11 + 3 \cdot 1$	$6(2 + 1)$
$\frac{40 + -6}{2}$	${}^{-}6 \cdot {}^{-}2 - {}^{-}2 \cdot {}^{-}13$	$2 + 3 \times 4$	7×2	$7(3 - 5)$
$.5 \times \lvert 1 - 35 \rvert$	$-8 + 3({}^{-}2)$	$8 \times 2 + 1$	50% of 28	
$\frac{30}{4} + \frac{19}{2}$	$-7(2)(-1)$	$2(5 + 2)$	$-\frac{1}{3} \cdot (-42)(-1)$	
$2 \cdot 5 + 2 \cdot 2$	$11 \cdot 2 - 16 \cdot \frac{1}{2}$	$-7 \cdot -2$	$(6 + 1)(9 - 7)$	
$87\frac{1}{2}\%$ of -16	$5(2 - 7) + 22$	$-\frac{13 \cdot 3 + 4 \cdot 3}{-3}$		
$\frac{6 + 2(10 + 1)}{-2}$	$3 \times 1 + 3 \times 2$	$-7 \cdot -(5 - 3)$	$5 \cdot -2 - 4$	
$7 \cdot 2 + 3$	$-8 \cdot -2 + 3 \cdot 11$	$\frac{-17}{2} - \frac{-11}{-2}$	$8(2 - \frac{7}{8})$	
$2 + (6 - 1)(2 + 1)$	$-1 \cdot 5 + -2 \cdot -1$	$6(8 - 6) - 24$		
$\frac{9 + 3(11 - 2\frac{2}{3})}{2}$	$\frac{-18}{5} - \frac{-42}{-5}$	$\frac{-5}{8} + \frac{5}{-2} + \frac{-1}{-8}$		
20% of 85	2% of 850	$.02\%$ of 85000		
$(2 + 5) \div 3 + 7 - \frac{1}{3}$	4% of -300	700% of 2		

D. State the generalization involved in each of the following descriptions and prove it.

Sample. If I multiply a first number by itself and add the first number to this product, I get the same result I would have gotten if I had multiplied the first number by a number which is 1 more than the first number.

Solution. [I try a special case first to get the "feel" of this generalization.

$$8 \times 8 + 8 = 8 \times (8 + 1).]$$

The generalization is:

$$\forall_x \; xx + x = x(x + 1).$$

Proof.

$$\begin{aligned} & xx + x \\ = \; & x \cdot x + x \cdot 1 && \text{pm1} \\ = \; & x(x + 1). && \text{ldpma} \end{aligned}$$

1. The square of the double of a number is 4 times the square of the number. [The square of 7 is 49, of 8 is 64, of -3 is 9; the double of 3 is 6, of 15 is 30, of -4 is -8.]

2. The product of the opposites of two numbers is the product of the two numbers.

3. If I subtract 8 times a first number from 13 times the first number, I get 5 times the first number.

4. Pick a number. Multiply it by 10. Add the number you started with. Divide the sum by 11. What number do you get?

5. Pick a number. Add 9 to it to get a second number. Subtract 9 from the first number to get a third number. Take the average of the second and third numbers. What number do you get?

6. I decrease each of a first number and a second number by a third number to get a fourth number and a fifth number, respectively. The difference of the fourth number from the fifth number is the difference of the first number from the second number.

7. I subtract a first number from a second number, and then subtract the second number from the first number. Then I pick a third number and multiply it by each of the differences. The sum of these products is 0.

8. I multiply a first number by a second number which is 3 less than the first. I multiply the first number by a third number which is 3 more than the first. I add the products and get twice the square of the first.

9. The sum of the reciprocals of two numbers is the sum of the numbers divided by their product.

10. Pick a first number. Add 1 to it to get a second number. Add 1 to the second number to get a third number. Add 1 to the third number to get a fourth number. Keep this up until you get a tenth number. The sum of the ten numbers is 5 times the sum of the first and tenth numbers.

11. Continue the process of adding 1 in Exercise 10 until you get a thousandth number. The sum of the thousand numbers is 500 times the sum of the first and thousandth numbers.

☆12. Continue the process of adding 1 in Exercise 10 until you get bored. What is the sum of the numbers you get?

E. Complete each sentence to a true one by writing the simplest expression you can in the blank.

1. For each x, the sum of 2x and 5 is ________________.

2. For each y, the product of 5 by 3y is ________________.

3. For each x, the difference of 7 from x + 7 is __________.

4. For each x, x + 7 exceeds 7 by ________________.

5. For each x, 3x exceeds 2x - 1 by ________________.

6. For each a, for each b, 3a + 2b exceeds 5a - 6b by ________________.

7. For each x, x decreased by 5 is ________________.

8. For each x, 7x decreased by 7 is ________________.

9. For each x, x increased by 9 is ________________.

10. For each x, 5x increased by 3x - 2 is ________________.

11. For each x, for each y, for each z, the sum of 5x - 4y + 6z and -3x + 2y - 2z is ________________.

12. For each x, the product of 3x by the sum of 4x + 1 and 1 - x is ________________.

13. For each x, for each y, for each z, the difference of x - y + 1 from the sum of x + z + 1 and x + y + 1 is ________________.

14. For each x, for each y, 3x - 2y exceeds 2x + 5y by ________________.

15. For each x, the difference of the product of x + 9 by x from the product of x + 1 by x is ________________.

16. For each a, for each b, the difference of ________________ from 3a - 2b is 2a - 3b.

17. For each x, for each y, for each z $\neq$ 0, the quotient of 3xz - 3yz by 3z is ________________.

18. For each a, for each $b \neq 0$, the quotient of $ab + b$ by b is __________________.

19. For each $\bigcirc$, for each $\square$, the sum of $3\bigcirc + 5\bigcirc$ and $9\square + 6\square + 3$ is __________________.

20. For each m, for each n, the product of $15m + 10n$ by $\frac{1}{5}m$ is __________________.

21. For each $\square$, for each $\bigtriangledown$, the difference of $45\square - 9 - 15\bigtriangledown$ from $3\bigtriangledown + 12 + \square - 18$ is

__________________.

22. For each r, for each $t \neq 0$, the quotient of $6rt + 4t - 2rt + 2r$ by 2t is __________________.

23. For each a, 13a increased by $9 + 14a$ is __________________.

24. For each $x \neq 0$, the sum of $\frac{3}{x}$ and $\frac{5}{7x}$ is __________________.

25. For each x other than 2 and 3, the sum of $\frac{5}{x - 2}$ and $\frac{7}{x - 3}$ is __________________.

26. For each a other than $\frac{3}{2}$ and $\frac{2}{3}$, the difference of $\frac{9}{2a - 3}$ from $\frac{8}{3a - 2}$ is __________________.

27. For each $x \neq 0$, for each $y \neq 0$, for each $z \neq 0$, the product of $\frac{-8xy}{3yz}$ by $\frac{-3yz}{8xx}$ is __________________.

28. For each $x \neq 0$ and $\neq 3$, the quotient of $\frac{3}{5} - \frac{4}{x}$ by $\frac{1}{x} - \frac{1}{3}$ is __________________.

F. Rearrange these numerals into columns with all numerals for the same number in the same column.

$-\frac{3}{5}$	$\frac{3-8}{-3}$	$-\frac{3/4}{5/4}$	$\frac{3\times 7}{5\times 7}$
$\frac{8-3}{-3}$	$\frac{3-8}{1-4}$	$-\frac{-2\times -3}{-2\times -5}$	$\frac{3}{-5}$
$\frac{4-9}{10-7}$	$-\frac{-5}{-3}$	$\frac{3\times -1}{5\times -1}$	$\frac{6}{10}$
$\frac{+3}{+5}$	$\frac{6-16}{-6}$	$-\frac{+3}{+5}$	$\frac{3/4}{5/4}$
$-\frac{5}{-3}$	$\frac{9-4}{7-10}$	$\frac{-3}{5}$	$\frac{-2\times -3}{2\times 5}$
$-\frac{3-8}{-3}$	$-\frac{3-8}{4-1}$	$\frac{3\times -7}{5\times 7}$	$-\frac{3}{-5}$
$\frac{4-9}{7-10}$	$-\frac{8-3}{1-4}$	$\frac{2\times 3}{2\times 5}$	$\frac{3}{5}$
$\frac{-3}{-5}$	$-\frac{-5}{3}$	$-\frac{5}{3}$	$\frac{3-8}{3}$
$-\frac{-3}{5}$	$-\frac{3-8}{3}$	$\frac{8-3}{3}$	$\frac{2\times -3}{5\times -2}$
$-\frac{-3}{-5}$	$\frac{5}{3}$	$\frac{-5}{-3}$	$\frac{-(7-2)}{-(9-6)}$

G. 1. Which of the following numerals name the opposite of -6?

(a) $--6$ (b) $+6$ (c) $-(7 - 13)$

(d) -3×-2 (e) -1×-6 (f) $-1 \times -2 \times -3$

(g) $-1 \times 2 \times -3$ (h) $1 \times 2 \times -3$ (i) 2×3

(j) $\frac{6}{-1}$ (k) $\frac{-6}{-1}$ (l) $-\frac{6}{-1}$

(m) $\frac{-18}{-3}$ (n) $(5 - 3) \times (5 - 2)$

2. Which of the following numerals name the reciprocal of $-\frac{3}{8}$?

(a) $\frac{3}{8}$ (b) $\frac{8}{3}$ (c) $-\frac{8}{3}$ (d) $\frac{8}{-3}$

(e) $\frac{-8}{3}$ (f) $\frac{-\frac{3}{8}}{-1}$ (g) $-1 \times -\frac{3}{8}$ (h) $-1 \times \frac{8}{3}$

(i) $\frac{-16}{6}$ (j) $\frac{16}{-6}$ (k) $-\frac{16}{6}$ (l) $1 \div -\frac{3}{8}$

3. Which of the following numerals name the opposite of $(6 - 3 + 7)$?

(a) $-(6 - 3 + 7)$ (b) $-6 + 3 - 7$ (c) $-6 - 3 + 7$

(d) $-1 \times (6 - 3 + 7)$ (e) $6 + 3 - 7$ (f) $(6 - 3 + 7) \div -1$

4. Which of the following numerals name the opposite of $\frac{3 - 8}{-2 + 7}$?

(a) $-1 \times \frac{3 - 8}{-2 + 7}$ (b) $\frac{3 - 8}{-2 + 7} \div -1$ (c) $\frac{8 - 3}{-2 + 7}$

(d) $\frac{3 - 8}{7 - 2}$ (e) $\frac{3 - 8}{2 - 7}$ (f) $\frac{3 + 8}{2 + 7}$

(g) $\frac{-2 + 7}{3 - 8}$ (h) $\frac{-3 + 8}{-2 + 7}$ (i) $\frac{-1 \times (3 - 8)}{-2 + 7}$

(j) $\frac{3 - 8}{-1 \times (-2 + 7)}$ (k) $-1 \times \frac{-2 + 7}{3 - 8}$

5. (a) Which number is its own opposite?

(b) Which number is its own reciprocal?

(continued on next page)

6. True or false?

   (a) If you multiply a number by -1, the product is the opposite of the given number.

   (b) If you divide a number by -1, the quotient is the reciprocal of the given number.

   (c) If you divide a number by -1, the quotient is the opposite of the given number.

   (d) If you divide -1 by a nonzero number, the quotient is the opposite of the reciprocal of the given number.

   (e) If you divide -1 by a nonzero number, the quotient is the reciprocal of the opposite of the given number.

H. Evaluate each of the following pronumeral expressions using the given values of the pronumerals. Answers should be in simplest form.

Sample. $a + (n - 1)d$; '7' for 'a', '13' for 'n', '4' for 'd'

Solution.
$$\begin{aligned} & 7 + (13 - 1)4 \\ = \; & 7 + 12 \cdot 4 \\ = \; & 7 + 48 \\ = \; & 55. \end{aligned}$$

1. $\frac{n}{2}(a + \ell)$; '19' for 'n', '71' for 'a', '35' for 'ℓ'

2. $\frac{n}{2}[2a + (n - 1)d]$; '20' for 'n', '6' for 'a', '-5' for 'd'

3. $\frac{r\ell - a}{r - 1}$; '2' for 'a', '$\frac{1}{2}$' for 'r', '$\frac{1}{512}$' for 'ℓ'

4. $\frac{a}{1 - r}$; '1.04' for 'a', '.01' for 'r'

5. $2\pi r$; '13' for 'r'

6. $\frac{1}{2}rC$; '5' for 'r', '10π' for 'C'

7. πrr; '7' for 'r'

8. $\frac{1}{2}bh$; '14' for 'b', '6' for 'h'

9. $\frac{(n - 2)180}{n}$; '17' for 'n'

10. $\frac{n(n - 3)}{2}$; '100' for 'n'

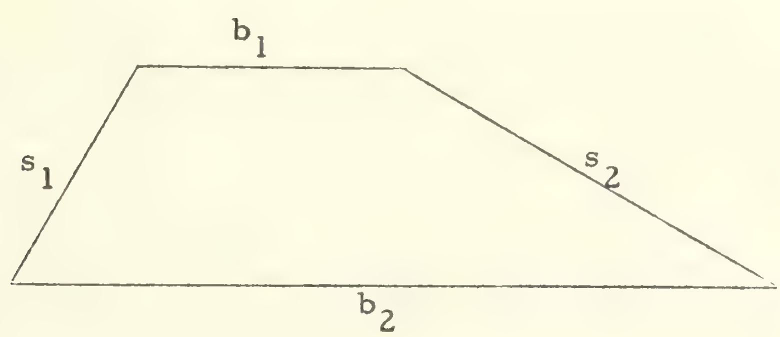
*

In some of the expressions which follow, you will see pronumerals like 'd_1' and 'd_2'. The small numerals written at the lower right of the letter are called <u>subscripts</u>, and they are used to indicate that 'd_1' and 'd_2' are different pronumerals. [Read 'd_1' and 'd_2' as 'dee sub one' and 'dee sub two'.]

Using subscripts enables you to manufacture an unlimited number of pronumerals from just one letter. They are especially helpful in writing and remembering formulas. For example, the formula:

$$P = b_1 + s_1 + b_2 + s_2$$

can be used to compute the perimeter of a trapezoid.



The values of 's_1' and 's_2' are the measures of the nonparallel <u>sides</u>, and the values of 'b_1' and 'b_2' are the measures of the <u>bases</u>.

*

11. $\frac{1}{2}d_1d_2$; '6' for 'd_1', '7' for 'd_2'

12. $\frac{h(b_1 + b_2)}{2}$; '3' for 'h', '6' for 'b_1', '15' for 'b_2'

13. $s(s - a)(s - b)(s - c)$; '15' for 's', '5' for 'a', '12' for 'b', '13' for 'c'

14. $\pi r(\ell + r)$; '4' for 'r', '12' for 'ℓ'

15. $\frac{1}{3}\pi rrh$; '7.5' for 'r', '8.3' for 'h'

(continued on next page)

16. $\frac{1}{2}(P_1 + P_2)\ell$; '19' for '$P_1$', '28' for '$P_2$', '6' for '$\ell$'

17. $\pi\ell(r_1 + r_2)$; '9' for 'ℓ', '3' for 'r_1', '8' for 'r_2'

18. aa + bb + cc; '1.5' for 'a', '2.0' for 'b', '3.5' for 'c'

19. 2(ab + ac + bc); '2.1' for 'a', '3.4' for 'b', '4.2' for 'c'

20. $\frac{E}{720} \cdot 4\pi rr \cdot \frac{r}{3}$; '36' for 'E', '2' for 'r'

21. $\frac{h}{6}(B_1 + B_2 + 4M)$; '7.1' for 'h', '20.3' for 'B_1', '31.2' for 'B_2', '28.1' for 'M'

22. aa + bb + 2abc; '5.1' for 'a', '3.2' for 'b', '0.87' for 'c'

23. $\frac{kr}{t}$; '1.3' for 'k', '6.2' for 'r', '2' for 't'

24. $\frac{v_1 - v_0}{t}$; '60' for 'v_1', '40' for 'v_0', '15' for 't'

25. $v_0t + \frac{1}{2}att$; '3' for 'v_0', '20' for 'a', '5' for 't'

26. $r\left(\frac{k_2 - k_1}{t}\right)$; '9' for 'r', '50' for 'k_2', '80' for 'k_1', '3' for 't'

27. $G \cdot \frac{m_1m_2}{dd}$; '0.0000000666' for 'G', '250000' for 'm_1', '8700000' for 'm_2', '20' for 'd'

28. $ma + Rv$; '41' for 'm', '25' for 'a', '0.12' for 'R', '38' for 'v'

29. $m \cdot \frac{vv}{r}$; '94' for 'm', '27' for 'v', '6' for 'r'

30. $\frac{1}{p} + \frac{1}{q}$; '8' for 'p' and '12' for 'q'

31. $\frac{1}{\frac{1}{p} + \frac{1}{q}}$; '7' for 'p' and '3' for 'q'

32. $\frac{pq}{p + q}$; '7' for 'p' and '3' for 'q'

33. $\frac{r_1t}{m(r_1 + r_2) - t(m - 1)}$; '10' for '$r_1$', '20' for '$r_2$', '1.0' for 't', '1.5' for 'm'

I. Complete each sentence to a true one by writing the simplest expression you can in the blank.

1. (a) If eggs cost 60 cents a dozen, 3 dozen eggs cost ______ cents.

   (b) For each $x > 0$, if eggs cost 60 cents a dozen, x dozen eggs cost ______ cents. [Why 'For each $x > 0$,' instead of just 'For each x,'?]

   (c) For each $x > 0$, if eggs cost 60 cents a dozen, x eggs cost ______ cents.

   (d) For each $x > 0$, if eggs cost 60 cents a dozen, $(x + 3)$ dozen eggs cost ______ cents.

2. (a) For each $x > 0$, if one pencil costs 2 cents, x pencils cost ______ cents.

   (b) For each $x > 0$, if one pencil costs 3 cents, $x + 5$ pencils cost ______ cents.

   (c) For each $x > 0$, if a dozen pencils cost $30x$ cents then 2 pencils cost ______ cents.

   (d) For each $x > 0$, for each $y > 0$, if 3 pencils cost y cents then x pencils cost ______ cents.

3. (a) For each $x > 0$, the perimeter of a square with one side $2x$ units long is ______.

   (b) For each $y > 0$, the perimeter of a square with one side $(3y + 7)$ units long is ______.

   (c) For each $x > 0$, for each $y > 0$, the perimeter of a square with one side $\left(\frac{y}{8} + \frac{x}{2}\right)$ units long is ______.

4. (a) If there are 100 sheets of paper in a pile 1 inch thick, there are ______ sheets of paper in a pile 9 inches thick, and 575 sheets in a pile ______ inches thick.

   (b) For each $x > 0$, if there are 75 sheets of paper in a pile 1 inch thick then there are ______ sheets of paper in a pile $2x$ inches thick.

(c) For each $a > 0$, for each $b > 0$, for each $c > 0$, if there are $20ab$ sheets of paper in a pile 1 inch thick, there are ______ sheets in a pile $15abc$ inches thick.

(d) For each $x > 0$, for each $y > 0$, if there are 125 sheets of paper in a pile y inches thick then there are ______ sheets of paper in a pile x inches thick.

5. (a) For each $k > 0$, the perimeter of an equilateral triangle with one side $7k$ units long is ______.

(b) For each $t > 0$, if the perimeter of an equilateral triangle is $9tt + 12t + 18$, one of its sides is ______ units long.

6. (a) For each $x > 0$, the perimeter of a rectangle whose dimensions are $3x$ units by $2x + 4$ units is ______.

(b) For each $x > 0$, if the perimeter of a rectangle is $18x + 4$, and if two of the sides are each $(7x + 2)$ units long, then each of the other two sides is ______ units long.

(c) For each $x > 0$, for each $y > 0$, if one dimension of a rectangle is $(2y + 4)$ units and the perimeter is $4[2(x + 3) + y]$, the other dimension is ______ units.

(d) For each $a > 0$, for each $b > 0$, the perimeter of a rectangle whose dimensions are $\frac{a}{4}$ units by $2b$ units is ______.

(e) For each $x > 0$, the perimeter of a rectangle whose dimensions are $\frac{x}{2}$ units by $4x$ units is ______ times the perimeter of one whose dimensions are $\frac{x}{4}$ units by $2x$ units.

7. (a) If a car travels 1 mile in $\frac{3}{2}$ minutes, it will travel 30 miles at this rate in ______ minutes.

(b) For each $x > 0$, if a car travels 1 mile in $\frac{3}{2}$ minutes, it will travel $10x$ miles at this rate in ______ minutes.

(c) For each $x > 0$, if a car travels 1 mile in $\frac{3}{2}$ minutes, it will travel ______ miles at this rate in $18x$ minutes.

(d) For each $x > 0$, if a car travels $13x$ miles in $\frac{x}{2}$ minutes, it will travel ______ miles at this rate in 1 minute!

(continued on next page)

8. (a) For each $x > 4$, if a side of an equilateral triangle is $(x - 4)$ units long, the perimeter is ______. [Why 'For each $x > 4$,' instead of 'For each $x > 0$,'?]

   (b) For each $x > \frac{3}{2}$, if a side of an equilateral triangle is $(2x - 3)$ units long, the perimeter is ______.

   (c) For each $x > \frac{28}{3}$, if a side of a square is $\left(\frac{3x}{4} - 7\right)$ units long, the perimeter is ______ units.

   (d) For each $t > 0$, for each $r > 0$, if the perimeter of a square is $16tr$, a side of the square is ______ units long.

9. (a) If a man borrows \$1200 at an annual interest rate of 3%, the total simple interest due at the end of 4 years is ______ dollars.

   (b) For each $x > 0$, for each $y > 0$, if a man borrows $30x$ dollars at an annual interest rate of 4%, the total simple interest due at the end of $2y$ years is ______ dollars.

   (c) For each $a > 0$, for each $b > 0$, for each $c > 0$, if a man borrows $15a$ dollars at an annual interest rate of b%, the total simple interest due at the end of $6c$ years is ______ dollars.

   (d) For each $x > 0$, for each $y > 0$, for each $z > 0$, if a man borrows ______ dollars at an annual interest rate of $4y$%, the total simple interest due at the end of $3z$ years is $6abc$ dollars.

   (e) For each $x > 0$, for each $y > 0$, if a person borrows x dollars at 4% per annum and y dollars at 5% per annum, the total simple interest due on these two loans at the end of 2 years is ______ dollars.

10. (a) For each $x > 0$, a pile of coins consisting of x nickels and $3x$ dimes is worth ______ cents.

   (b) For each $x > 0$, a pile of coins consisting of $2x$ nickels, $(3x + 2)$ dimes, and $(2x + 7)$ quarters is worth ______ cents.

   (c) For each $y > 0$, a pile of coins consisting of ______ nickels, $(7y + 2)$ dimes, and $(5y + 3)$ quarters is worth $(200y + 100)$ cents.

J. True or false?

1. $5 \geq -5$
2. $-1.4 \leq -1.3$
3. $-9 \not> -7$
4. $7 \neq |-7|$
5. $-6 \geq |-6|$
6. $7 - 3 = |3 - 7|$
7. $\frac{2}{73} < \frac{3}{110}$
8. $-.062 = \frac{-31}{500}$
9. $0 \nless |2 - 100|$
10. $(3 - 5)(6 - 7) = (5 - 3)(6 - 7)$
11. $(859 - 384)(7842 - 9257) = (384 - 859)(9257 - 7842)$
12. $\frac{8 - 2}{5 - 6} = \frac{2 - 8}{6 - 5}$
13. $\frac{58 - 72}{69 - 93} = \frac{58 - 72}{93 - 69}$
14. $\frac{(583 - 729)(864 - 275)}{(421 - 593)(684 - 275)} = \frac{(275 - 864)(583 - 729)}{(421 - 593)(275 - 684)}$
15. $(785 - 359)(621 - 256)(16 - 34) = (58 - 97)(42 - 42)(83 - 75)$

K. Solve these problems.

1. A television set is advertised for $97.39. If the set costs the store 75% of this price, what is the store's margin?

2. A recipe includes 2 cups of sugar, 3 tablespoons of citron, and a pinch of salt. If the recipe is to be increased to take care of 8 people, how many more cups of sugar will be needed?

3. A train travels from station A to station B in 2 hours and 20 minutes. If the distance between the two stations is 97 miles, what is the average rate of the train? [Round to the nearest mile per hour.]

4. A baseball player's batting average at the end of a season is .302. If he was "at bat" 473 times during the season, how many hits did he get?

5. Mr. Alexander has a $15,000 life insurance policy. If the annual premium rate is $11.47 per thousand dollars of insurance, what is his annual premium?

6. If Mrs. Smith buys a sofa selling at $99.50 at a discount of 18%, how much does she pay for the sofa?

7. How many hours of baby sitting at 55 cents per hour will it take to accumulate $13.75?

(continued on next page)

8. Mrs. Ashton buys a radio selling at $112. She gives $20 as a down payment, and agrees to pay $8.25 each month for a year. How much is she paying for the privilege of buying the radio on the installment plan?

9. A cement sidewalk is placed around a flower bed. If the flower bed has a circumference of 45 feet and the sidewalk is 2.9 feet wide, what is the circumference of the outer edge of the sidewalk?

10. If the wholesale price of an article is 75% of the retail price, what percent of the wholesale price is the retail price?

L. [Make a careful sketch for each problem.]

1. What is the perimeter of a rectangle whose smaller dimension is 3 units and whose larger dimension is 3 units more than 3 times the smaller dimension?

2. A semicircle is drawn on each side of the rectangle in Exercise 1 and the sides of the rectangle are erased leaving a figure with four bulges. What is the perimeter of this figure?

3. What is the perimeter of a parallelogram if each of its longer sides measures 2 more than either of its shorter sides, and if the measure of one of its shorter sides is 1 less than the average of the measures of the four sides?

4. The measure of one of the longer sides of a rectangle is 3.5 times the measure of one of the shorter sides. The sum of the measures of the four sides is 6 times the measure of a shorter side. What is the perimeter?

5. The midpoints of the four sides of a square are connected to form another square. If the perimeter of the smaller square is 8, estimate the perimeter of the larger square.

M. Simplify.

1. $\triangle + 2\triangle$
2. $3m + 5m$
3. $\triangle + \triangle + \square$
4. $3\square - \square$
5. $5\triangle + 6\triangle$
6. $15x - x - 7x$
7. 5⬡ - 3⬡ - 2⬡
8. $15x + 11x$
9. $3c + d + c + d$
10. $2x + 3x$
11. $60a + 10a + 2a$
12. $3g + 12 + g + 8$
13. $7x - 5 + 4x$
14. $45\square - 15\square$
15. $5t + 8 - 2t + 6$
16. $16 + 6y - 13 + 2y$
17. 56⬡ - 16⬡
18. $6u + 4y - 2u + 2y$
19. $5n + 6 + 7n$
20. $5x + 4x + x$
21. $9e + 6e + 3$
22. $2h + 3j + h + j$
23. $8\triangle + 2\triangle + \triangle$
24. $4a + (9a + 3)$
25. $p + (n + 2p)$
26. $9r + 10r - 3r$
27. $8y + (6y - 4z)$
28. $8 + [5 + (6 - a)]$
29. $b + 4b - 5b$
30. $\square - (\square - 2)$
31. $4a + (9 - a)$
32. $d + (c + 2d)$
33. $8r + (7 - 9r)$
34. $7x + (4x - 6)$
35. $4g - (-5h + 4g)$
36. $3(6b + 3c)$
37. $\frac{1}{2}(14c + 16d)$
38. $\frac{1}{3}(21x - 24y)$
39. $12(\frac{2}{3}m - 4m)$
40. $-8(4r - \frac{1}{4}s)$
41. $3(y)(2)(\frac{1}{3})$
42. $2\frac{1}{3}\square + 4\frac{2}{3}\square$
43. $3(a + 2) - 4a$
44. $2(b - 1) - b$
45. $3(c + 1) - 3c$
46. $4x + 7x - 2x$
47. $7g - 5 + 2g$
48. $8 + 6g - 5$

(continued on next page)

49. $7a - 2a + 3a$

50. $0.8n - 0.1n + 0.7$

51. $7w - 15 + w$

52. $5h + 3 - 2h$

53. $10d - 7d - 19$

54. $7j + 2 - 5j$

55. $8 - 9t + 3t$

56. $2p - 5 + 8p$

57. $11r - 7 - 4r$

58. $2s - 8 + 5s$

59. $7f - 4 + f$

60. $13 - 5k + 7$

61. $9 + a - 6a$

62. $2e - 2 + 2e$

63. $2r + (5 - r)$

64. $5x - (2x + 3)$

65. $13d - (d + 21)$

66. $3g - (5 - 2g)$

67. $5p + (2 - 7p)$

68. $9z - (3z - 6)$

69. $6n - (4n - 8)$

70. $3m + 4 - (m + 8)$

71. $5k + 7 - (k + 2)$

72. $7t - (-t - 9)$

73. $2x(x + y) - 3y(x - y)$

74. $a(3a + 3b) - b(a + 2b)$

75. $4a(9a - 3b) + 3b(6a - 2b)$

76. $2(5r - 5r) + 3(3r + 3s)$

77. $3x(3y - 2) - 2x(2y - 3)$

78. $5(2x + 2) + x(3x - 3) - 3xx$

79. $xx(x - y) - yy(x + y) - (x - y)$

80. $3(x - 4) + 5(x + 2)$

81. $2(y + 3) + 3(b - 4)$

82. $2(2x + 1) + (3x - 4)$

83. $3(y + 5) + 2(y - 2)$

84. $5(y + 2) - 4(y - 5)$

85. $6(y + 8) - 4(y - 4)$

86. $5(y + 3) + 4(y - 3)$

87. $3(y - 2) - 2(y + 2)$

88. $2(2a + b) - (a - b)$

89. $2(2a + 3b) - (a + b)$

90. $4(a + b) - 3a$

91. $\frac{56n}{14}$

92. $\frac{-132ab}{12}$

93. $\frac{21cd}{-\frac{1}{7}}$

94. $\frac{15n - 60r}{15}$

95. $\frac{38xy}{2x}$

96. $\frac{-46stt}{-2t}$

97. $\frac{52nnr}{-13nrr}$

98. $\frac{35abd - 5d}{5d}$

99. $\frac{8uv - 12}{\frac{1}{4}}$

100. $\frac{(r - 6)(r - 7)}{\frac{1}{2}(r - 7)}$

101. $\frac{5xy}{3ab} \times \frac{2xa}{15yb}$

102. $\frac{-3xxy}{-2yyz} \times \frac{6yz}{-9xyy}$

103. $\frac{4ab}{7bcc} \div \frac{3ac}{14bc}$

104. $\frac{5abbc}{3xyyz} \div \frac{-5axby}{9bzxy}$

105. $28\left(\frac{a}{4} + \frac{a}{14}\right)$

106. $63\left(\frac{2b + 1}{7} + \frac{b - 1}{9} - \frac{3b}{3}\right)$

107. $2y\left(\frac{5}{2y} + 7\right)$

108. $30\left(\frac{c + 5}{5} - \frac{2c - 1}{10} + \frac{3c + 2}{-6}\right)$

109. $12e\left(\frac{7}{e} + \frac{6}{4e}\right)$

110. $42\left[\frac{1}{7}(m + 4) - \frac{1}{2}(m - 2) + \frac{1}{6}\right]$

111. $5(n + 2)\left(\frac{8}{5} - \frac{n}{n + 2}\right)$

112. $16t\left(\frac{2t - 1}{2t} + \frac{3t + 2}{4} - \frac{5t - 2}{8}\right)$

113. $\frac{b}{3} - \frac{b}{8}$

114. $\frac{5r}{10} - \frac{3}{20}$

115. $\frac{h + 7}{4} - \frac{h - 2}{6}$

116. $\frac{4}{7x} - \frac{1}{2x} + 8 \cdot \frac{5}{21x}$

117. $\frac{d - 3}{5} - \frac{d + 2}{2}$

118. $\frac{k + 9}{9} + \frac{1}{3} - \frac{k - 7}{18} + 2$

119. $\frac{c + 6}{c - 2} + \frac{5}{c - 2} - 4$

120. $\frac{13}{2a - 3} + \frac{5}{a} - \frac{6}{2a - 3}$

121. $\frac{f + 4}{f - 5} + \frac{3}{7} + \frac{2}{f - 5}$

122. $\frac{g}{g + 6} - \frac{2g}{g + 7} - \frac{3g}{g + 6}$

123. $\frac{\frac{1}{3} + \frac{2}{x}}{\frac{1}{5} - \frac{3}{x}}$

124. $\frac{1 + \frac{5}{y}}{1 - \frac{5}{y}}$

125. $\frac{\frac{2}{k} - \frac{3}{k}}{\frac{1}{3k} - \frac{3}{k}}$

126. $\frac{\frac{1}{5r} + \frac{2}{3r}}{\frac{7}{2r} - \frac{1}{15r}}$

127. $\frac{1}{\frac{3}{t} - \frac{5}{r}}$

128. $\frac{\frac{a}{b} - \frac{c}{d}}{\frac{a}{b} + \frac{c}{d}}$

TEST

I. In these exercises make the substitutions indicated and tell whether the resulting sentence is true or false.

'-1' for 'a'   '4' for 'b'

'-3' for 'c'   '$\frac{1}{-5}$' for 'r'

'2' for 'y''

1. $2a + 5 = 7a - 5$
2. $3c + 2c = 5c$
3. $y + 3y = 25$
4. $b + 8 = 6 - b$
5. $8r + 6 = 4 - 2r$

II. In these expressions make the substitutions listed below and simplify the resulting numeral.

'2' for 'x'   '4' for 'z

'-3' for 'y'   '0' for 'a'

1. $(3 - 2y)4x$
2. $\dfrac{4y - 2ax}{3y}$
3. $\dfrac{3x - 2yz}{3x + 4y}$
4. $\dfrac{\frac{1}{3}az}{-\frac{2}{5}xy}$
5. $\dfrac{3xy + 4xz}{5ax - \frac{1}{2}yz}$

III. Use single quotes to punctuate the following paragraph so that it makes sense.

I wanted to have a dog as a pet because a dog is a nice animal for a pet. But I could not have a dog for a pet, because a dog is not and cannot be a pet. A dog is not even an animal. If I had a dog for a pet, I could have a lot of fun playing with it. But I could not even walk a dog!

IV. Simplify.

1. $y + y + y + y$

2. $3x + 2 + x$

3. $7n + 12n - 28$

4. $5a + 4 - 9a - 12$

5. $\frac{x}{2} + 2(x + y) - 3y$

6. $\frac{4}{5}x \cdot \frac{3}{2}xy \cdot -\frac{1}{3}y$

7. $-2(4x + 3y) + 7(2x - 5y)$

8. $12r - 3(7 + 2r) + 5(-3r + 2)$

9. $\left(\frac{1}{3}n\right)\left(\frac{1}{3}n\right)\left(\frac{1}{3}n\right)$

10. $\left(-\frac{1}{5}c\right)\left(-\frac{2}{3}d\right)$

11. $\frac{2}{3}e - \frac{5}{6}f - 3e$

12. $-\frac{1}{7}g - \frac{3}{8}h - \frac{1}{2}g - \frac{5}{8}h$

13. $\frac{-169ab}{-13a}$

14. $\frac{176abc}{8a}$

15. $\frac{-6xy}{5ab} \times \frac{5baz}{12xyu}$

16. $\frac{35nx}{3a} \times \frac{12az}{7n}$

17. $\frac{a + b}{2} - \frac{3a + b}{5}$

18. $\frac{3n - 2}{2n} + \frac{n - 1}{n}$

19. $\frac{-4c}{-5d} \div \frac{-20cd}{-5ab}$

20. $7c\left(\frac{6}{c} \div \frac{7}{c}\right)$

21. $\frac{10x - 35}{5}$

22. $\frac{4aq - 15\,aqx}{-aq}$

23. $\frac{3}{d - 6} - \frac{4d}{d - 2}$

24. $\frac{r + s}{\frac{4}{r} - \frac{5}{s}}$

V. In these exercises, write in each blank the principle or theorem [<u>not just its name</u>] which justifies the step.

1. $\forall_x \forall_y \forall_z \; (y + z)z = xz + yz.$

$$
\begin{aligned}
&(y + x)z \\
&= z(y + x) \quad \underline{\hspace{8cm}} \\
&= z(x + y) \quad \underline{\hspace{8cm}} \\
&= zx + zy \quad \underline{\hspace{8cm}} \\
&= xz + yz. \quad \underline{\hspace{8cm}}
\end{aligned}
$$

2. $\forall_x \forall_y \forall_z \; x + y - z = (y - z) + x.$

$$
\begin{aligned}
&x + y - z \\
&= x + y + -z \quad \underline{\hspace{8cm}} \\
&= x + [y + -z] \quad \underline{\hspace{8cm}} \\
&= [y + -z] + x \quad \underline{\hspace{8cm}} \\
&= (y - z) + x. \quad \underline{\hspace{8cm}}
\end{aligned}
$$

3. $\forall_a \forall_b \; -(a - b) = b - a.$

$$
\begin{aligned}
&-(a - b) \\
&= -1(a - b) \quad \underline{\hspace{8cm}} \\
&= -1(a + -b) \quad \underline{\hspace{8cm}} \\
&= -1a + -1 \cdot -b \quad \underline{\hspace{8cm}} \\
&= -1a + 1b \quad \underline{\hspace{8cm}} \\
&= 1b + -1a \quad \underline{\hspace{8cm}} \\
&= 1b + -a \quad \underline{\hspace{8cm}} \\
&= 1b - a \quad \underline{\hspace{8cm}} \\
&= b - a. \quad \underline{\hspace{8cm}}
\end{aligned}
$$

VI. In each blank at the right, indicate with a 'T' or an 'F' whether the statement is true or false.

1. For each x, $+x$ is a positive number. (1) ______

2. For each a, $a = {}^{+}|a|$. (2) ______

3. For each x, for each y, the product of x by y is xy. (3) ______

4. For each c, for each d, $c \cdot -d \leq 0$. (4) ______

5. For each $\square$, for each $\triangle$, if $\square + \triangle = 0$, then $\square = -\triangle$. (5) ______

6. For each m, for each n, $-m \cdot -n = mn$. (6) ______

7. For each x, for each y, $|x + y| \geq |x| + |y|$. (7) ______

8. For each r, for each s, $rs > 0$. (8) ______

9. For each m, $8m + m = 8mm$. (9) ______

10. For each s, for each t, $-(t - s) = s - t$. (10) ______

VII. Examine the pairs listed below, and put a '$\surd$' in the blank alongside each one whose first component is greater than or equal to its second component.

1.	$(.5,\ .5 \times .5)$	______	2.	$(-|42|,\ -|-42|)$	______
3.	$(-12,\ 10)$	______	4.	$(-5,\ -8)$	______
5.	$(-10,\ -26)$	______	6.	$(|-5|,\ |-6|)$	______
7.	$(-\frac{1}{4},\ -\frac{1}{2})$	______	8.	$(\frac{-23}{7},\ \frac{-7}{23})$	______
9.	$(-1000,\ -999)$	______	10.	$(-200,\ 0)$	______

VIII. 1. List 3 pairs of real numbers which belong to the operation multiplying by -2.

2. List 3 pairs of real numbers which belong to the operation which is the inverse of adding $^{-}5$.

3. List 3 pairs of real numbers which belong to the operation multiplying by zero.

4. List 3 pairs of real numbers which belong to the operation which is the inverse of multiplying by $^{-}3$.

5. List 3 pairs of real numbers which belong to the operation dividing by $-\frac{1}{2}$.

IX. Complete with the simplest expression which makes the statement true.

1. For each n, the sum of 3n and 10 is ________.

2. For each r, the product of 8 by $-3r$ is ________.

3. For each c, the difference of 12 from $c + 12$ is ________.

4. For each y, y increased by 9 is ________.

5. For each a, for each b, 5a decreased by $2a + 7b$ is ________.

6. For each x, for each $y \neq 0$, the quotient of $3xy - 6y + 9xy - 3y$ by $3y$ is ____________.

7. For each $x > 0$, for each $y > 0$, the perimeter of a rectangle with short side 3x units long and long side 2y units long is ______________.

8. For each $a > 0$, if 3a objects are to be distributed equally among 5 persons, then each person will receive __________ objects.

9. For each $m > 0$, a car traveling at a steady rate of 3m miles an hour will travel 150 miles in _________ hours.

X. Each of the following statements is a consequence of one of the principles for real numbers. Below them are the names of these principles, each being preceded by a letter. In the blank at the left of each statement, write the letter corresponding to the principle of which the statement is a consequence.

_______ 1. $4 + (5 + 8) = (4 + 5) + 8$

_______ 2. $(-3 + 0) + 7 = -3 + 7$

_______ 3. $-4 \cdot 12 + 7 \cdot 12 = (-4 + 7)12$

_______ 4. ${}^{-}13 \cdot 2 \cdot 1 = {}^{-}13 \cdot 2$

_______ 5. $(9 \cdot 8)5 + 10 = 9(8 \cdot 5) + 10$

_______ 6. $15 + (6 + 2) = (6 + 2) + 15$

_______ 7. $(-19 + 7) - 15 = (-19 + 7) + (-15)$

_______ 8. $-3[(5 + 2)6] = -3[6(5 + 2)]$

_______ 9. $9(8 \cdot 0) = 9 \cdot 0$

_______ 10. $578 + -578 = 0$

_______ 11. $63 + 2 = \frac{63 + 2}{-13} \cdot -13$

A. Commutative principle for addition

B. Commutative principle for multiplication

C. Associative principle for addition

D. Associative principle for multiplication

E. Distributive principle for multiplication over addition

F. Principle for adding 0

G. Principle for multiplying by 0

H. Principle for multiplying by 1

I. Principle of opposites

J. Principle of quotients

K. Principle for subtraction

SUPPLEMENTARY EXERCISES

A. Find the value of each of the following pronumeral expressions for the given values of the pronumerals.

Pronumeral	'x'	'y'	'u'	'v'
Value	5	3	2	-4

1. $x + y$
2. $u + v$
3. $u - v$
4. $2x$
5. $7y$
6. $8u$
7. $63 + u$
8. $6 \cdot 3 + y$
9. $6.3 - u$
10. $11 - x$
11. $x - 11$
12. $-3x$
13. $-4v$
14. $y - x$
15. $x - y$
16. $6y - 2$
17. $4u + 5$
18. $5 + 4u$
19. $3 + 7x$
20. $(3 + 7)x$
21. $3x + 7x$
22. $3 + 3u$
23. $3(1 + u)$
24. $3 + u$
25. $1 + 5y$
26. $6 + y$
27. $6y$
28. xx
29. $xx - 55$
30. vv
31. $1 + uu$
32. $(1 + u)u$
33. $u + uu$
34. $2x + y$
35. $2x + 3y$
36. $2(x + 3y)$
37. $2uxy$
38. $3uvx$
39. $-5xvy$
40. $x + 5y + u$
41. $7x - 2y - v$
42. $3u - 2v + 4x$
43. $6x + 2v - 7$
44. $19 - 3x - 3x$
45. $12 + 2u - 7$
46. $\frac{3x}{y}$
47. $\frac{7y}{3u}$
48. $\frac{9v}{8y}$
49. $\frac{-2v}{7u}$
50. $\frac{x - y}{5}$
51. $\frac{x + y}{y - x}$
52. $\frac{xx - 1}{yy + 2}$
53. $-\frac{u + v}{2v}$
54. $\frac{5x - 3u}{7v + y}$
55. $2(x + 3) + 5(x - 7)$
56. $3(u + 2) + 7(8 + u)$
57. $4(7 - y) - x + y$
58. $5(2x + 4) - 3(1 - 2y)$
59. $8(u - v + x) + 5(u - v + x)$
60. $7u - 3x(2 - 5y + v)$

B. In each of the following pronumeral expressions make the substitutions listed below and simplify the resulting numeral.

'7' for 'm',	'0' for 'y',
'2' for 'x',	'-3' for 'z',
'$\frac{3}{4}$' for 'v',	'-8' for 'u'.

1.	$m + x - y$	2.	$3m + 2x$
3.	$6x - 3z + 59y$	4.	$5x - 2u + 3m$
5.	$6u + 8v \div x$	6.	$-x - z - 2u$
7.	$12v - 3y + 8z$	8.	$-v + 3x + 2z$
9.	$11m - 2x + 3u$	10.	$3(x + z) + 5$
11.	$x(2m - x) + 11$	12.	$5z(x - 3u) + v$
13.	$\frac{x + z + y}{m}$	14.	$\frac{m}{x} \cdot \frac{m}{z}$
15.	$\frac{z}{m} \cdot m$	16.	$(m + y)(m - y)$
17.	$(m + x)(m - x)$	18.	$2m + 3x + 4.2y$
19.	$2(m + 2x) + z$	20.	$mx + my + mx$
21.	$z(m + x) + mx$	22.	$4xv + 8mv + xu$
23.	$-5zu + 3xuzy$	24.	$3m - (x - 2u)$
25.	$\frac{m + (z + x) + 4z}{2x}$	26.	$\frac{m}{z} \div \frac{y}{m}$
27.	$\frac{2m}{3x} \cdot \frac{x}{mz}$	28.	$\frac{mx + yz}{m - x}$
29.	$\frac{x - z}{2m + 3y}$	30.	$\frac{xz + mz}{3x + z}$
31.	$\frac{2}{3}x + \frac{4}{3}v + m$	32.	$(\frac{1}{3}z)(12v) - 5x$
33.	$(\frac{2}{7}xm)(\frac{4}{7}xv)$	34.	$2y + 4z - 3.5m$
35.	$8z + 5y - 3x$	36.	$5mx - 4yz + 8xz$
37.	$\frac{m}{z} + \frac{y}{x} \div \frac{z}{m}$	38.	$\frac{4mz}{3xz}$
39.	$\frac{25xz + 2my}{5z - 5x}$	40.	$3x[4(z - 2x) - 117y]$

C. Use each of the following open sentences to generate a statement. Tell whether the statement is true or false. Try to generate some true statements and some false statements.

1. $4y - 6 = 30$
2. $10m - m + 1 = 10$
3. $4 = 4b - b + b$
4. $x - \frac{1}{2}x - 2 = 1$
5. $a - 3b = -3b + a$
6. $7.5r - 22.5 = (3 - r)7.5$
7. $c - \frac{1}{3}c - 7 = 1$
8. $18 + 3(y - 5) = 12$
9. $4ww - 8.5 = 7.5$
10. $(2m + 3)(2m - 3) = 4mm - 9$
11. $10(g - 5) + 30 = 20$
12. $1.005 + d < d$
13. $16 - (s - 10) > 0$
14. $5(n - 1) - 15 \leq 5$
15. $7d - (d - 10) \geq 0$
16. $p = p - 1$
17. $|x - 1| = 7$
18. $|x + 2| + 8 = 5$
19. $rr - 26r + 169 = (r - 13)(r - 13)$
20. $\frac{4}{9}kk - 64 = (\frac{2}{3}k - 8)(\frac{2}{3}k + 8)$

D. Here are test-patterns for generalizations. Your job is to give the reasons for the steps in the proof.

1. For each x, $4x(2x) = 8(xx)$.

$4x(2x) = 4x2x$ [__________] (a)

$4x2x = 4(x2)x$ [__________] (b)

$4(x2)x = 4(2x)x$ [__________] (c)

$4(2x)x = 4 \cdot 2xx$ [__________] (d)

$4 \cdot 2xx = 8xx$ [__________] (e)

$8xx = 8(xx)$. [__________] (f)

2. For each y, $2y + 5 + 3y + 8 = 5y + 13$.

$2y + 5 + 3y + 8 = 2y + (5 + 3y) + 8$ [__________] (a)

$2y + (5 + 3y) + 8 = 2y + (3y + 5) + 8$ [__________] (b)

$2y + (3y + 5) + 8 = (2y + 3y) + 5 + 8$ [__________] (c)

$(2y + 3y) + 5 + 8 = (2 + 3)y + 5 + 8$ [__________] (d)

$(2 + 3)y + 5 + 8 = 5y + 5 + 8$ [__________] (e)

$5y + 5 + 8 = 5y + (5 + 8)$ [__________] (f)

$5y + (5 + 8) = 5y + 13$. [__________] (g)

3. For each k, $2(7k + 1) + 3(2k + 5) = 20k + 17$.

$2(7k + 1) + 3(2k + 5) = 2(7k) + 2 \cdot 1 + [3(2k) + 3 \cdot 5]$ [__________](a)

$2(7k) + 2 \cdot 1 + [3(2k) + 3 \cdot 5] = (2 \cdot 7)k + 2 \cdot 1 + [(3 \cdot 2)k + 3 \cdot 5]$ [____, ____](b)

$(2 \cdot 7)k + 2 \cdot 1 + [(3 \cdot 2)k + 3 \cdot 5] = 14k + 2 \cdot 1 + [6k + 15]$ [____, ____, ____](c)

$14k + 2 \cdot 1 + [6k + 15] = 14k + 2 + [6k + 15]$ [__________](d)

$14k + 2 + [6k + 15] = 14k + 2 + 6k + 15$ [__________](e)

$14k + 2 + 6k + 15 = 14k + (2 + 6k) + 15$ [__________](f)

$14k + (2 + 6k) + 15 = 14k + (6k + 2) + 15$ [__________](g)

$14k + (6k + 2) + 15 = 14k + 6k + 2 + 15$ [__________](h)

$14k + 6k + 2 + 15 = (14 + 6)k + 2 + 15$ [__________](i)

$(14 + 6)k + 2 + 15 = (14 + 6)k + (2 + 15)$ [__________](j)

$(14 + 6)k + (2 + 15) = 20k + 17$. [____, ____](k)

E. Each of the following is a generalization about real numbers. Some are true and some are false. Your job is to decide which, and in each case to give either a counter-example or a proof.

1. For each x, $1 + 3x + 5 = 3x + 6$.
2. No matter what number you pick, if you add 4 to it, and multiply 2 by this sum, the result is the product of 2 by the chosen number, plus 8.
3. For each t, $3t + 9t = 12t$.
4. For each k, $3(5k) + 6(2k) = 27k$.
5. For each m, $9 + m + 7 = 16 + m$.
6. For each y, $3 + 8y = 11y$.
7. For each r, $5 + 10r = 5(1 + 2r)$.
8. For each k, $3k + 5 + 9k = 17k$.
9. For each t, $7t + 6 + 8t = 15t + 6$.
10. For each q, $2q + 9 + 7q = 9(q + 1)$.
11. For each a, $3a(7a) = 21(aa)$.
12. For each r, $7r + 1 + 3r + 5 = 10r + 6$.
13. For each n, $n(n + 2) + 6n = nn + 8n$.

F. Each of the following generalizations is a consequence of one of the principles of real numbers. Tell which principle.

1. For each x, $3x(x + 5) = (x + 5)(3x)$.
2. For each y, $7 + (y + 3) + 5 = (y + 3) + 7 + 5$.
3. For each r, $(2r + 1)(3r + 7) = 2r(3r + 7) + 1(3r + 7)$.
4. For each k, $7k + 3 - (8k + 5) = 7k + 3 + -(8k + 5)$.
5. For each y, $3(y + 12) + 3(y + 11) = 3[(y + 12) + (y + 11)]$.
6. For each m, $(m + 4)(m + 3)(m + 1) = (m + 4)[(m + 3)(m + 1)]$.
7. For each x, $3(x - 5) + 6(x + 9) + 12(x - 2) = 3(x - 5) + [6(x + 9) + 12(x - 2)]$.

G. Simplify.

1. $3a + 5a$
2. $7b + 12b$
3. $2x + 9x$
4. $8t + 17t$
5. $9r + 41r$
6. $x + 3x$
7. $9x + 5x + 3x$
8. $4y + 7y + 11y$
9. $2a + 3 + 12a$
10. $7a + 1 + 15a$
11. $6b + 4b + 115$
12. $31 + k + 7k$
13. $7r + 2 + 3r$
14. $6p + p + 12 + 3p$
15. $6x + 3x + 7$
16. $x + 2x + 5 + x$
17. $5p + 0.5p + 0.7p$
18. $11w + w + 4$
19. $4r + 9 + 5r$
20. $12 + 7d + 19 + 9d + 10d$
21. $6a + 2b + 8a + 2c$
22. $4m + 9n + 15m + 11n$
23. $2x + 5 + 3y + 7x$
24. $11a + 2b + 5 + a + 6$
25. $4t + 13s + 7t + 5s$
26. $4m + 2m + 7 + 10m$
27. $14s + r + 3r + 9s$
28. $9x + 7y + 5 + 5x + 6y$
29. $\frac{1}{5}a + \frac{1}{4}b + \frac{3}{4}b + \frac{3}{5}a$
30. $\frac{1}{3}b + \frac{1}{6}b + 5 + \frac{1}{9}b$
31. $(-9)k + (-6)k + 12k$
32. $M + 6M + (-5)M + (-1)M$
33. $u + (-1)v + (-2)u$
34. $7t + 4 + (-10)t + -12$
35. $x + (x + 1)$
36. $2(b + 4) + b$
37. $3(7k + 7) + 5k$
38. $10(5a) + 5$
39. $2(j + 4) + 3j$
40. $3x + 4(.25x)$
41. $x + 2(-2x)$
42. $7x + (8x + 54)$
43. $9f + (7f + 67)$
44. $3x + 5(x + 2)$
45. $4x + 5(x + 3)$
46. $2(a + 5) + 5(a + 3)$
47. $2a(ab) + 3b(ab) + 5$
48. $b(2a + 1) + a(3b + 5)$
49. $(-1)t + (-5)t + (-9)t + (-6)t$
50. $15c + (-9)d + 13 + (-18)c + (-20)d$
51. $17m + (-7)n + 8p + (-5)n + (-10)m$

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52. $\frac{1}{5}x + \left(-\frac{1}{4}\right)y + \frac{4}{5}x + \left(-\frac{2}{7}\right)y$

53. $2.5s + 7.5r + 9.4s + (-11.6)r$

54. $12j + -5 + 11k + (-7)h + 19 + (-7)h$

55. $(-1)c + (-4)c + (-9)c + (-3)c + 6c$

56. $9u + (-2)v + 12 + (-19)u + (-12)v$

57. $9d + (-7)e + 6d + (-2)d + e$

58. $\frac{3}{8}y + -\frac{1}{3} + \left(-\frac{7}{8}\right)c + 1$

59. $\frac{1}{3}t + \left(-\frac{2}{5}\right)s + \frac{3}{4}t + \left(-\frac{1}{2}\right)s$

60. $2.6g + 4.5h + 3.7g + (-5.4)h$

61. $4.8e + (-9.6)f + (-5.4)e + f$

62. $\frac{4}{9}c + -\frac{1}{3} + \left(-\frac{5}{9}\right)c + 3$

63. $7\frac{1}{5}x + \left(-14\frac{2}{5}\right)y + \left(-1\frac{4}{5}\right)x + y$

64. $3(3n + 2y) + 5(n + 4y)$

65. $4[r + (-5)s] + (-7)[r + (-3)s]$

66. $\left(-\frac{1}{5}\right)(25z + 15w) + \frac{2}{3}(3z + 12w)$

67. $(-10)(5c + 6d) + 8(8c + 9d)$

68. $4(3m + 2n) + 5(3m + 7n)$

69. $(-1)(10y + 3) + (-2)(5y + 1)$

70. $7(6 + 5w) + (-4)(5 + w)$

71. $8(5 + 3t) + 5(2 + t)$

72. $4(3p + 7) + 2(4 + p)$

73. $7(a + b) + (-14)(a + b)$

74. $6(w + 3u) + 4(w + v)$

75. $4b(b + 3) + 6b(3b + 4)$

76. $5a(a + 4) + 10a(3a + 2)$

77. $3x(x + 9) + 7x(3 + 4x)$

78. $2xy(5x + 36) + 4xy(6y + 9x)$

79. $6(2a + 4b) + 10a + 18b$

80. $\frac{1}{5}(10c + 5d) + \frac{1}{3}(3d + 6c)$

81. $\frac{1}{6}(3w + 6v) + \left(-\frac{1}{3}\right)(3v + 9u)$

82. $3rt(2r + 5t + 4) + (-2)rt(6 + 3r + 4t)$

83. $6(m + n) + (-3)(2m + n) + (-2)(n + m) + 4(m + n)$

84. $2(sst + tt) + 4(ss + 2tt) + (-3)(t + s)$

85. $2(3aaa + ab) + (-1)(5aab + ab) + (-6)(bb + abb) + 2(bb + aaa)$

86. $\left(\frac{1}{2}t\right)\left(\frac{1}{2}t\right)\left(\frac{1}{2}t\right)$

87. $\left[\left(-\frac{1}{4}\right)g\right]\left[\left(-\frac{2}{3}\right)h\right]$

88. $1.2(5e)(6.5e)$

89. $9r(3s)(2t)$

90. $c(8b)(-9)d$

91. $5n(7nn)(3nn)$

92. $(4.5ef)(2e)(-7)f$

93. $(-3)r(-1.5)s$

94. $7a(-3)b(-4)ac$

95. $3c(-2)d(5c)$

96. $4\triangle(3\square)(-6)\triangle$

97. $-2.3(5\square)(-3)\bigcirc$

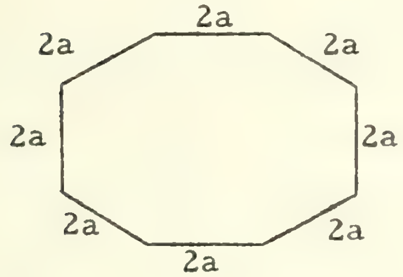
98. $[(6p)(-3)q][(-2)3p(-5)q]$

99. $7ab[(2a + 5b) + (2b + 6a)] + 4ab[(9a + 4b) + (3b + 4a)]$

100. $\frac{1}{6}[(12d + 18) + (9d + 30c)] + \frac{1}{8}[(4d + 2c) + (16 + 12c)]$

H. Write the simplest formula you can for the perimeter of each figure pictured below.

1.



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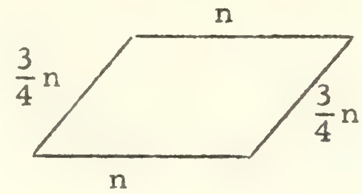

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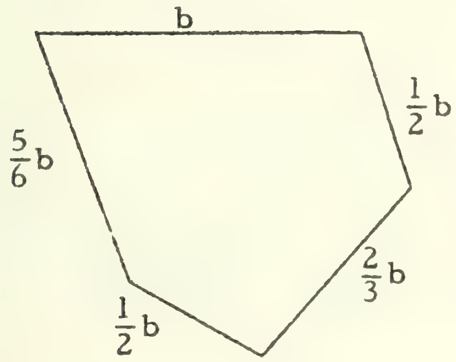

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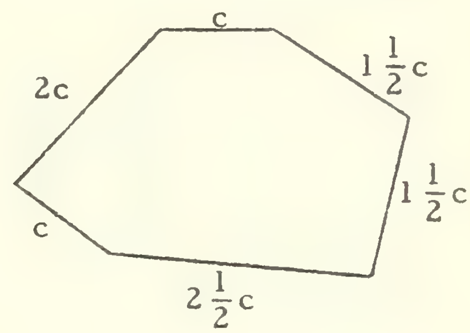

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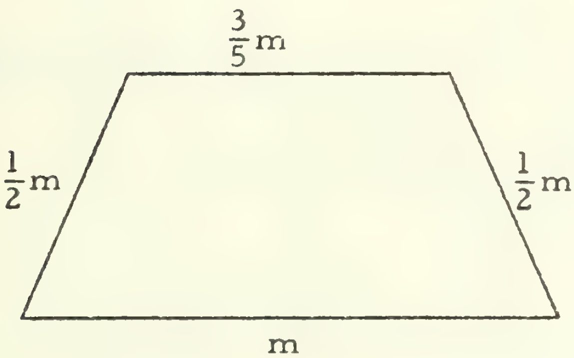

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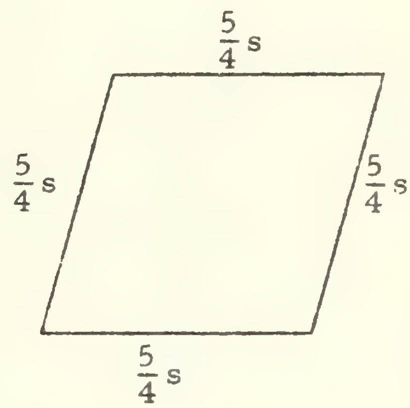

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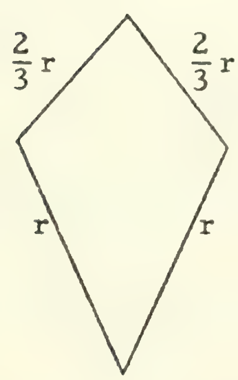

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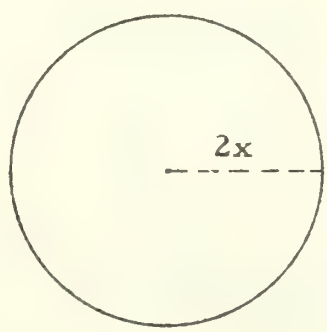

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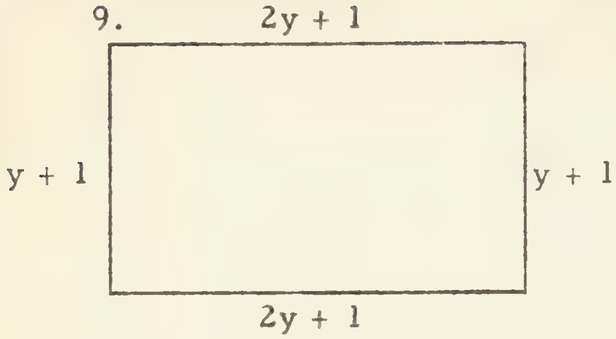

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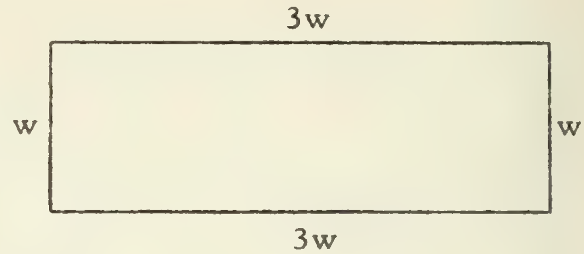

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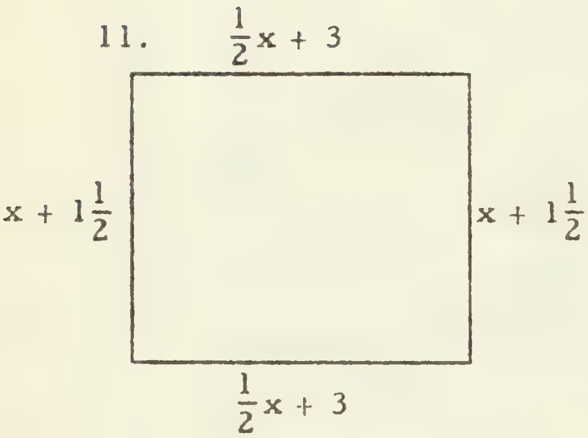

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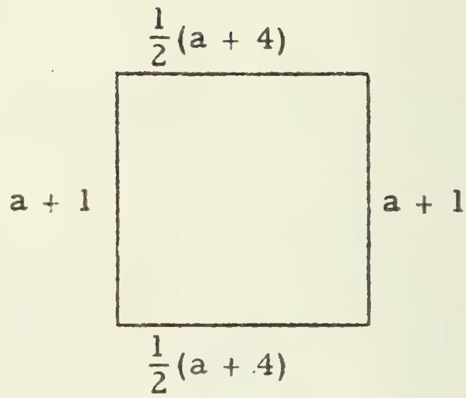

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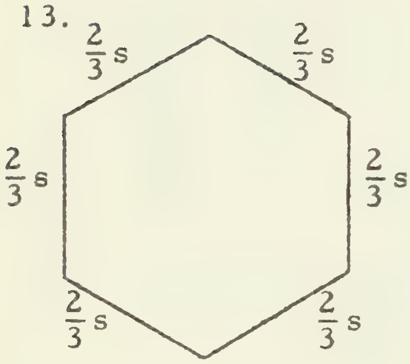

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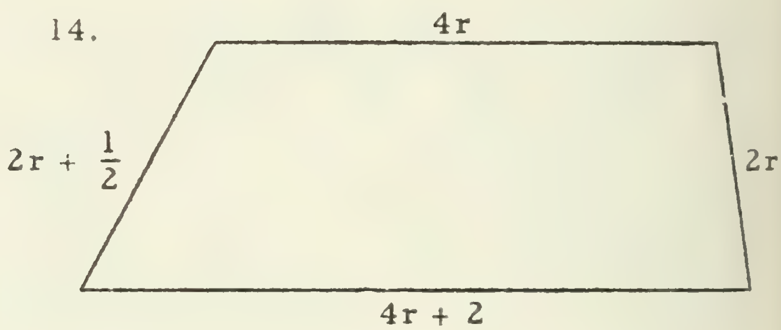

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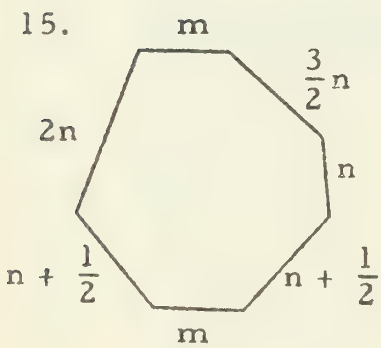

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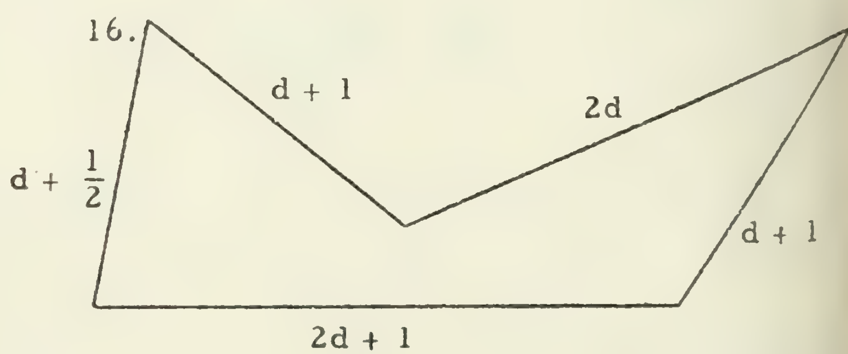


Write the simplest formula you can for the perimeter of each figure described below.

17. A parallelogram whose longer side measures 3 more than 5 times the measure of its shorter side.

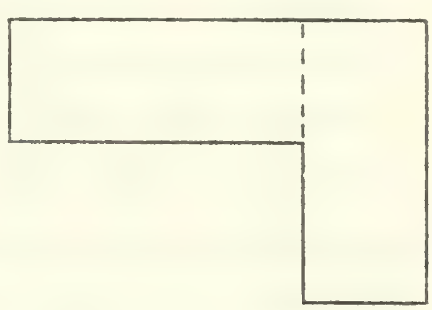
18. An isosceles trapezoid with its non-parallel sides measuring $\frac{2}{3}$ as long as the shorter of the parallel sides, and with the longer of its parallel sides measuring 5 more than the shorter one.

19. A rectangle whose length measures 8 more than $\frac{3}{4}$ the sum of the measures of its two shorter sides.

20. A pentagon whose longest and shortest sides differ in measure by 12; of the three remaining sides, one is three times as long as the shortest side of the pentagon; another is 7 units longer than $\frac{2}{3}$ the length of the shortest side; the last is 1 unit longer than 5 times the length of the shortest side.

21. A hexagon which has
    first side measuring 1 more than $\frac{1}{2}$ the fourth side;
    second side measuring 2 more than $\frac{1}{3}$ the fourth side;
    third side measuring 1 more than $\frac{3}{4}$ the fourth side;
    fifth side measuring twice the fourth side, and
    sixth side measuring 5 more than $\frac{2}{3}$ the fourth side.

22. A hexagon made by placing two rectangles together like this: The rectangles have the same perimeter, and the length of each rectangle is one unit more than twice its width.



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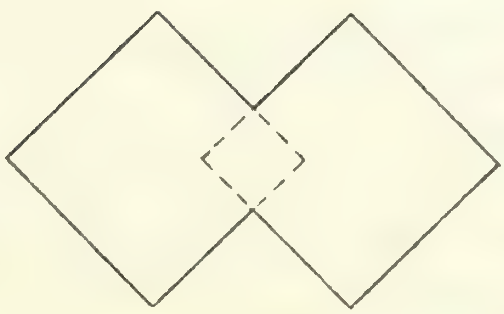

An octagon formed by placing two squares with equal perimeters in this position. The intersecting sides divide their lengths in a 2 to 1 ratio.

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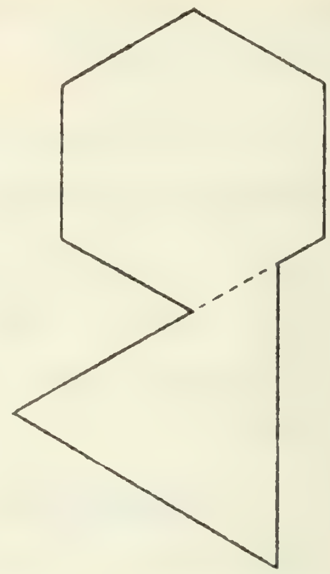
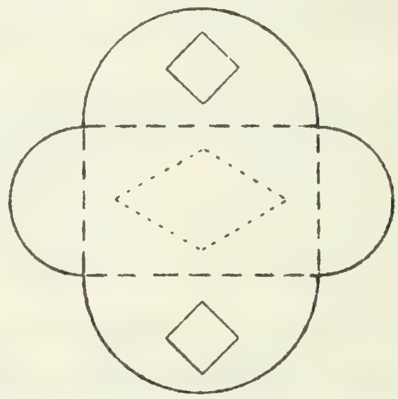
24. A nonagon made by placing an equilateral triangle next to a regular hexagon so that $\frac{1}{3}$ the length of a side of the triangle overlaps a side of the hexagon. A side of the triangle is twice as long as a side of the hexagon.

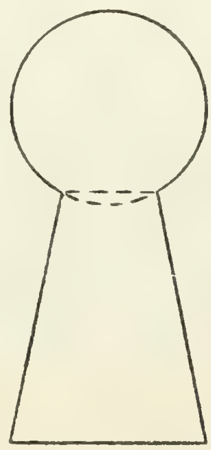


25. A trapezoid the shorter of whose parallel sides measures 3 more than $\frac{2}{3}$ the measure of the longer parallel side, one of whose non-parallel sides is $\frac{1}{6}$ as long as the longer parallel side, and whose other non-parallel side is $\frac{1}{2}$ unit more than $\frac{1}{6}$ the longer parallel side.

26. A formal flower garden is laid out as in this sketch. The rectangular part is $1\frac{3}{5}$ times as long as it is wide; each of the curved edges is a semicircle; the large diamond-shaped part is a rhombus whose side is $\frac{2}{3}$ of the width of the rectangular part; the small squares each have a side measuring $\frac{1}{3}$ the width of the rectangle. Find the formulas for the perimeter of each of the following: the rectangle, the large rhombus, each square, and the figure formed by the four semicircles.



27. A keyhole is of the shape pictured here. The lower part is trapezoidal in shape, with a base edge of the same measure as the diameter of the circular part, and with its non-parallel edges each $1\frac{1}{4}$ times as long as the base edge. The trapezoidal part intersects the circular part in such a way that $\frac{1}{6}$ of the circle is missing. Find a formula for the perimeter of the keyhole.



I Simplify.

1. $7x + 3y - 2x$
2. $9y - 5z - 6y - z$
3. $6a + 5b + 1$
4. $7c + 3x - 2c - 5x$
5. $3x - 13y - 7y - x$
6. $8r + 2r - 5t + 9t$
7. $\frac{1}{3}x + \frac{1}{4}y - \frac{2}{3}x + \frac{1}{4}y$
8. $\frac{1}{5}a + \frac{1}{3}b - \frac{1}{3}a - \frac{1}{5}b$
9. $3ab + 7c - 2ab + 5c$
10. $8xy - 3zy - 7yz - 9xy$
11. $4x - 3y - 17y - \frac{1}{5}x$
12. $-x - x - \frac{2}{3}y + \frac{7}{5}x$
13. $2r + 1 + 7s - 3$
14. $8t - 15 - 6 - 3t$
15. $-3m - 2mn + 5m$
16. $-2xy - 3yx - 5x - 6y$
17. $\frac{1}{2}x - \frac{3}{4}y - 2x$
18. $-\frac{1}{5}a - \frac{3}{5}b - \frac{1}{4}a + \frac{1}{5}b$
19. $(3 + k)x - kx$
20. $5 + 3r(s + r) + 5sr$
21. $3 - 2j + 4(2 - j)$
22. $1 + \frac{1}{2}x - \frac{1}{4}x + \frac{1}{8}x - x$

23. $7(a - 3k) + 9k$
24. $5(8r - 2s) - 7s$
25. $2(j - 5m) + j - m$
26. $2(-5r - 8t) - 3r - 7t$
27. $9(X - 2Y) + 3(2X - Y) + 4(-X - Y)$
28. $2(6p - 7r) + 9(r - 2s) + 7(1 + r - s - p)$
29. $-5(1 + 3k) + 5k - 7$
30. $-2(5x + 3y) - y - 7x$
31. $2k(k - 5m) + 5mk + 8$
32. $-3y(-y - 8x) - x(2y + 1 - x)$

33. $9a + 11b - (a - b)$
34. $12r + 7s - (-r - s)$
35. $6n - 11p - (10n - 12p)$
36. $15c - 10d - (13c - 7d)$
37. $25y - (9 - 3z + 8x)$
38. $37g - (8g - 3h - 6) + 5h$
39. $f - (6q + 3f) - 4f$
40. $t - (9t - 7s) - (10s - 8t)$

41. $5x - 3(16 - 4x)$
42. $5(x + 1) - 3x$
43. $3a + 2(3a - 3)$
44. $7x - 4(3 - 2x)$

(continued on next page)

45. $2x + 3(5 - 3x)$

46. $7b - 9(3b + 24)$

47. $2x + 5(-2x)$

48. $3(2x - 2) + 2x$

49. $3z - 2(-5 - 2z)$

50. $4x - (2x + 2)$

51. $3(-1 - 2y) - 2y$

52. $2(y + 4) - 5y$

53. $7x - 3(2x + 1)$

54. $5x + 3(3 - x)$

55. $2(8 + 2b) + 3b$

56. $2(5 - s) + 7s$

57. $2(8 - 3q) - 10q$

58. $3(x - 4) + x$

59. $5(6 - 3y) + 7y$

60. $r - (7 - 3r)$

61. $8x - (5 - 2x)$

62. $7x - 9(3x + 24)$

63. $4a - 3(8 - a)$

64. $3x + 2(19 - 3x)$

65. $x - (11 - 2x)$

66. $4y - 2(3y - 4)$

67. $2(3y) - 5y$

68. $k - [-3k - (-2k - 1)]$

69. $a + b + 2(a - b)$

70. $2(2a + b) - 3(3a - 2b)$

71. $3(3x - 5) + 4(5x - 3)$

72. $4(4n - 1) - 5(n + 2)$

73. $2(5x - 3) - 3(3x + 11)$

74. $4(3s + 5) - 3(s - 3)$

75. $3xx - 2[x - (3 - 2x)]$

76. $2xx - [x - (1 - 2x)]$

77. $7 - [x + (3 - 2x)]$

78. $-(2a - 4) - 18 + a$

79. $5k - 4(k - 1)$

80. $4(z + 1) - 7z$

81. $5 - 3(x - 3)$

82. $4y - (5y - 1)$

83. $8n - 9(3p - 4n)$

84. $11r - 7(-2s - 6r)$

85. $12j - 3(4h - 5j)$

86. $21t - 8(9u - 2t)$

87. $5a - 2b - 7(-4a - 3b)$

88. $6(3c + 8d) - 12(-c - d)$

89. $1.5e - (.5e - 2.5f) - 4(4e - 10f) - 6(-3e - 5f) - 8e$

90. $-7p - q(p - 3n) - 6(-p - 2 - 5n) - q(2 - p - 4n)$

91. $7C - 2(C - D + 3) - 8(3C + 4 - 4D) - 7(2D - 2C)$

92. $-x(13z - 3u) - 3x(6z - 5u)$

93. $-3r(7r - 3s) - 9s(r - 2s)$

94. $-14g - 8(-h + 3g) - h$

95. $17d + 19e - 13(d - 2e - 2)$

96. $-4(8 - 2c + 7b) - 6(1 - 3c + 3b) - (-c - b - 5)$

97. $-8(7 - v - 2u) - 5[3 - 4v - 3u - (3v - 2u)]$

98. $-5[4(k - n) - 9(3k - 4n)] - 2[-8(-k - 4n) - (-k - n)]$

99. $-2a(5b)(-3z)$

100. $4a(-b)(-2c)$

101. $6r(-\frac{1}{6}r)(7r)$

102. $7r(-\frac{2}{7}r)(-3r)$

103. $-A \cdot -3B \cdot -8C$

104. $-X \cdot --Y \cdot ---3Z$

105. $2rs \cdot -3st \cdot -8tr$

106. $-abc \cdot -3ab \cdot -2bc \cdot -ca$

107. $(5 \cdot -4 \cdot -t)(-5 \cdot 4 \cdot -t)$

108. $8ab \cdot 1 \cdot 7.53a \cdot 0 \cdot 4ab$

109. $-8r(2 - 5s + 6r) - 3r(5 - s - r) - 4(rr + ss)$

110. $-2j(3k - 5j) + 7kj(1 - 3k) - 5jk(-2 - j)$

111. $-3w(u - 2v) - v(-w - 2u) - (u - v - w)$

112. $8a[5 - 3(1 - 2a - b)] - [4a - 1 - b(5 - a + 2b)]$

113. $9(-3x) - 2(-4x - 5) - 3x[2 - 5(-1 - x)]$

114. $4y(3 - 2y) + 7(1 - yy) - y[-(3 - y)] + 8$

115. $3(xx - 2x - 3) - 2x(-5x - 3) - 3(-x - 2) - 1$

116. $-(x - 3)(-2x) - (-xx - x + 1) - (3 - 2x - xx)$

117. $\frac{1}{2}(-4x)(-8y)z - \frac{2}{9}(3y)(-2z)(-3x) - \frac{1}{7}(-14xy)(-z)$

118. $2.5(3 - 1.2x) - 7.8(4.1 - 6.3x) - 2.9(-x - 5.8)$

119. $\frac{2}{3}(12 - .6n) - \frac{1}{9}(18 - 2.7) - \frac{5}{6}(18 - 2.4n)$

120. $.3(2a)(-2b)c - .2(-3c)(-8b)(-a) - .4(-2b)(3a)(-4c)$

J. Simplify.

1. $\frac{3}{5} \times 5$	2. $\frac{^-3}{^+4} \times {}^+4$	3. $\frac{6}{7} \times (3 + 4)$
4. ${}^-5 \times \frac{4}{^-5}$	5. $81 \times \frac{17}{100 - 19}$	6. ${}^+5 \times \frac{4}{7 - 12}$
7. $\frac{1}{2} \times \frac{9}{1/2}$	8. $\frac{3}{4} \times \frac{5}{7}$	9. $\frac{2}{5} \times \frac{8}{3}$
10. $\frac{^-7}{^-5} \times \frac{^+2}{^-3}$	11. $-\frac{4}{9} \cdot -\frac{4}{9}$	12. $\frac{17}{9} \times \frac{101}{102}$
13. $4 \div \frac{3}{5}$	14. ${}^-7 \div \frac{5}{8}$	15. ${}^-2 \div \frac{^-3}{^-7}$
16. $\frac{4 \times 9}{9}$	17. $\frac{^-8 \times {}^-5}{^-5}$	18. $\frac{^+6 \times 2/5}{2/5}$
19. $\frac{7 \times 4}{9 \times 4}$	20. $\frac{38}{42}$	21. $\frac{^-160}{^-970}$
22. $7 \times \frac{5}{3}$	23. $4 \times \frac{1}{5}$	24. $7 \times \frac{1/7}{15}$
25. $\frac{4}{5} \div 3$	26. $\frac{^+5}{^+17} \div {}^-2$	27. $\frac{^-1}{^-3} \div 2$
28. $\frac{3}{4} \div \frac{2}{7}$	29. $\frac{^-8}{5} \div \frac{1}{^-3}$	30. $\frac{7}{^-3} \div \frac{4}{^-7}$
31. $-\frac{5}{4} \div \frac{31}{49}$	32. $\frac{2 - 7}{4} \div \frac{3}{4 + 7}$	33. $\frac{12}{5} \div \frac{61}{5}$
34. $\frac{13}{33} \div \frac{^-2}{33}$	35. $\frac{6 \cdot 4}{5 \cdot 7} \div \frac{11}{5 \cdot 7}$	36. $\frac{15}{5 + 3} \div \frac{17}{2 \cdot 4}$
37. $\dfrac{\frac{16}{3}}{\frac{15}{3}}$	38. $\dfrac{\frac{18}{7}}{\frac{^-43}{5 + 2}}$	39. $\dfrac{\frac{^-65}{12}}{\frac{^+65}{4 \cdot 3}}$
40. $\frac{4}{5} + \frac{3}{5}$	41. $\frac{17}{32} \div \frac{28}{32}$	42. $\frac{14 - 3}{19} + \frac{14 + 3}{19}$

43. $\frac{65}{17} + \frac{^-32}{17}$

44. $\frac{65}{17} - \frac{32}{17}$

45. $\frac{43}{15} - \frac{13}{15}$

46. $\frac{2}{3} + \frac{5}{7}$

47. $\frac{3}{4} + \frac{7}{13}$

48. $\frac{5}{6} + \frac{9}{14}$

49. $\frac{7}{8} + \frac{^-3}{5}$

50. $\frac{7}{8} - \frac{3}{5}$

51. $\frac{1}{3} - \frac{1}{2}$

52. $\frac{-10}{2}$

53. $\frac{-4}{3} \div \frac{4}{3}$

54. $\frac{5 \times -2}{5 \times 2}$

55. $\frac{10}{^-2}$

56. $\frac{3}{4} + \frac{5}{-4}$

57. $\frac{3}{8} - \frac{5}{^-8}$

58. $\frac{-10}{-2}$

59. $\frac{3}{4} + \frac{-5}{-4}$

60. $\frac{3}{8} - \frac{-5}{-8}$

61. $\frac{7 \cdot 5 + 7 \cdot 2}{7}$

62. $\frac{60 + 6}{6}$

63. $\frac{3 \cdot 9378 + 3}{3}$

64. $\frac{9 + 3}{9 + 2}$

65. $\frac{16}{64}$

66. $\frac{10 + 3}{5 \cdot 2 + 3}$

Sample $5\frac{1}{3} \div 3\frac{1}{5}$

Solution.

$$5\frac{1}{3} \div 3\frac{1}{5}$$

$$= \frac{16}{3} \div \frac{16}{5}$$

$$= \frac{16}{3} \times \frac{5}{16}$$

$$= \frac{5}{3}.$$

67. $4\frac{1}{5} \div 7\frac{1}{8}$

68. $9\frac{1}{2} \div 2\frac{3}{4}$

69. $^-2\frac{1}{5} \div {}^-7\frac{1}{4}$

70. $2\frac{1}{7} \times 9\frac{1}{3}$

71. $5\frac{1}{4} \times 4\frac{1}{5}$

72. $^-2\frac{1}{3} \times {}^+5\frac{1}{9}$

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73. 82% of 424
74. 62.5% of 848
75. 0.375% of 16

76. 15% of ? is 9
77. 27% of ? is 51
78. 6.3% of ? is 126

79. 35 is ? % of 140
80. 12 is ? % of 75
81. 94 is ? % of 23.5

82. $\frac{1}{5} + 0.6$
83. $9.35 + \frac{3}{2}$
84. $8.375 - \frac{5}{8}$

85. 4.87×0.02
86. 3.92×2.81
87. 5.31×0.00025

88. $28.8 \div 1.2$
89. $6250 \div 0.025$
90. $0.00441 \div 30$

91. $9.02 \times 7\frac{1}{4}$
92. $0.038 \times 2\frac{1}{9}$
93. $8\frac{1}{7} \times 7.077$

94. $8.3(2\frac{1}{3} + 5.04)$
95. $8.3 \div (2\frac{1}{3} + 5.04)$
96. $15\frac{1}{7} - 6.21$

97. $\dfrac{8 + \frac{1}{3}}{4 + \frac{1}{5}}$
98. $\dfrac{\frac{1}{2} + \frac{1}{7}}{2}$
99. $\dfrac{3 - \frac{1}{4}}{\frac{1}{2} - \frac{3}{8}}$

100. $\dfrac{\frac{2}{9} \div \frac{1}{4}}{\frac{8}{3} - \frac{1}{2}}$
101. $\dfrac{\frac{5}{12} - \frac{1}{6}}{\frac{2}{3} + \frac{1}{4}}$
102. $\dfrac{\frac{1}{2} - \frac{1}{3} \div \frac{5}{8} - \frac{7}{12}}{\frac{3}{8} - \frac{1}{6} + \frac{5}{12} - \frac{7}{2}}$

103. $\dfrac{-\frac{3}{4}}{8}$
104. $\dfrac{1}{\frac{3}{5}}$
105. $\dfrac{\frac{5}{6}}{3}$

106. $\frac{5}{6} \div \dfrac{-1}{\frac{2}{3}}$
107. $\frac{1}{2} \div \dfrac{4}{\frac{5}{12}}$
108. $\dfrac{\frac{2}{3}}{\frac{4}{9}} \div \dfrac{\frac{1}{5}}{\frac{3}{10}}$

109. $\dfrac{-\frac{2}{5} \times \frac{3}{4} \times \frac{2}{3}}{-\frac{1}{5} \times -\frac{2}{3} \times -\frac{6}{7}}$
110. $\dfrac{\frac{5}{9} \times -\frac{3}{10}}{\frac{5}{9} + -\frac{3}{10}}$
111. $\dfrac{\frac{1}{3} + \frac{1}{7}}{\frac{1}{3} \times \frac{1}{7}}$

112. $\dfrac{1}{\frac{1}{5} + \frac{1}{7}}$
113. $\dfrac{1}{\frac{2}{3} - \frac{6}{7}}$
114. $\dfrac{\frac{2}{9} - \frac{3}{5}}{\frac{2}{9} + \frac{3}{5}}$

K. Simplify.

1. $\frac{39a}{13}$ 2. $\frac{63n}{9}$ 3. $\frac{42c}{7}$ 4. $\frac{-18d}{-3}$

5. $\frac{-34r}{-2}$ 6. $\frac{44rs}{4}$ 7. $\frac{96xyz}{-12}$ 8. $\frac{-41ab}{-1}$

9. $\frac{9ef}{\frac{1}{3}}$ 10. $\frac{-11r}{-3}$ 11. $\frac{-98n}{7}$ 12. $\frac{-205ab}{-5}$

13. $\frac{-960d}{-.8}$ 14. $\frac{-33.3x}{3}$

15. $\frac{16np}{4n}$ 16. $\frac{24rs}{3r}$ 17. $\frac{12aad}{4ad}$ 18. $\frac{15ad}{5ac}$

19. $\frac{8rst}{8rst}$ 20. $\frac{-52eef}{-4f}$ 21. $\frac{21w}{-w}$ 22. $\frac{28ccde}{7cdde}$

23. $\frac{-18xu}{2xuu}$ 24. $\frac{11nn}{-11nn}$ 25. $\frac{-48dee}{-3dde}$ 26. $\frac{-26cdx}{-2cuv}$

27. $\frac{8x}{3y} \times \frac{9k}{4j}$ 28. $\frac{3ab}{7cd} \times \frac{14ac}{9bb}$ 29. $\frac{-8r}{-4s} \times \frac{6xs}{5yr}$

30. $\frac{-5ab}{9xc} \times \frac{3cxy}{-5abz}$ 31. $\frac{-2k}{-3j} \times \frac{5rj}{-2sk}$ 32. $\frac{abcc}{-rs} \times \frac{-sr}{bc}$

33. $\frac{1}{3y} \div \frac{9}{2y}$ 34. $\frac{x}{2a} \div \frac{-5xb}{-2ac}$ 35. $\frac{-3a}{-2b} \div \frac{-3ab}{8cd}$

36. $\frac{x - 9}{3} \times \frac{12}{x - 9}$ 37. $\frac{x + 2}{3(y - 5)} \times \frac{9(y - 5)}{2(x + 2)}$

38. $\frac{18(a + 3)}{5(a + 4)} \times \frac{25(a + 4)}{6(a + 3)}$ 39. $\frac{2(y - 1)(y - 3)}{9(y - 2)} \times \frac{27(y - 2)}{4(y - 3)(y - 5)}$

40. $\frac{7(x + 2)}{9(x - 3)} \div \frac{14(x - 3)}{3}$ 41. $\frac{8(3 - x)}{5(3 + x)} \div \frac{4(x - 3)}{15(x + 3)}$

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42. $\frac{6a - 18}{6}$

43. $\frac{5x - 5}{-5}$

44. $\frac{3.9b - 7.8}{1.3}$

45. $\frac{2b + 50c}{2}$

46. $\frac{3}{5}(20a - 5b)$

47. $\frac{1}{9}(-9n - 63t)$

48. $\frac{2}{7}(28c - 49d)$

49. $(8 - 32s)\frac{3}{8}$

50. $\frac{7gh - 11}{1/3}$

51. $\frac{6c + 6d + 6e}{-6}$

52. $\frac{14nr + 2ns}{-2}$

53. $(12g - 26k) \div \frac{1}{2}$

54. $\frac{8xy + 8}{8}$

55. $\frac{-5 - 5kj}{-5}$

56. $\frac{\frac{1}{3}a + \frac{2}{3}y}{3}$

57. $\frac{6rs - 11rt}{r}$

58. $\frac{5nq - 8nqr}{-nq}$

59. $\frac{9effg + 27efg}{3efg}$

60. $\frac{45abc + 5c}{5c}$

61. $\frac{21cde - 7c}{7c}$

62. $\frac{24g - 8e}{-4}$

63. $\frac{7(a + 5)}{a + 5}$

64. $\frac{(n - 3)(n + 5)}{n - 3}$

65. $\frac{(a - 4)(a + 7)}{-(a - 4)}$

66. $\frac{(c - 8)(c + 9)}{8 - c}$

67. $(x - 3) \cdot \frac{3x}{x - 3}$

68. $5(a + 7) \cdot \frac{a}{a + 7}$

69. $15\left(\frac{n}{3} + \frac{n}{5}\right)$

70. $14\left(\frac{3c}{7} - \frac{9c}{2}\right)$

71. $10\left(\frac{4}{5}a - \frac{7}{2}b\right)$

72. $21\left(\frac{5r}{3} + \frac{5r}{7}\right)$

73. $32\left(\frac{3x}{4} - \frac{5y}{2}\right)$

74. $18\left(\frac{n}{9} - \frac{n}{6}\right)$

75. $12\left(\frac{d + 1}{4} + \frac{d + 2}{6}\right)$

76. $16\left(\frac{e + 4}{2} - \frac{e + 1}{8}\right)$

77. $8\left(\frac{3n}{2} - \frac{5n}{4}\right)$

78. $24\left(\frac{3k + 2}{4} - \frac{4k + 1}{6} - \frac{2k - 3}{8}\right)$

79. $28\left(\frac{3u}{7} - \frac{5u}{2}\right)$

80. $15\left[\frac{1}{5}(d + 3) - 5 - \frac{d - 6}{3} + \frac{1}{5}\right]$

81. $9\left(\frac{d + 13}{9} - \frac{12 - d}{3}\right)$

82. $36\left[\frac{t - 5}{9} - \frac{8 - t}{4} + \frac{t}{12}\right]$

83. $r\left(\frac{5}{r} - 1\right)$

84. $3x\left(\frac{7}{3x} - 4\right)$

85. $5(s - 3)\left(\frac{4}{s - 3} + 2\right)$

86. $7x\left(\frac{3}{x} - \frac{1}{7}\right)$

87. $10d\left(\frac{5}{d} + \frac{4}{5d}\right)$

88. $9g\left(\frac{7}{9} - \frac{5}{g}\right)$

89. $4(a + 7)\left(\frac{a}{a + 7} - \frac{1}{4}\right)$

90. $30r\left(\frac{3r + 1}{3r} - \frac{5r - 3}{5r} - \frac{11}{15}\right)$

91. $4(d + 5)\left(\frac{7}{4} - \frac{d}{d + 5}\right)$

92. $9(z + 2)\left(\frac{z - 1}{z + 2} - \frac{5}{9}\right)$

93. $6k(k + 5)\left(\frac{7}{6k} - \frac{3}{k + 5}\right)$

94. $7e(e - 5)\left(\frac{e}{e - 5} - \frac{9}{e}\right)$

95. $b(b + 2)\left(\frac{3}{b + 2} - \frac{1}{b}\right)$

96. $12v(2v - 5)\left(-\frac{1}{4v} + \frac{3v}{2v - 5}\right)$

97. $\frac{a}{7} - \frac{a}{4}$

98. $\frac{3x}{5} + \frac{4x}{3}$

99. $\frac{11r}{3} - \frac{1}{2} + \frac{r}{2}$

100. $\frac{5d}{3} + \frac{d}{6}$

101. $\frac{3h}{6} - \frac{1}{12}$

102. $\frac{m}{4} + \frac{m}{6} + m - 1$

103. $\frac{9}{g} - \frac{5}{3g}$

104. $\frac{10}{p} + \frac{7}{2p}$

105. $\frac{13}{s} - \frac{1}{3s} + 5$

106. $\frac{3}{5b} - \frac{1}{3b} + 9 - \frac{4}{30b}$

107. $1 - \frac{3}{5c} + \frac{1}{3} - \frac{4}{c}$

108. $\frac{5j + 1}{9} - \frac{j - 7}{5}$

109. $\frac{1}{9}(5j + 1) - \frac{1}{5}(j - 7)$

110. $\frac{t + 6}{3} - \frac{t + 4}{7}$

111. $\frac{5n + 1}{3} - \frac{3n + 4}{4} - \frac{6n - 1}{5}$

112. $\frac{z + 1}{3} - \frac{z - 7}{4}$

113. $\frac{5u - 2}{4} - \frac{4u - 3}{3}$

114. $\frac{e - 5}{4} + 3 - \frac{e + 7}{8}$

115. $\frac{A - 3}{8} - \frac{A - 3}{6} - \frac{3}{4}$

116. $7 + \frac{B - 2}{5} - \frac{B + 12}{10}$

117. $\frac{f + 8}{8} + \frac{1}{4} - \frac{f - 5}{2} + 1$

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118. $\frac{3}{n - 4} - \frac{5}{n + 2}$

119. $\frac{10}{a - 3} - \frac{9}{a - 5}$

120. $\frac{6}{c - 3} + \frac{3}{c - 6}$

121. $\frac{y + 5}{y - 3} + \frac{4}{y - 3} - 5$

122. $\frac{7}{d} + \frac{5}{d - 3}$

123. $\frac{12}{3y + 2} - \frac{7}{y} + \frac{6}{3y + 2}$

124. $\frac{10}{g - 5} + \frac{4}{g + 7}$

125. $\frac{5}{8} - \frac{e}{e + 3} + \frac{e}{5}$

126. $\frac{r + 2}{r - 7} + \frac{4}{5} + \frac{3}{r - 7}$

127. $\frac{t}{t + 5} - \frac{3t}{t + 6} - \frac{4t}{t + 5}$

128. $\dfrac{\frac{5}{3} - \frac{1}{5x}}{\frac{1}{5x} + \frac{1}{6}}$

129. $\dfrac{\frac{1}{f} - \frac{1}{g}}{\frac{1}{f} - \frac{1}{g}}$

130. $\dfrac{5 - \frac{3}{4r}}{6 + \frac{1}{2r}}$

131. $\dfrac{2}{\frac{1}{k} + \frac{1}{m}}$

132. $\dfrac{x - \frac{1}{3y}}{y - \frac{1}{5x}}$

133. $\dfrac{a + b}{\frac{2}{a} - \frac{3}{b}}$

134. $\dfrac{\frac{2}{x}}{\frac{3}{y} + \frac{4}{z}}$

135. $\dfrac{1}{3 - \dfrac{1}{3 - \dfrac{1}{3 - 1}}}$

HIGH SCHOOL MATHEMATICS

Unit 3.

EQUATIONS AND INEQUATIONS

UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS

MAX BEBERMAN, *Director*

HERBERT E. VAUGHAN, *Editor*

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TABLE OF CONTENTS

A report on a ticket sale.--Betty Morris, who is chairman of the committee in charge of selling tickets for the play at Zabranchburg High School, gave a financial report to the Student Council:

> The ticket committee sold 167 tickets to the school play. We charged $.66 for an adult ticket and $.39 for a student ticket and we collected a total of $79.17.

John Sanders, a member of the Student Council, had been critical of the expensive advertising used to interest adults in coming to the play. After Betty gave her report, he said:

> I wonder if all of our advertising was worth the money. I didn't see many adults at the play. Just how many adults did attend the play, Betty?

The Council president asked Betty if she could tell them the number of adult tickets sold. Betty said that she couldn't give this information immediately because her records were at home. If the Council didn't mind waiting, she would call home and ask her mother to read the figures to her over the telephone. Bill Smith, another Council member, said:

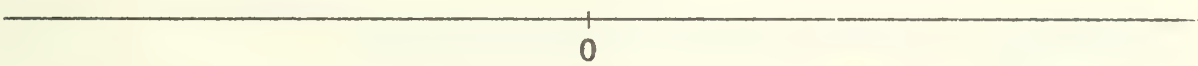
> That won't be necessary, Betty. As soon as John asked his question, I used the information you gave us and computed the number of adult tickets sold. It was easy to find the answer.

Can you tell how many adult tickets were sold?

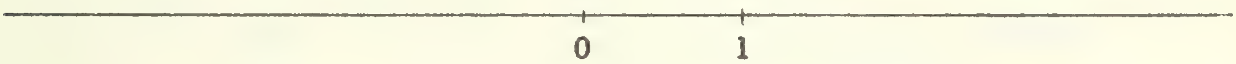
The problem Bill solved is not an easy one if you try to solve it by methods you have used in earlier grades. It can be solved by those methods, but there is a faster method which makes this problem a very simple one. You will learn this faster method later in this unit and be able to solve even more complicated problems.

3.01 Graph and coordinates. --In Unit 1 you learned that you could think of the real numbers as being "lined up". You can make a picture of this "line-up" by drawing a straight line,

marking any one of the points on it and labeling it with a '0',



marking another point and labeling it with a '1',



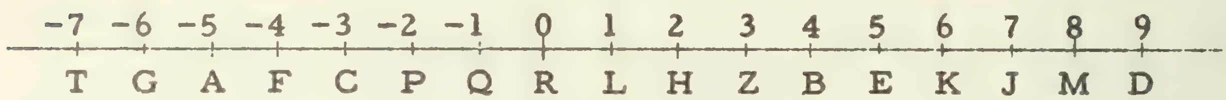
and using a uniform scale to assign points to the rest of the real numbers.

This diagram is a picture of [part of] the number line. Each point you mark on the picture corresponds with a real number, and this real number is the coordinate of the point. Each mark is the graph of the real number which is its coordinate.

[Once you have drawn a picture of the number line, you can make a mark corresponding with any real number whose graph falls within the boundaries of the picture. But, for a given picture, no matter what the scale, there are bound to be two real numbers whose difference is so small that you cannot mark different dots. So, no picture of the number line can be "accurate" in the sense that there is a one-to-one correspondence between the real numbers and their graphs. Nevertheless, pictures of the number line are helpful in visualizing properties of real numbers.]

EXERCISES

A. Here is a picture of the number line. Some of the points of the picture are labeled with letters as well as with numerals. [Notice, for example, that the point labeled 'Z' is the graph of the real number 3, and that the coordinate of point Z is 3.]



Sample 1. Which point is the graph of +6?

Solution. Point K.

Sample 2. Which point is the graph of $-\frac{10}{3}$?

Solution. The point which is $\frac{1}{3}$ of the way from point C to point F.

Sample 3. What is the coordinate of the point A?

Solution. −5.

Sample 4. What is the coordinate of the point halfway between L and P?

Solution. The coordinate of L is 1 and the coordinate of P is −2. The number we are looking for is the midpoint of the number line segment whose end points are 1 and −2. Since the distance between the end points is 3, the distance between the midpoint and either end point is 1.5. Hence, the midpoint is −0.5, and this is the coordinate of the point halfway between L and P.

What are the graphs of the listed numbers?

1. 3	2. +7	3. −5
4. 0	5. 2.5	6. $-5\frac{1}{4}$
7. $\frac{49}{10}$	8. −4.8	9. 4.99
10. −0.1	11. $4 + -3$	12. $-2 \div -\frac{1}{2}$

*

−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9
T	G	A	F	C	P	Q	R	L	H	Z	B	E	K	J	M	D

Give the coordinates of the points described in these exercises.

13. The point D.

14. The point T.

15. The point R.

16. The point 1 unit to the right of A.

17. The point 2 units to the left of M.

18. The point halfway between D and F.

19. The point halfway between D and T.

20. The point halfway between F and H.

21. The point 40% of the way from A to L.

22. The point one third of the way from M to J.

23. The point one fourth of the way from P to A.

24. The point which is as far from C as it is from H.

25. The point between G and Z which is twice as far from G as it is from Z.

26. A point such that the distance between it and L is twice the distance between it and Q.

27. A point which is three times as far from P as it is from H.

B. Complete into true sentences.

1. If the coordinate of a point A on a picture of the number line is 7, and the coordinate of a point B on this picture of the number line is −3, then the coordinate of the point halfway between A and B is ________.

2. If the coordinate of a point T is 9, and the coordinate of a point V is 12, then the coordinate of the point halfway between T and V is ________.

(continued on next page)

3. The midpoint of the number line segment whose end points are 9 and 12 is _____.

4. The midpoint of the number line segment whose end points are 11 and -10 is ____.

*

Use the simplest expression you can to complete the table.

	end point	end point	midpoint
5.	9	31	________
6.	-2	-16	________
7.	8	12	________
8.	200	400	________
9.	-10	10	________
10.	$10 + x$	$20 + x$	________
11.	$3x$	$9x$	________
12.	$2k + 7$	$6k - 7$	________
13.	$3a + 2b$	$7a - 16b$	[Answer: $5a - 7b$]
14.	$2(5 - x)$	$7(2 + 4x)$	________
15.	$3 - 5r + 7t$	$11 - 7r - 21t$	________
16.	$5 - (7 + 3n)$	$9n - (6 - 8n)$	________
17.	$-(3 - x + 2y)$	$-(5 + 7x - 12y)$	________
18.	-6	________	-10
19.	________	1	-3
20.	$2x$	________	$7x$
21.	________	$3t$	$8t + 5$
22.	$a + 3b$	________	$7a - 2b$
23.	$2x - 3y + 5$	________	$-7x + 8y - 6$
24.	$2 - 3(x - y)$	________	$5 - 7(3x - 4y)$

3.02 Solution set of a sentence. --Consider the open sentence:

$$x + 3 > 1.$$

We know that this open sentence is neither true nor false, but that we can generate a statement from it by substituting a numeral for 'x'. Here are some of the true statements and some of the false ones.

True	False
$5 + 3 > 1$	$-2 + 3 > 1$
$9 + 3 > 1$	$-5 + 3 > 1$
$172 + 3 > 1$	$-17 + 3 > 1$
$-1.4 + 3 > 1$	$-2.5 + 3 > 1$

Each value of 'x' can be used to convert the open sentence '$x + 3 > 1$' into a true statement or into a false statement. A value of 'x' which converts it into a true statement is said to satisfy the open sentence or to be a solution of the open sentence. The set of all real numbers which satisfy the sentence is called the solution set of the sentence. So, the solution set of the sentence:

$$x + 3 > 1$$

is the set of all real numbers which are greater than -2.

EXERCISES

A. Describe the solution set of each of the following sentences.

Sample 1. $3t = 12$

Solution. There is just one real number which satisfies this sentence. This is the number 4. So, we write:

the solution set is the set of real numbers which consists of just the number 4.

Sample 2. $k > k + 1$

Solution. There are no numbers which satisfy this sentence. We express this fact by writing:

the solution set is the empty set.

Sample 3. $5x < 30$

Solution. Each real number less than 6 is a solution of this sentence, and each number which is not less than 6 does not satisfy it. So, our answer is:

the solution set is the set of all real numbers which are less than 6.

Sample 4. $2 < x$ and $x < 6$

Solution. To find solutions for this sentence is to find numbers which are both greater than 2 and less than 6. These are the numbers which are between 2 and 6. So, we write:

the solution set is the set of all real numbers between 2 and 6.

[Note: An abbreviation of the sentence '$2 < x$ and $x < 6$' is '$2 < x < 6$', which is read as '2 is less than x is less than 6'.]

Sample 5. $2 > x$ or $x > 3$

Solution. Here we are looking for numbers which are smaller than 2 or greater than 3. Such a number is 1, for $1 < 2$ even though it is not larger than 3. Also, 6 is such a number. So, the answer is:

the solution set is the set of all real numbers which are less than 2 and all real numbers which are greater than 3.

[Note: Would it make sense to abbreviate '$2 > x$ or $x > 3$' to '$2 > x > 3$'? Why? Does the sentence '$1 < x$ and $x < 5$' have the same solution set as the sentence '$1 < x$ or $x < 5$'?]

1.	$x + 5 > 7$	2.	$y - 3 > 4$	3.	$3m \leq 6$
4.	$10 + x = 7$	5.	$\lvert y - 2 \rvert = 12$	6.	$5t + 1 > 21$
7.	$8y > 0$	8.	$x(x - 1) = 0$	9.	$y + y < 5$
10.	$x = x + 3$	11.	$x + 4 > 4$	12.	$x + 4 = 4 + x$
13.	$1 < y$ and $y < 3$	14.	$-2 < x < 0$	15.	$1 > x$ or $x > 4$

B. Each exercise describes a set of numbers. For each exercise, write three sentences which have the given set as solution set.

Sample 1. the set of all numbers greater than 5

Solution. (1) $x > 5$

(2) $x + 4 > 9$

(3) $y - 7 > -2$

[Note: Suppose someone writes '$x > 10$' as one of the answers, and defends this answer by saying that each number in the solution set of '$x > 10$' is a number which is greater than 5. How would you show him that '$x > 10$' is not a correct answer?]

Sample 2. $\{4\}$

Solution. '$\{4\}$' is a short way of describing the set which consists of just the number 4. So, three sentences which have $\{4\}$ as solution set are:

(1) $x = 4$

(2) $3y = 12$

(3) $7z + 91 = 119$.

1. the set of all numbers less than 2
2. the set of all numbers less than 0
3. the set of all numbers not less than 3
4. the set of all numbers between (but not including) 1 and 7
5. the set of all numbers between (and including) 1 and 7
6. the set of all numbers greater than 5 and all numbers less than 4
7. the set of all numbers greater than 1 and all numbers less than 3
8. $\{2\}$ 9. $\{-2\}$ 10. $\{1, 3\}$
11. the set of all numbers
12. the empty set

C. Each of the following 12 exercises describes a set of numbers. Altogether, just 4 sets are described. Tell which descriptions refer to the same set.

(1) the set of all numbers such that each is greater than 3

(2) the set of all numbers such that each is less than 5

(3) the set of all numbers such that each is 7

(4) the set of all numbers such that the product of 2 by each is greater than 6

(5) the set of all numbers such that 1 more than the product of 3 by each is 7

(6) the set of all numbers such that 4 more than each is less than 9

*

We can simplify these descriptions of sets of numbers by using pronumerals. For example, here is a restatement of the description given in Exercise (6):

(a) the set of all numbers x such that each of them plus 4 is less than 9.

Now, let's abbreviate this to:

(b) the set of x such that $x + 4 < 9$,

and, finally, to:

(c) $\{x: \ x + 4 < 9\}$.

[Read (c) as you do (b).]

*

Now, continue sorting the descriptions.

(7) $\{x: \ 9x - 1 = 17\}$

(8) $\{x: \ 10 - x > 5\}$

(9) $\{y: \ 3y = 21\}$

(10) $\{a: \ 6a + 1 = 13\}$

(11) $\{z: \ 7 - z < 4\}$

(12) $\{k: \ 33 > 51 - 6k\}$

<u>D</u>. Which of the following describe the set $\{5, -5\}$?

<u>Sample 1</u>. $\{x: \ 2xx + 13 = 63\}$

<u>Solution</u>. To determine whether this describes the set $\{5, -5\}$, we must decide whether the solution set of the sentence '$2xx + 13 = 63$' consists of 5 and -5, and nothing else. We replace the 'x's in the sentence by '5's:

$$2 \cdot 5 \cdot 5 + 13 = 63$$

$$50 + 13 = 63.$$

Since 5 gives us a true statement, 5 is a solution of the

sentence. Next, we try substituting '−5' for 'x':

$$2(-5)(-5) + 13 = 63$$

$$50 + 13 = 63.$$

Again we get a true statement, so we know that −5 is a solution. Can you think of any other number which would satisfy '2xx + 13 = 63'? [Could a number greater than 5 be a solution?]

Having found that both 5 and −5 are solutions of the sentence '2xx + 13 = 63', and believing that there are no other solutions, we would conclude that '{x: 2xx + 13 = 63}' describes the set {5, −5}.

<u>Sample 2</u>. $\{n: \ |n| > 4.8\}$

<u>Solution</u>. We substitute '5' for 'n':

$$|5| > 4.8,$$

and note that we get a true sentence because $|5| = 5$, and $5 > 4.8$. So, 5 is a solution of the sentence '$|n| > 4.8$'.

Substituting '−5' for 'n', we get:

$$|-5| > 4.8,$$

and we know that this is true since $|-5| = 5$. So, −5 is a solution.

But, are there other numbers which might be solutions also? Yes, 6 is such a number; so is −4.9.

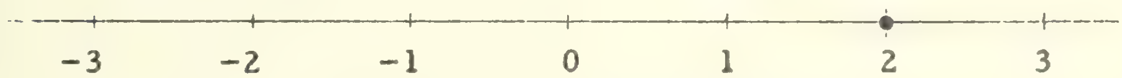
Therefore, we conclude that '$\{n: \ |n| > 4.8\}$' does <u>not</u> describe the same set as does '{5, −5}' because the solution set of '$|n| > 4.8$' contains numbers other than 5 and −5.

1. $\{x: \ xx = 25\}$
2. $\{y: \ yy - 25 = 0\}$
3. $\{k: \ 25 + kk = 0\}$
4. $\{m: \ m + 5 = 0\}$
5. $\{t: \ (t - 5)(t + 5) = 0\}$
6. $\{x: \ x \text{ is the opposite of } -x\}$
7. $\{\square: \ 4\square\square = 100\}$
8. $\{\triangle: \ 5\triangle = -5\}$
9. $\{x: \ x + 3 \neq 9\}$
10. $\{y: \ 5 + y \neq 10\}$

3.03 Graph of a sentence.--Suppose you mark on a picture of the number line the points corresponding with the numbers in

$$\{x:\ \ 3x + 1 = 7\}.$$

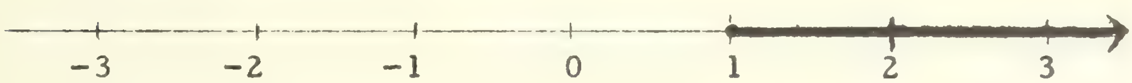
The picture you get is this:



If you mark the graphs of the numbers in

$$\{x:\ \ x + 4 \geq 5\},$$

the picture you get is this:



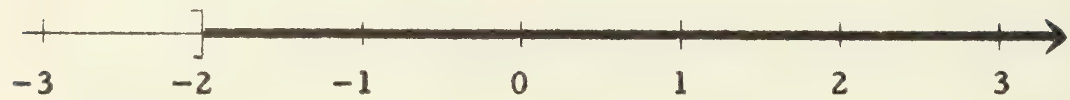
The set of numbers described by '$\{x:\ \ 3x + 1 = 7\}$' is the solution set of the sentence '$3x + 1 = 7$', and the set of numbers described by '$\{x:\ \ x + 4 \geq 5\}$' is the solution set of the sentence '$x + 4 \geq 5$'.

> The picture made up of the graphs of the numbers in the solution set of a sentence is called the graph of the sentence.

So, the graph of '$3x + 1 = 7$' consists of a single point, the heavy dot in the first picture. The graph of '$x + 4 \geq 5$' is the shaded portion of the second picture. [Notice the arrowhead in the second picture. What does it tell you?]

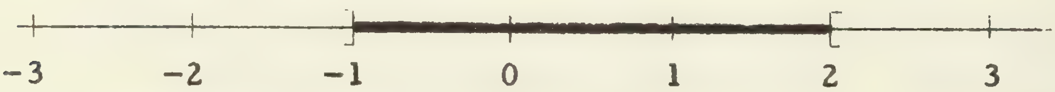
Here are several sentences and their graphs.

(I) $2y > -4$

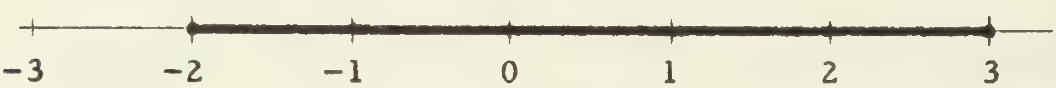


[Is -2 a solution of '$2y > -4$'? What does the '] ' at the graph of -2 show?]

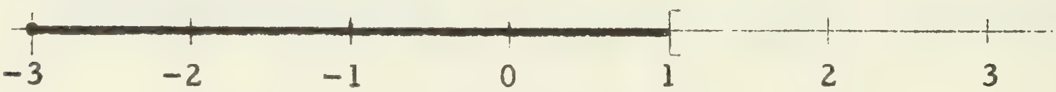
(II) $-1 < x$ and $x < 2$



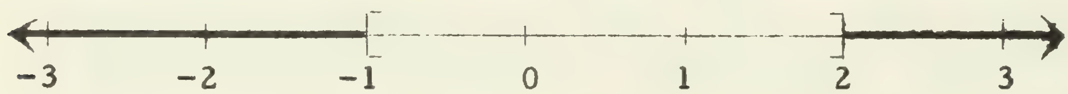
(III) $-2 \leq y \leq 3$



(IV) $-3 \leq k < 1$

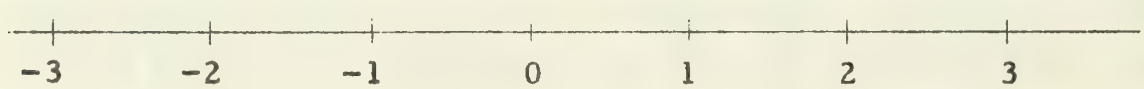


(V) $t < -1$ or $t > 2$



(VI) $x(x - 3) = 0$

(VII) $a + 1 < a$



[Note: Since the solution set is the empty set, the graph of '$a + 1 < a$' contains no points. Therefore, there is nothing to picture.]

<u>A</u>. Sketch the graph of each of the following sentences.

1. $3x > 6$

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

2. $2x < -4$

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

3. $t + 5 \geq 5$

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

4. $-\square < 1$

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

5. $x = -x$

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

6. $2k \geq 0$

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

7. $-1 \leq x \leq 3$

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

8. $x > 1$ or $x < 0$

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

9. $x > 0$ or $x < 1$

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

10. $-1 \geq x \geq 3$

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

11. $x(x - 2)(x - 3) = 0$

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

12. $|x - 1| = 1$

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

[More exercises are in <u>Part A</u>, <u>Supplementary Exercises</u>.]

B. Each exercise contains a picture of the number line with a graph marked on it. For each exercise, give three descriptions of the set of numbers which are the coordinates of the points on the graph.

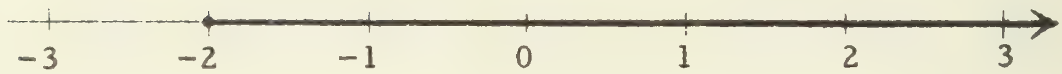
Sample.

Solution. This graph consists of all the points with coordinate not less than −1. So, here are three descriptions of the set of numbers which are the coordinates of these points.

(1) $\{x: \ x \geq -1\}$

(2) $\{k: \ 3k \geq -3\}$

(3) $\{y: \ 2y + 1 \not< -1\}$

1.


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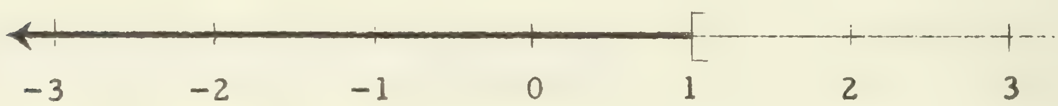

3.


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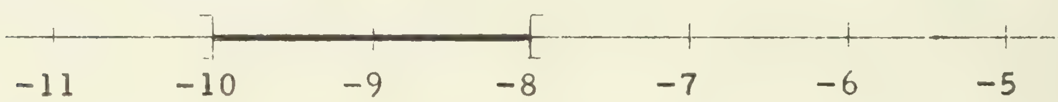

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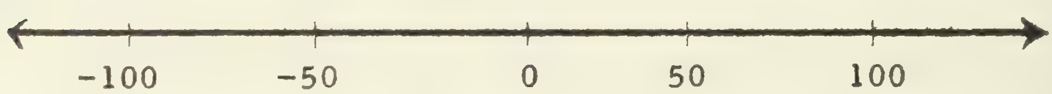

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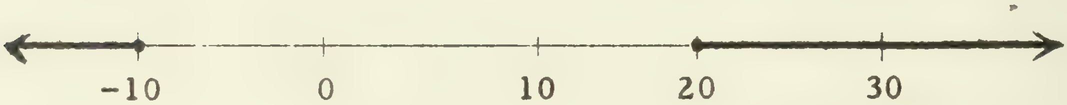


7.


8.


9.


LOCUS OF A SENTENCE

When people think of the set of real numbers in geometric terms, they call it 'the number line'. And, they often use another geometric term, <u>locus</u>, in place of 'solution set'. The locus of a sentence is the solution set of the sentence.

Consider the sentence '$1 < x < 3$'. The locus of this sentence is an <u>interval of the number line</u>, that is, it is the set of all those numbers which are between two numbers, in this case, between the two numbers 1 and 3. The graph of the sentence '$1 < x < 3$' is a picture of this interval.

One name for this interval is:

$$\{x\colon\ 1 < x < 3\}.$$

Shorter names for this interval are:

$$\overline{1,\ 3} \qquad \text{and:} \qquad \overline{3,\ 1}\ .$$

[Read as 'interval one three' and as 'interval three one'.]

Consider, next, the sentence '$1 \leq x \leq 3$'. Here is a graph of this sentence.

The locus of this sentence is a <u>segment of the number line</u>. It is an interval together with its <u>end points</u>. One name for this segment is:

$$\{x\colon\ 1 \leq x \leq 3\}.$$

Shorter names are:

$$\overset{\bullet\!-\!-\!\bullet}{1,\ 3} \qquad \text{and:} \qquad \overset{\bullet\!-\!-\!\bullet}{3,\ 1}\ .$$

[Read as 'segment one three' and as 'segment three one'.]

The locus of '$1 \leq x < 3$' is a <u>half-open interval of the number line</u>. A name for it is '$\overset{\bullet\!-\!-}{1,\ 3}$' which is read as 'half-open interval one three'. The locus of '$1 < x \leq 3$' is another half-open interval, $\overset{\bullet\!-\!-}{3,\ 1}$. Make a picture of $\overset{\bullet\!-\!-}{-2,\ 3}$ and a picture of $\overset{\bullet\!-\!-}{2,\ -3}$.

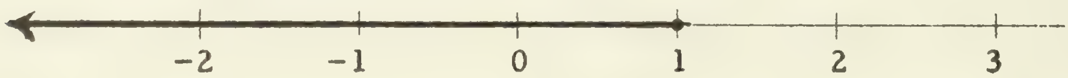
Here is a graph of the sentence '$2y > 2$'.

Its locus is a <u>half-line of the number line</u>. Short names for this half-line are:

$\overrightarrow{1, 2}$, $\overrightarrow{1, 3}$, $\overrightarrow{1, 7.3}$, $\overrightarrow{1, 86}$, etc.

[Read as 'half-line one two', etc.] Write three other short names for this half-line. Draw a picture of $\overrightarrow{2, 1}$. Do you see that $\overrightarrow{1, 2} \neq \overrightarrow{2, 1}$?

Let's look at a graph of '$2y \leq 2$'.



The locus of this sentence is a <u>ray of the number line</u> [a half-line together with its <u>vertex</u>], and is named by:

$\overset{\bullet\!\!\longrightarrow}{1, 0}$, $\overset{\bullet\!\!\longrightarrow}{1, -1}$, $\overset{\bullet\!\!\longrightarrow}{1, -2.3}$, $\overset{\bullet\!\!\longrightarrow}{1, -581}$, etc.

[Read as 'ray one zero', etc.]

If the solution set of a sentence consists of just one number, the locus of the sentence is a <u>unit set</u> or a <u>singleton</u>. For example, the locus of '$3x + 1 = 7$' is $\{2\}$. The locus of '$x^2 = 9$' is the <u>couple</u> $\{3, -3\}$.

The locus of '$x = x + 1$' is the empty set. A short name for the empty set is '$\emptyset$'. Hence we could write 'the locus of '$x = x + 1$' is $\emptyset$'. The locus of '$x = x$' is the number line itself. [So, we can use '$\{x\colon\ x = x\}$' as a name for the number line.]

EXERCISES

<u>A.</u> Sketch pictures of these sets.

1. $\overset{\bullet\!\!-\!\!-\!\!\bullet}{-2, 2}$ 2. $\overset{\bullet\!\!-\!\!-}{2, -2}$ 3. $\overset{\bullet\!\!-\!\!-}{-2, 2}$

4. $\overline{-2, 2}$ 5. $\overset{\bullet\!\!\longrightarrow}{-2, 5}$ 6. $\overset{\bullet\!\!\longrightarrow}{5, -2}$

7. $\overrightarrow{2, 1}$ 8. $\overline{3, 3}$ 9. $\overset{\bullet\!\!-\!\!-\!\!\bullet}{3, 3}$

B. Name the loci of the following sentences using geometric language wherever possible. ['loci' is the plural of 'locus'.]

Sample. $x + 2 < 5$

Solution. $\overrightarrow{3, -8}$

[Other correct answers are: $\overrightarrow{3, 0}$, $\overrightarrow{3, -100}$, $\overrightarrow{3, 2.999}$.]

1. $x > -4$
2. $y \leq 3$
3. $x + 1 > 5$
4. $3 < y < 5$
5. $-1 \leq k \leq 4$
6. $|x| = 2$
7. $|x| < 1$
8. $|y| \leq 4$
9. $xx \leq 4$
10. $-y > 4$
11. $-y \leq -2$
12. $y > -4$ or $y < 4$
13. $-2 < x < 3$
14. $x > -5$ and $x < -2$
15. $0 \leq x < 4$
16. $-6 < n < +5$
17. $1 > t > -1$
18. $1 > s > 5$
19. $-3 < x > 4$
20. $-5 < x > -5$
21. $6 < x < 6$
22. $-x < 3$
23. $-4 \leq x < 5$
24. $-4 < x \leq 5$
25. $-3 \leq x \leq 3$
26. $6 \leq x \leq 6$
27. $xx = 9$
28. $xx \leq 9$
29. $xx < 0$
30. $xx \leq 0$
31. $kk = -4$
32. $k(-k) = -25$
33. $|x| < 5$
34. $|x| \geq 0$
35. $|x| < -5$
36. $|x| > -5$
37. $|k - 2| = 3$
38. $|3 - m| = 2$

(continued on next page)

39. $x + 1 = 1 + x$

40. $x - 1 = 1 - x$

41. $-x = 0$

42. $0 \cdot x = 5$

43. $t = 2t$

44. $x - 2 = x - 3$

45. $\boxed{} + 2\boxed{} = 3\boxed{}$

46. $5g - 3g = 2g$

47. $3t \times 5t = 15tt$

48. $6 \times 3y = 9y$

49. $\boxed{}$ is a positive number

50. $\boxed{}$ is a negative number

51. $-\boxed{}$ is a positive number

52. $-\boxed{}$ is a negative number

53. $-\triangle\triangle$ is a nonpositive number

54. $-\triangle\triangle$ is not a negative number

55. $\triangle\triangle\triangle$ is a negative number

C. Each exercise contains a geometric name of a set. Write a sentence which has this set for its locus, and then use brace-notation together with the sentence to write a name for the set.

Sample. $\overset{\bullet\!\!\longrightarrow}{-3, 6}$

Solution. [This is the ray which consists of -3 and all numbers greater than -3.] A sentence whose locus is this ray is:

$$x \geq -3.$$

Using brace-notation, a name for the ray is:

$$\{x: \; x \geq -3\}.$$

Answer. $x \geq -3$, $\{x: \; x \geq -3\}$

1. $\overset{\bullet\!\!\longrightarrow}{-5, 4}$

2. $\overset{\bullet\!\!\longrightarrow}{1, -8}$

3. $\{-2\}$

4. $\overrightarrow{-6, 6}$

5. $\overrightarrow{4, -8}$

6. $\overline{4, -4}$

7. $\overset{\bullet\!\!-\!\!-\!\!\bullet}{-6, 6}$

8. $\emptyset$

9. $\overrightarrow{100, 101}$

10. $\overrightarrow{101, 100}$

11. $\overset{\bullet\!\!-\!\!-}{0, 1}$

12. $\overset{\bullet\!\!-\!\!-}{1, 0}$

13. $\{0\}$

14. $\{-1, 0\}$

15. $\{-1, 0, 1\}$

3.04 Equations. --An equation is a sentence obtained by connecting expressions by an equality sign. Examples of equations are:

(1) $3 + 5 = 4 \times 2$, (3) $x + 4 = 9 - x$,

(2) $9 + 7 = 7 \times 3$, (4) $x = x - 1$.

An equation can be a statement such as (1) and (2), or open sentences such as (3) and (4). [Sentences obtained by connecting expressions by inequality signs like '$>$', '$\neq$', '$<$', '$\not>$', and '$\geq$' are called inequations.]

The numbers which satisfy an equation are called the roots of the equation [or: the solutions of the equation].

To solve an equation is to find all of its roots.

EXERCISES

A. Solve.

Sample. $x + 3 = 8$

Solution. To solve '$x + 3 = 8$' is to find each number whose sum with 3 is 8. 5 is such a number. So, a root is 5. All you need write is:

5.

[How do we know that 5 is the only root? Does the cancellation principle for addition help you answer this question?]

1. $x + 9 = 15$
2. $15 = y + 17$
3. $a + 4 = 0$
4. $t + 12 = 9$
5. $9 + m = 10.4$
6. $18 + p = 8.6$
7. $\square + 2 = 28$
8. $12 + \triangle = 1$
9. $x = -1$
10. $y = 1.5$
11. $\square = 17$
12. $-\square = 5$

(continued on next page)

13. $7 = x - 9$

14. $y - 4 = 3$

15. $A - 2 = -3$

16. $K - 9 = 0$

17. $3 - x = 1$

18. $26 - y = 18$

19. $4 - q = 6$

20. $-8 = -5 - t$

21. $15 = 5\triangle$

22. $3\square = 12$

23. $9x = 16$

24. $8y = -24$

25. $2x = 1 + 2x$

26. $2r = -9$

27. $-4\triangle = 0$

28. $+27 = -3\square$

29. $\square \div 8 = 5$

30. $\triangle \div -2 = 3$

31. $\frac{x}{5} = 2$

32. $\frac{y}{4} = -2$

33. $0 = \frac{t}{17}$

34. $\frac{y}{-3} = 5$

35. $\frac{-\square}{8} = 1$

36. $-\frac{\triangle}{6} = 1$

37. $\frac{1}{2}\square = 4$

38. $\frac{3}{5}\triangle = 6$

39. $\frac{5}{3}t = 15$

40. $\frac{7}{2}z = -35$

41. $\frac{3}{\square} = 1$

42. $2 = \frac{6}{\triangle}$

43. $\frac{21}{x} = 7$

44. $24 = \frac{8}{y}$

45. $5 = 8 + x$

46. $20 = y - 10$

47. $y - 17.8 = 25.2$

48. $18.3 = z + 97.6$

49. $12x = 93$

50. $18z = 38$

51. $17.3 = 8.5k$

52. $x - 22\frac{1}{2} = 48\frac{3}{4}$

53. $xx = 100$

54. $-81 = tt$

55. $2AA = 98$

56. $yy - 12 = 24$

57. $2 + y = 3 + y$

58. $2y = 3y$

59. $2 - y = 3 - y$

60. $2 \div y = 3 \div y$

__B__. Find the roots of the following equations.

1. $3x = 9$	2. $3x + 4 = 16$
3. $1 = 2x - 5$	4. $7x + 8 = 43$
5. $5 + 8x = 29$	6. $2 + 3q = 29$
7. $5x + 4 = -11$	8. $-12 = 3y + 12$
9. $7k + 9 = 2$	10. $15 + 12m = -9$
11. $2x - 12 = -16$	12. $7 + 3x = 7$
13. $5 - 2x = 1$	14. $9 - 2x = 13$
15. $5x + 7 = 18$	16. $9 = 3 + 4x$
17. $6y - 5 = 17$	18. $9z - 8 = 5$
19. $\frac{1}{2}x - 7 = 12$	20. $\frac{2}{3}y + 2 = 16$
21. $\frac{2}{5}x - 8 = -2$	22. $\frac{1}{8}k - 1 = 1$
23. $9 = \frac{1}{2}(t + 6)$	24. $\frac{1}{3}(s - 4) = 2$
25. $\frac{x + 2}{5} = 3$	26. $\frac{y + 7}{2} = 9$
27. $\frac{m - 5}{8} = 3$	28. $\frac{n - 2}{4} = 6$
29. $2 = \frac{4 - \square}{7}$	30. $\frac{8 - \triangle}{10} = 8$
31. $\frac{2a + 7}{3} = 5$	32. $\frac{5z - 4}{-3} = -2$
33. $3 = 2\frac{1}{2}x + 1\frac{3}{4}$	34. $3\frac{1}{4}y - 7\frac{1}{5} = 8\frac{4}{5}$
35. $5.2y - 4.3 = 8.6$	36. $9.2b - 3.8 = -5.7$
37. $7.8 - 3.9z = 0$	38. $5.2 = 8.7 - 5.3u$
39. $\lvert x - 5 \rvert = 5$	40. $\lvert 3 - 2y \rvert = 1$
41. $2 + \lvert 3 - 3x \rvert = 8$	42. $5 + \lvert 7 - 4x \rvert = 2$
43. $15 = 2ZZ - 3$	44. $9kk - 3 = -2$

[More exercises are in __Part B__, __Supplementary Exercises__.]

C. Each exercise contains descriptions of sets of numbers. For each exercise, tell which descriptions refer to the same set as the first description. [The first description is underlined.]

1. $\underline{\{x:\ 2x + 3x + 7 = 17\}}$
   (a) $\{x:\ 2x + 10 = 17\}$
   (b) $\{x:\ 5x + 7 = 17\}$
   (c) $\{x:\ 12x = 17\}$

2. $\underline{\{r:\ 5r + 3 - 2r - 12 = 12\}}$
   (a) $\{r:\ 5r + r - 12 = 12\}$
   (b) $\{r:\ 3r - 9 = 12\}$
   (c) $\{r:\ 5r - 11 = 12\}$

3. $\underline{\{x:\ 3x + 4(2x + 7) = 83\}}$
   (a) $\{x:\ 7x + 2x + 7 = 83\}$
   (b) $\{x:\ 3x + 8x + 7 = 83\}$
   (c) $\{x:\ 3x + 8x + 28 = 83\}$
   (d) $\{x:\ 11x + 28 = 83\}$

4. $\underline{\{k:\ 2(k - 3) + 7(k + 1) = 46\}}$
   (a) $\{k:\ 2k - 6 + 7k + 1 = 46\}$
   (b) $\{k:\ 2k - 6 + 7k + 7 = 46\}$
   (c) $\{k:\ 9k - 13 = 46\}$
   (d) $\{k:\ 9k + 1 = 46\}$

5. $\underline{\{y:\ 5y - 3 + 4y + 5 = 20\}}$
   (a) $\{y:\ 2y + 4y + 5 = 20\}$
   (b) $\{y:\ 5y + y + 5 = 20\}$
   (c) $\{y:\ 5y - 7y + 5 = 20\}$
   (d) $\{y:\ 9y - 3 + 5 = 20\}$
   (e) $\{y:\ 9y + 2 = 20\}$

6. $\underline{\{t:\ 3(t - 4) - 5(3 - 2t) = 38\}}$
   (a) $\{t:\ 3t - 4 - 15 - 10t = 38\}$
   (b) $\{t:\ 3t - 4 - 15 + 10t = 38\}$
   (c) $\{t:\ 3t - 19 + 10t = 38\}$
   (d) $\{t:\ 3t - 9t = 38\}$
   (e) $\{t:\ 13t - 27 = 38\}$

7. $\underline{\{y:\ 6y - 5(2 - 3y) = 32\}}$
   (a) $\{y:\ y - 2 - 3y = 32\}$
   (b) $\{y:\ 6y - 10 - 15y = 32\}$
   (c) $\{y:\ 6y - 10 + 15y = 32\}$
   (d) $\{y:\ 21y - 10 = 32\}$

8. $\underline{\{z:\ 8z + 3 + 2z + 9 = -28\}}$
   (a) $\{x:\ 8x + 2x + 3 + 9 = -28\}$
   (b) $\{a:\ 10a + 12a = -28\}$
   (c) $\{k:\ 10k + 12 = -28\}$
   (d) $\{z:\ 8z + 3 + 11z = -28\}$

9. $\underline{\{y:\ 8(9 - 3y) + 5(6 - 7y) - 2y = 15\}}$
   (a) $\{y:\ 72 - 3y + 30 - 7y - 2y = 15\}$
   (b) $\{y:\ 72 - 24y + 30 - 35y - 2y = 15\}$
   (c) $\{y:\ 102 - 61y = 15\}$

10. $\underline{\{k: 5k - 2(k - 3) - 5(4 - 5k) = 144\}}$

(a) $\{k: 5k - 2k - 3 - 20 - 5k = 144\}$

(b) $\{k: 5k - 2k + 6 - 20 + 25k = 144\}$

(c) $\{k: 5k + -2k + 25k + 6 + -20 = 144\}$

(d) $\{k: 28k - 14 = 144\}$

11. $\underline{\{x: 8xx - 2x(3 + 4x) = -12\}}$

(a) $\{x: 8xx - 6x - 8xx = -12\}$

(b) $\{x: 2x - 8xx = -12\}$

(c) $\{x: 8xx - 8xx - 6x = -12\}$

(d) $\{x: -6x = -12\}$

<u>D</u>. Solve.

<u>Sample 1</u>. $5x - 3 + 4x + 5 = 20$

<u>Solution</u>. If we simplify the pronumeral expression '$5x - 3 + 4x + 5$', we get '$9x + 2$'. Does the given equation:

(1) $5x - 3 + 4x + 5 = 20$

have the same roots as the equation:

(2) $9x + 2 = 20$?

Suppose you pick a value of 'x' and substitute a numeral for it in both (1) and (2). If the new sentence you get from (1) is true. will the sentence you get from (2) be true? If the sentence you get from (1) is false, will the sentence you get from (2) be false? Which equation is easier to solve, (1) or (2)?

Equation (2) has the root 2. So, equation (1) should have the root 2. We check this by substituting '2' for 'x' in (1):

$$5(2) - 3 + 4(2) + 5 = 20. \quad \text{True?}$$

$$10 - 3 + 8 + 5 \mid 20$$

$$20 = 20 \checkmark \text{ Yes!}$$

<u>Sample 2.</u> $3(x - 4) - 5(3 - 2x) = 38$

<u>Solution.</u> Derive a new equation which has the same roots but is easier to solve by simplifying the left side of '$3(x - 4) - 5(3 - 2x) = 38$'.

$$\begin{aligned} 3(x - 4) - 5(3 - 2x) &= 38 \\ 3x - 12 - 15 + 10x &= 38 \\ 13x - 27 &= 38 \end{aligned}$$

This last equation has the root 5. So, the given equation has the root 5.

<u>Check.</u> $3(5 - 4) - 5(3 - 2 \cdot 5) = 38$ True?

$$\begin{array}{ccc|c} 3 \cdot 1 & - & 5 \cdot -7 & 38 \\ 3 & + & 35 & 38 \end{array}$$

$$38 = 38 \checkmark \quad \text{Yes!}$$

<u>Sample 3.</u> $2x - 3(4 - x) = 7$

<u>Solution.</u>

$$\begin{aligned} 2x - 3(4 - x) &= 7 \\ 2x - 12 + 3x &= 7 \\ 5x - 12 &= 7 \end{aligned}$$

The root is $\frac{19}{5}$.

<u>Check.</u>

$$2 \cdot \frac{19}{5} - 3(4 - \frac{19}{5}) = 7 \ ?$$

$$\begin{array}{ccc|c} \frac{38}{5} & - & 3 \cdot \frac{1}{5} & 7 \\ & & \frac{35}{5} & 7 \end{array}$$

$$7 = 7 \checkmark$$

1. $7y - 2 + 5y = 10$
2. $8a - 3 - 2a = 15$
3. $5z + 4 - 3z + 2 = 2$
4. $1 = m + 2m + 1 + 4m$
5. $3k - 2 - k = 8$
6. $-21 = 4x + 5 + 9x$
7. $3x + 2(x - 2) = 11$
8. $8(7 - k) + 12(3 + 2k) = 12$
9. $x - (2 - x) = 36$
10. $3y - (12 - 2y) = 3$
11. $x - (x - 1) - (x - 2) + 2 = 0$
12. $5m - 2(m - 3) - 5(4 - 5m) = -7$
13. $5(2r - 3) - 3(r + 7) = -15$
14. $7(3 - 5s) - 2(2s - 4) = -127$
15. $2a - 2(3a - 1) = 22$
16. $3b - 5(2 - 4b) = -10$
17. $4(B - 6) + 3(2B + 1) = -41$
18. $26 = 3y + 5(4 - y)$
19. $5 - 7(2 - x) + 4(2x - 5) = 31$
20. $224 = 8(9 - 3y) + 5(6 - 7y) - 2y$
21. $5(3 - 2z) - 6(5z - 2) + 8(3z - 5) = -61$
22. $5.3(4 + 3m) - 8.2(2m - 1) = 7.9$
23. $4x + 2(5 - 2x) = 10$
24. $2x - 2(x - 3) = 7$
25. $3(x + 2) + 2(5 - x) + 8x = 19$
26. $2(y - 3) + 3(3 + 2y) = 7$
27. $\frac{1}{2}(2x - 6) + \frac{1}{3}(3x + 9) = 16$
28. $\frac{1}{3}(9x + 12) + \frac{1}{5}(5x - 15) = 25$
29. $x(x - 5) + 2x(3 - x) + xx = 5$
30. $3y(2 - y) - 4y(3 - y) + 6y - 5 = 116$
31. $6(a - 3) + 7(a - 3) - 8(a - 3) - 4(a - 3) = 7$
32. $5(bb - 3) - 8(bb - 3) + 3(bb - 3) = 6$

[More exercises are in <u>Part C</u>, <u>Supplementary Exercises</u>.]

EXPLORATION EXERCISES

The solution set of the sentence '$x + 4 > 6$' is the set of numbers greater than 2. Here is its graph.

The solution set of the sentence '$x + 4 > 5$' is the set of numbers greater than 1, and its graph is:

It is easy to see that each member of the solution set of '$x + 4 > 6$' is a member of the solution set of '$x + 4 > 5$'. A quick way to express this fact is to say:

the solution set of '$x + 4 > 6$'
is a <u>subset</u> of
the solution set of '$x + 4 > 5$',

which can be abbreviated to:

$$\{x:\ x + 4 > 6\} \subseteq \{x:\ x + 4 > 5\}.$$

Is the solution set of '$x + 4 > 5$' a subset of the solution set of '$x + 4 > 6$'? The answer to this question is 'no' because the solution set of '$x + 4 > 5$' contains at least one member which is not a member of the solution set of '$x + 4 > 6$'. For example, ___ is such a member.

<u>A</u>. True or false?

<u>Sample 1</u>. $\{7, 9, 13\} \subseteq \{7, 9, 13, 17\}$

<u>Solution</u>. True, because each member of $\{7, 9, 13\}$ is a member of $\{7, 9, 13, 17\}$.

<u>Sample 2</u>. $\{5, 8, 71\} \subseteq \{1, 2, 5, 7, 8, 69, 70\}$

<u>Solution</u>. False, because 71 is a member of $\{5, 8, 71\}$ but is not a member of $\{1, 2, 5, 7, 8, 69, 70\}$.

Sample 3. $\{4, 31\} \subseteq \{4, 31\}$

Solution. True. [Why?]

Sample 4. $\{x:\ xx = 1\} = \{x:\ |x| = 1\}$

Solution. $\{x:\ xx = 1\}$ contains just the numbers 1 and -1. Also, $\{x:\ |x| = 1\}$ contains just the numbers 1 and -1. So, the given sentence is true.

1. $\{1, 2\} \subseteq \{2, 3\}$
2. $\{3, 5, 6\} \subseteq \{3, 5, 6, 7\}$
3. $\{2, 5\} \subseteq \{4, 9\}$
4. $\{8, 9, 10\} \subseteq \{7, 9, 11\}$
5. $\{x:\ x > 5\} \subseteq \{x:\ x > 3\}$
6. $\{x:\ x < 5\} \subseteq \{x:\ x < 3\}$
7. $\{t:\ 3t = 1\} \subseteq \{t:\ tt = \frac{1}{9}\}$
8. $\{y:\ yy = 16\} \subseteq \{y:\ y = 4\}$
9. $\{a:\ 8a - 6 = 2a\} \subseteq \{a:\ 8a - 6 - 2a = 0\}$
10. $\{x:\ 4x + 2 = 3x\} \subseteq \{x:\ 4x + 2 - 3x = 0\}$
11. $\{x:\ x = -5\} \subseteq \{x:\ 3xx = 75\}$
12. $\{z:\ zz = 100\} \subseteq \{z:\ |z| = 10\}$
13. $\{a:\ 9aa = 36\} = \{a:\ aa = 4\}$
14. $\{x:\ xx = x\} \subseteq \{x:\ x = 1\}$
15. $\{x:\ x = x + 1\} \subseteq \{x:\ x = x + 2\}$
16. $\{x:\ 2 \cdot |x| = |2x|\} \subseteq \{x:\ x = 8\}$
17. $\{x:\ x = x + 1\} \subseteq \{x:\ 2x = 160\}$ [Hint: Is there a member of $\{x:\ x = x + 1\}$ which is not a member of $\{x:\ 2x = 160\}$? You must believe that the answer to this question is 'yes' if you claim that the sentence is false.]
18. $\{x:\ xx = x \text{ and } x \neq 0\} = \{x:\ \frac{xx}{x} = \frac{x}{x} \text{ and } x \neq 0\}$
19. $\{x:\ 3xx - 2xxx + 5x - 7 = 8x - 9xx + 5\} \subseteq \{x:\ x + 1 = 1 + x\}$

☆20. $\{y:\ y + \frac{1}{y - 8} = 8 + \frac{1}{y - 8} \text{ and } y \neq 8\} = \{y:\ y = 8 \text{ and } y \neq 8\}$

B. Each exercise contains descriptions of solution sets of sentences. For each exercise, tell which descriptions refer to the same set as the first description. [The first description is underlined.]

1. $\underline{\{x:\ 4x + 7 + 2x = 1\}}$

   (a) $\{x:\ [4x + 7 + 2x] + 3 = [1] + 3\}$
   (b) $\{x:\ [4x + 7 + 2x] - 7 = [1] - 7\}$
   (c) $\{y:\ 6y + 7 = 1\}$
   (d) $\{k:\ 6k + 7 - 7 = 1 + 7\}$

2. $\underline{\{b:\ 11b - 8 = 7b\}}$

   (a) $\{b:\ [11b - 8] - 7b = [7b] - 7b\}$
   (b) $\{b:\ 4b - 8 = 0\}$
   (c) $\{b:\ b = -2\}$
   (d) $\{b:\ b[11b - 8] = b[7b]\}$

3. $\underline{\{x:\ 5x - 6 = 3x + 8\}}$

   (a) $\{x:\ [5x - 6] - 3x = [3x + 8] - 3x\}$
   (b) $\{x:\ 2x - 6 = 8\}$
   (c) $\{x:\ [5x - 6] - 5x = [3x + 8] - 5x\}$
   (d) $\{x:\ [5x - 6] - 3x = [3x + 8] - 5x\}$

4. $\underline{\{y:\ 3y - 2 = 2y - 7\}}$

   (a) $\{y:\ [3y - 2] - 2y = [2y - 7] - 2y\}$
   (b) $\{y:\ y - 2 = -7\}$
   (c) $\{y:\ [3y - 2] - 3y = [2y - 7] - 3y\}$
   (d) $\{x:\ 5(3x - 2) = 5(2x - 7)\}$

5. $\underline{\{t:\ 7t - 8 = 10 - 2t\}}$

   (a) $\{t:\ [7t - 8] - 2t = [10 - 2t] - 2t\}$
   (b) $\{t:\ [7t - 8] + 2t = [10 - 2t] + 2t\}$
   (c) $\{t:\ 9t + 8 = 10\}$
   (d) $\{t:\ -8 = [10 - 2t] - 7t\}$

6. $\underline{\{s: 6s + 9 = -2 - 5s\}}$

(a) $\{s: 6s + 9 + 5s = -2\}$

(b) $\{s: 6s + 9 - 6s = -2 - 5s + 6s\}$

(c) $\{s: s(6s + 9) = s(-2 - 5s)\}$

(d) $\{s: 6s + 9 + 5s - 9 = -2 - 5s + 5s - 9\}$

C. In each exercise pick out the equations which have the same roots as the equation in (a).

1. (a) $3x + 4 = 10$
   (b) $[3x + 4] + 2 = [10] + 2$
   (c) $3x + 5 = 15$
   (d) $[3x + 4] + 2x = [10] + 2x$
   (e) $[3x + 4] - 3x = [10] - 3x$
   (f) $9(3x + 4) = 90$

2. (a) $7y + 3 = 31$
   (b) $[7y + 3] + 87 = [31] + 87$
   (c) $[7x + 3] - 159 = [31] - 159$
   (d) $[7y + 3] + 16y = [31] + 16$
   (e) $7y + 3 - y = 31 - y$
   (f) $783(7y + 3) = 783 \cdot 31$

3. (a) $5t - 9 = 5 - 2t$
   (b) $16(5t - 9) = 16(5 - 2t)$
   (c) $[5t - 9] + 10 = [5 - 2t] + 15$
   (d) $[5t - 9] + 4 = [5 - 2t] - 4$
   (e) $[5x - 9] + 2x = [5 - 2x] + 2x$
   (f) $[5t - 9] + 2t = 5$

4. (a) $7 - 3x = 5 - 9x$
   (b) $[7 - 3x] + 3x = [5 - 9x] + 4x$
   (c) $7 = 5 - 6x$
   (d) $700 - 300x = 500 - 900x$
   (e) $[7 - 3x] + 9x = [5 - 9x] + 9x$
   (f) $7 + 6x = 5$

5. (a) $9 - 6x = 8x + 9$
   (b) $[9 - 6x] + 15 = [8x + 9] + 15$
   (c) $[9 - 6x] + 71x = [8x + 9] + 83x$
   (d) $10(9 - 6y) = 100(8y + 9)$
   (e) $[9 - 6x] + 6x = [8x + 9] + 6$
   (f) $[9 - 6x] + 6x = [8x + 9] - 8x$

D. For each of the following equations, write 3 equations which have the same roots as the given equation.

1. $5x + 7 = 17$
2. $3 - 2y = 1$
3. $6x + 1 = 9 - 2x$
4. $8y + 7 = 5 - 7y$
5. $5 - 3x = 8x + 9$
6. $11y + 3 = 8 + 5y$

E. Do the equations:

$$(1) \quad 2x + 4 = 46 - x$$

and:

$$(2) \quad [2x + 4] + x = [46 - x] + x$$

have the same roots?

1. (a) Suppose you know a root of (1). If you substitute for 'x' a numeral for this root in equation (1), will the new sentence you get be true? Will the numerical expressions on both sides of the equality sign in the new sentence be names for the same number?

   (b) Suppose you substitute for 'x' in equation (2) a name of this root of equation (1). Will the numerical expressions you get <u>in the brackets</u> be names for the same number? How can you tell? Will the new sentence you get from (2) be a true sentence?

2. Your answers to parts (a) and (b) of Exercise 1 should tell you that each root of (1) is a root of (2). Another way of saying this is [complete the sentence]:

   $\{x: 2x + 4 = 46 - x\}$ _______ $\{x: [2x + 4] + x = [46 - x] + x\}$.

3. (a) Suppose you know a root of (2), and you substitute one of its numerals for 'x' in (2). Will the new sentence you get be true? Will the numerical expressions on both sides of the equality sign be names for the same number? Will the numerical expressions you get <u>in the brackets</u> be names for the same number? What principle learned in Unit 2 tells you this?

   (b) Suppose you substitute for 'x' in (1) a name for this root of (2). Will you get a true sentence from (1)? How can you use your answers to part (a) to show this?

4. Your answers to parts (a) and (b) of Exercise 3 should tell you that each root of (2) is a root of (1). In other words, [complete the sentence]:

$\{x\colon\ [2x + 4] + x = [46 - x] + x\}$ ______ $\{x\colon\ 2x + 4 = 46 - x\}$.

* * *

In Exercise 2 you found that

(*) $\{x\colon\ 2x + 4 = 46 - x\} \subseteq \{x\colon\ [2x + 4] + x = [46 - x] + x\}$,

and, in Exercise 4 that

(**) $\{x\colon\ [2x + 4] + x = [46 - x] + x\} \subseteq \{x\colon\ 2x + 4 = 46 - x\}$.

These statements tell you that each member of a first set is a member of a second set, and that each member of the second set is a member of the first set. Do you see that this is just another way of saying that the first set is the same as the second set? So, it follows from (*) and (**) that

$$\{x\colon\ 2x + 4 = 46 - x\} = \{x\colon\ [2x + 4] + x = [46 - x] + x\}.$$

In other words, the equations:

(1) $2x + 4 = 46 - x$

and: (2) $[2x + 4] + x = [46 - x] + x$

have the same roots.

* * *

F. 1. Tell why $\{x\colon\ 7x + 1 = 3x - 6\} \subseteq \{x\colon\ 5(7x + 1) = 5(3x - 6)\}$.

2. Tell why $\{x\colon\ 5(7x + 1) = 5(3x - 6)\} \subseteq \{x\colon\ 7x + 1 = 3x - 6\}$.

3. Tell why $\{x\colon\ 7x + 1 = 3x - 6\} \subseteq \{x\colon\ x(7x + 1) = x(3x - 6)\}$, and $\{x\colon\ x(7x + 1) = x(3x - 6)\} \not\subseteq \{x\colon\ 7x + 1 = 3x - 6\}$.

3.05 Equivalent equations. --You have solved many equations so far in this unit. The method you used may have been something like this.

To solve the equation:

$$3x + 4 = 19,$$

I must find a number such that 4 more than its product with 3 is 19. But, 15 + 4 is 19. So, it follows [from what principle?] that the product of this number with 3 is 15. By the principle of division, $\frac{15}{3}$ is the number whose product with 3 is 15. So, 5 is the solution of the equation.

Now, let's try to solve the equation:

$$(1) \quad 5x + 9 = 13 - 2x.$$

You must find a number such that 9 more than its product with 5 is the difference of twice the number from 13! This is a much more difficult job than the first example.

We could make some guesses.

(1) Try 7.

$$5(7) + 9 = 13 - 2(7)?$$

$$44 \neq -1.$$

(2) Try 2.

$$5(2) + 9 = 13 - 2(2)?$$

$$19 \neq 9.$$

(3) Try 1.

$$5(1) + 9 = 13 - 2(1)?$$

$$14 \neq 11.$$

(4) Try 0.

$$5(0) + 9 = 13 - 2(0)?$$

$$9 \neq 13.$$

It seems that the root is between 0 and 1 [Why?]. Is it closer to 0 than to 1? You could get to the root by continuing this process of guessing and "closing in" on it. But, there is a faster procedure.

What we should like to do is to find another equation which is easier to solve than:

$$(1) \quad 5x + 9 = 13 - 2x$$

but which has the <u>same</u> roots as (1). An equation whose roots are the same as the roots of (1) is said to be <u>equivalent</u> to (1) [and (1) is <u>equivalent</u> to it].

Now, there are many equations which are equivalent to equation (1). Here are just a few of them:

$$(a) \quad [5x + 9] + 15 = [13 - 2x] + 15,$$

$$(b) \quad [5x + 9] - 78 = [13 - 2x] - 78,$$

$$(c) \quad 10[5x + 9] = 10[13 - 2x],$$

$$(d) \quad [5x + 9] + 4x = [13 - 2x] + 4x.$$

But, even if you simplify the sides of these equations to get:

$$(a') \quad 5x + 24 = 28 - 2x,$$

$$(b') \quad 5x - 69 = -65 - 2x,$$

$$(c') \quad 50x + 90 = 130 - 20x,$$

$$(d') \quad 9x + 9 = 13 + 2x,$$

none of the simplified ones seems to be easier to solve.

What is there about these equations and equation (1) which makes them harder to solve than the equations you solved on pages 3-21 and 3-25? Look at the equations on those pages. Notice that in none of them do you find a pronumeral in <u>both</u> sides. Equation (1) [as well as (a'), (b'), (c'), and (d')] has the pronumeral occurring in both sides:

$$(1) \quad 5x + 9 = 13 - 2x,$$

and this is what makes it harder to solve than the equations you worked with earlier. So, what we need to do is to get an equation equivalent to (1) with the pronumeral in one side only. Do you see how this can be done?

Here is one possibility. If we write a '+ 2x' on both sides of equation (1), we get:

$$(2)\quad 5x + 9 + 2x = 13 - 2x + 2x,$$

which is equivalent to (1). Will (2) be of more help to us than any of (a), (b), (c), and (d)? We can simplify the sides of (2) to get:

$$(3)\quad 7x + 9 = 13.$$

Are (2) and (3) equivalent? Since (1) is equivalent to (2), and (2) is equivalent to (3), do you think that (1) is equivalent to (3)?

Equation (3) is just like those we solved earlier. Its root is $\frac{4}{7}$, and since (3) and (1) are equivalent, we know that $\frac{4}{7}$ is a root of (1).

Instead of stopping with equation (3), we could continue getting equivalent equations which are easier to solve. We write a '$+ -9$' on both sides of (3) to get:

$$(4)\quad 7x + 9 + -9 = 13 + -9.$$

Is (4) equivalent to (3)? Simplify the sides of (4) to get:

$$(5)\quad 7x = 4.$$

Is (5) equivalent to (4)? It is easy to see that the root of (5) is $\frac{4}{7}$.

We can continue this process of getting easier equivalent equations. Write a '$\times \frac{1}{7}$' on both sides of (5) to get:

$$(6)\quad 7x \times \frac{1}{7} = 4 \times \frac{1}{7}.$$

Is (6) equivalent to (5)? Finally, simplify the sides of (6) to get:

$$(7)\quad x = \frac{4}{7}.$$

Is (7) equivalent to (6)? Equation (7) is the easiest to solve!

Let us summarize the steps involved in solving the given equation '5x + 9 = 13 - 2x'.

$$\begin{array}{ll} (1) & 5x + 9 = 13 - 2x \\ (2) & 5x + 9 + 2x = 13 - 2x + 2x \\ (3) & 7x + 9 = 13 \\ (4) & 7x + 9 + {}^-9 = 13 + {}^-9 \\ (5) & 7x = 4 \\ (6) & 7x \times \frac{1}{7} = 4 \times \frac{1}{7} \\ (7) & x = \frac{4}{7} \end{array}$$

Equations (1) and (2) are equivalent, equations (2) and (3) are equivalent, (3) and (4) are equivalent, (4) and (5) are equivalent, (5) and (6) are equivalent, and (6) and (7) are equivalent. So, (1) and (7) are equivalent.

[Suppose you substitute a numeral for 'x' in all these equations. If one of these equations is converted into a true sentence, what can you say about the sentences obtained from the rest of the equations? If you substitute a numeral for 'x' in all these equations and find that one of the sentences obtained is false, what can you say about the rest?]

Notice that the left sides of equations (2) and (3) are equivalent <u>expressions</u>, and that the right sides of (2) and (3) are equivalent expressions. Do these facts tell you that (2) and (3) are equivalent equations? What tells you that (4) and (5) are equivalent equations? What tells you that (6) and (7) are equivalent equations?

Are the left sides [and right sides] of (1) and (2) equivalent expressions? How do we know that (1) and (2) are equivalent equations? That (3) and (4) are equivalent equations? That (5) and (6) are equivalent? Your work in Part E on page 3-30 and in Part F on page 3-31 should help you answer these questions.

EXERCISES

A. Solve these equations by transforming the given equation to an equivalent one whose root is obvious.

Sample 1. $7t - 8 = 3 + 4t$

Solution.

$$\begin{aligned} 7t - 8 &= 3 + 4t \\ 7t - 8 + {}^-4t &= 3 + 4t + {}^-4t \\ 3t - 8 &= 3 \\ 3t - 8 + 8 &= 3 + 8 \\ 3t &= 11 \\ \frac{1}{3} \times 3t &= 11 \times \frac{1}{3} \\ t &= \frac{11}{3} \end{aligned}$$

The root is $\frac{11}{3}$.

[Since we believe that each of these equations is equivalent to each of the others, we believe that the root of the equation '$t = \frac{11}{3}$' is the root of the equation '$7t - 8 = 3 + 4t$'. However, sometimes in transforming equations we may make an error in simplification or computation [especially if in a hurry--or quite sleepy!]; so, it is a good idea to check the root of the last equation by substituting in the given equation.]

Check.

$$7\cdot\frac{11}{3} - 8 = 3 + 4\cdot\frac{11}{3} \quad ?$$

$$\begin{array}{c|c} \frac{77}{3} - 8 & 3 + \frac{44}{3} \\ \frac{77 - 24}{3} & \frac{9 + 44}{3} \end{array}$$

$$\frac{53}{3} = \frac{53}{3} \quad \checkmark$$

Sample 2. $8 - 4x = 7 - 9x$

Solution.

$$\begin{aligned} 8 - 4x &= 7 - 9x \\ 8 - 4x + 4x &= 7 - 9x + 4x \\ 8 &= 7 - 5x \\ 8 + {}^-7 &= 7 - 5x + {}^-7 \\ 1 &= {}^-5x \\ -\frac{1}{5}\cdot 1 &= -\frac{1}{5}\cdot {}^-5x \\ -\frac{1}{5} &= x \end{aligned}$$

The root is $-\frac{1}{5}$. [Check this!]

[Note: In doing the exercises which follow, it is a good idea to put in all of the steps as shown in the samples. Later, you will discover and use many short cuts.]

1. $3m - 2 = 8 - 2m$

2. $7t - 5 = 15 + 3t$

3. $4 + 5s = 3s$

4. $8 - k = 5k - 10$

5. $3 + 2x = 7 + 6x$

6. $5y - 3 = 9 + 2y$

7. $4m - 3 = 3 - 4m$

8. $3k - 10.5 = 4.5 - 12k$

Sample 3. $\frac{a}{2} + 2 + \frac{a}{4} = 7 + \frac{a}{3}$

Solution. This equation is complicated because it contains several fractions. We can transform it into an equivalent equation which does not contain any fractions by enclosing the sides in parentheses, writing a '12·' on both sides [Why '12'?], and then simplifying.

$$\begin{aligned}
\frac{a}{2} + 2 + \frac{a}{4} &= 7 + \frac{a}{3} \\
12\cdot\left(\frac{a}{2} + 2 + \frac{a}{4}\right) &= 12\cdot\left(7 + \frac{a}{3}\right) \\
12\cdot\frac{a}{2} + 12\cdot 2 + 12\cdot\frac{a}{4} &= 12\cdot 7 + 12\cdot\frac{a}{3} \\
6a + 24 + 3a &= 84 + 4a \\
9a + 24 &= 84 + 4a \\
-4a + 9a + 24 &= 84 + 4a + -4a \\
5a + 24 &= 84 \\
5a + 24 - 24 &= 84 - 24 \\
5a &= 60 \\
\frac{1}{5} \times 5a &= 60 \times \frac{1}{5} \\
a &= 12
\end{aligned}$$

The solution is 12. [Check this!]

9. $\frac{x}{4} - 5 = \frac{x}{3}$

10. $\frac{t}{4} + \frac{t}{8} = 5 + \frac{t}{3}$

11. $\frac{3y}{4} + 6 = \frac{8y}{7}$

12. $\frac{k}{2} + \frac{3k}{5} - 1 = \frac{k}{10} - \frac{2k}{3}$

13. $2(x + 1) = 2 + 2x$

14. $3 + x = x$

* * *

In doing Exercise 13 on the preceding page, you probably recognized immediately that since the two sides of the equation are equivalent pronumeral expressions, it has each real number as a root. And in Exercise 14, you no doubt knew at once that the equation had no roots. But, what would happen if you tried to solve these equations by the method of equivalent equations?

Let's try Exercise 13.

$$\begin{array}{lrcl} (1) & 2(x + 1) & = & 2 + 2x \\ (2) & 2(x + 1) + -2x & = & 2 + 2x + -2x \\ (3) & 2x + 2 + -2x & = & 2 \\ (4) & 2 & = & 2 \end{array}$$

Equations (1), (2), (3), and (4) are supposed to be equivalent equations. This means that these equations have the same roots. Now, it may be easy to see that (1) and (2) have the same roots [each substitution for 'x' in (1) and (2) gives you a pair of sentences which are both true], and that (2) and (3) have the same roots, but what does it mean to say that (3) and (4) have the same roots? Equation (4) doesn't even have a pronumeral in it!

In order to make sense out of this we shall extend our idea of what a root is. We said earlier that a root of an equation is a number which satisfies the equation, and we used the word 'satisfy' just in connection with <u>open sentences</u>. We said nothing about whether <u>statements</u> [such as: $3 > 1 + 0$, $3 \neq 17 - 14$, $2 = 2$, $18 = 19$] can be satisfied. It seems strange to think about whether sentences like '$5 = 4$' and '$2 = 2$' can be satisfied. But, let's stretch our meaning of the word 'satisfy' and agree that <u>every</u> number satisfies a true statement and that <u>no</u> number satisfies a false statement. Under this agreement, we can say that

'$2(x + 1) = 2 + 2x$' is equivalent to '$2 = 2$'.

Now, let's consider the equation in Exercise 14 on page 3-37, '$3 + x = x$'. We proceed according to the familiar method.

$$(1') \qquad 3 + x = x$$

$$(2') \qquad 3 + x + -x = x + -x$$

$$(3') \qquad 3 = 0$$

Under the agreement we just made, (3′) has no roots because it is a false sentence. We observed at the outset that (1′) does not have a root. So, under our agreement, we say that

'$3 + x = x$' is equivalent to '$3 = 0$'.

* * *

B. Solve these equations. Look for short cuts.

Sample 1. $4x - 3 + 2x = 5 - 10x$

Solution.

$$4x - 3 + 2x = 5 - 10x$$
$$6x - 3 = 5 - 10x$$
$$10x + 6x - 3 = 5 - 10x + 10x$$
$$16x - 3 = 5$$
$$16x = 8$$
$$x = \frac{1}{2}$$

The root is $\frac{1}{2}$.

Check. $4 \cdot \frac{1}{2} - 3 + 2 \cdot \frac{1}{2} = 5 - 10 \cdot \frac{1}{2}$?

$$2 - 3 + 1 \;\Big|\; 5 - 5$$
$$0 = 0 \quad \checkmark$$

1. $4x + 3 = 2x + 7$	2. $6k - 2 = 8 - 14k$
3. $3 - 7r = 17 + 7r$	4. $8t - 3 = 12 - 2t$
5. $10k + 3 = 16 - 3k$	6. $4 - 7y = y - 20$
7. $18 + 4x = x$	8. $17 + 2y = 9$
9. $5m = 5 - 5m$	10. $7y = 3y + 8$
11. $17b = 2b - 120$	12. $4x = 3x + 4$
13. $360 + 36t = 30t$	14. $17r = -5r + 66$
15. $-20A = 208 + 6A$	16. $15 - 3z = 12z$
17. $25t = 16 - 7t$	18. $5b - 1 = 3b + 3$
19. $d + 2 = 10 - d$	20. $3x - 6 = 14 - x$
21. $2c + 4 = c - 2$	22. $3a + 2 = 7 - 2a$
23. $80 + x = 79$	24. $500 + 15s = 5s$
25. $5n + 6 = 3n + 7$	26. $15k = 25k + 65$
27. $7p - 2 = 4p + 10$	28. $2 - 7m = 11 - 7m$
29. $x + 3x = 3x + 1$	30. $3 - x = 3 + x$
31. $3x = 5x + 1$	32. $3x = 5x$
33. $-6x + 11 = 2x + 43$	34. $5c - 3 = 8c - 16$
35. $3n - 7 = 2n + 7$	36. $4n - 10 = 3n + 5$
37. $8x + 3 = 5x + 30$	38. $12y - 1 = 4y + 3$
39. $7t - 3 = 6t + 1$	40. $11k - 3 = 7k + 9$
41. $8r - 7 = 6r + 1$	42. $4x - 11 = x + 4$
43. $4y - 5 = 3y + 5$	44. $103 - x = x + 3$
45. $s + 10 = 7s - 20$	46. $8n - 5 = 23n - 35$
47. $4s - 4 = s + 20$	48. $13n + 5 = 8n + 40$
49. $S = 14.70 + .30S$	50. $S = 13.20 + .20S$

[More exercises are in Part D, Supplementary Exercises.]

<u>Sample 2</u>. $7(x - 3) + 4 = 3x + 3$

<u>Solution</u>.

$$\begin{aligned} 7(x - 3) + 4 &= 3x + 3 \\ 7x - 21 + 4 &= 3x + 3 \\ 7x - 17 &= 3x + 3 \\ 4x &= 20 \\ x &= 5 \end{aligned}$$

The root is 5.

<u>Check</u>. $7(5 - 3) + 4 = 3 \cdot 5 + 3$?

$$14 + 4 \mid 15 + 3$$

$$18 = 18 \checkmark$$

51. $3(y - 2) + 5 = 3 + 5y$
52. $5(2x + 9) = 3(4x + 17)$
53. $9(3 - 2z) = 2(12 - 6z)$
54. $4x + 4 = 4(x + 1)$
55. $5(3 - x) - 1 = 13 - 5x$
56. $4(x - 5) + 3x + 1 = 2(x - 2) + 5(x - 3)$
57. $7(2 - x) + 4x = 2 + 3x$
58. $8 - 5x + 6 = 7(x + x) - 12x$
59. $8x = 2(3x + 4) + 6$
60. $3x + 5 - x = 2(2 + x) + 1$
61. $80(60 - n) + 100n = 88(60)$
62. $50p + 60(75) = 52(p + 75)$
63. $(5x - 24) - 6 = 5(x - 6)$
64. $2x + 3(5 - x) = 6(3 - x) + 5x$
65. $7(x - 2) - 2(3 + x) = 0$
66. $-7a + 4(2a - 3) = 16$
67. $7r - 2(1 - 4r) = 1$
68. $-3y + 6(y - 4) = 9$
69. $3(2a - 9) = 5(10 - a)$
70. $3(2x - 5) + 2(5 - x) = 4(x - 1) - 1$
71. $20x + 10(2x) + 5(2x + 6) + (5x + 6) = 806$
72. $25y + 10y + 5(y + 11) + (y + 17) = 359$
73. $-4x + 2(5x + 1) - 5 = 6 - 3(2x - 1)$
74. $10(3b - 4) - 5(4b + 7) = 10(5b - 6) - 25$
75. $18(c + 7) - 14(7c + 10) = 7c(4c - 7) - 4c(7c - 1)$

[More exercises are in <u>Part E</u>, <u>Supplementary Exercises</u>.]

Sample 3. $\frac{x - 7}{5} + 2 = \frac{x + 8}{10}$

Solution. Transform to an equation which has no fractions.

$$10 \times \left(\frac{x - 7}{5} + 2\right) = \left(\frac{x + 8}{10}\right) \times 10$$

$$10 \cdot \frac{x - 7}{5} + 10 \cdot 2 = x + 8$$

$$\begin{aligned} 2(x - 7) + 20 &= x + 8 \\ 2x - 14 + 20 &= x + 8 \\ 2x + 6 &= x + 8 \\ x &= 2 \end{aligned}$$

The solution is 2.

Check. $\frac{2 - 7}{5} + 2 = \frac{2 + 8}{10}$?

$$\begin{array}{r|l} -1 + 2 & 1 \\ 1 = & 1 \checkmark \end{array}$$

Sample 4. $.25x + .50(70 - x) = 25.00$

Solution.

Method I

$$\begin{aligned} .25x + .50(70 - x) &= 25.00 \\ .25x + 35 - .50x &= 25.00 \\ -.25x + 35 &= 25.00 \\ -.25x &= -10 \\ x &= \frac{-10}{-.25} \\ x &= 40 \end{aligned}$$

The root is 40.

Method II

$$\begin{aligned} .25x + .50(70 - x) &= 25.00 \\ 100[.25x + .50(70 - x)] &= 100[25.00] \\ 25x + 50(70 - x) &= 2500 \\ 25x + 3500 - 50x &= 2500 \\ -25x + 3500 &= 2500 \\ -25x &= -1000 \\ x &= 40 \end{aligned}$$

The root is 40.

Check. $.25(40) + .50(70 - 40) = 25.00$?

$$\begin{array}{r|l} 10 + 15 & 25 \\ 25 = & 25 \checkmark \end{array}$$

76. $\frac{a - 1}{3} = 2$

77. $m = \frac{1}{10} + \frac{1}{15}$

78. $\frac{1}{7}n = \frac{1}{2}n - 10$

79. $\frac{1}{2}y + \frac{1}{10}y + \frac{1}{5}y = y - 6$

80. $\frac{x}{200} + \frac{x}{160} = \frac{9}{2}$

81. $a + \frac{3}{5}a + 500 = 12500$

82. $c - .70c = 67.5$

83. $.05x + .04(4000 - x) = 175$

84. $.04n = .05(n - 300)$

85. $2(100 - p) + 1.6p = 1.75(100)$

86. $2n - 6 = \frac{1}{3}n$

87. $\frac{k}{4} - 3 = 12 + \frac{k}{5}$

88. $\frac{1}{2}(970 - a) + \frac{3}{4}a = 550$

89. $10x + (-3 + x) = \frac{7}{4}(-30 + 10x + x)$

90. $\frac{x}{3} - 2 = 5 + \frac{x}{2}$

91. $\frac{2x + 3}{2} = -5$

92. $\frac{y + 11}{6} - \frac{10 - y}{3} + 1 = 0$

93. $\frac{x + 9}{9} + \frac{1}{3} = \frac{x - 7}{2} - 1$

94. $\frac{a + 2}{3} + \frac{1}{2}(a + 3) = 3$

95. $\frac{x + 5}{2} - \frac{x + 1}{4} = 3$

96. $\frac{2x}{3} + \frac{3x}{2} = \frac{13}{3}$

97. $\frac{3x}{5} = \frac{19}{5} + \frac{5x}{2}$

98. $\frac{8y}{7} + \frac{2}{21} = 1 - \frac{2y}{3}$

99. $\frac{4m}{3} + \frac{3m}{5} = \frac{7m}{2}$

100. $\frac{1}{3}(5 - 7x) + \frac{1}{2}(4x + 7) = \frac{1}{6}(3x + 31)$

101. $\frac{2}{5}(8 - 3x) + \frac{5}{2}(6 - 7x) = \frac{3}{10}(4x + 15) - \frac{31}{5}$

102. $\frac{3}{a} = 2 - \frac{2 + a}{a}$

103. $\frac{5}{x} = 4 - \frac{3x - 5}{x}$

[More exercises are in Part F, Supplementary Exercises.]

*

Let's reconsider the equations in Exercises 102 and 103 on page 3-43. Both of these equations contain fractions. So, as we have seen, a useful first step in solving them is to transform them into equations which do not contain fractions.

$\frac{3}{a} = 2 - \frac{2 + a}{a}$	$\frac{5}{x} = 4 - \frac{3x - 5}{x}$
$a\left(\frac{3}{a}\right) = \left(2 - \frac{2 + a}{a}\right)a$	$x\left(\frac{5}{x}\right) = \left(4 - \frac{3x - 5}{x}\right)x$
$3 = 2a - \frac{2 + a}{a} \cdot a$	$5 = 4x - \frac{3x - 5}{x} \cdot x$
$3 = 2a - (2 + a)$	$5 = 4x - (3x - 5)$
$3 = 2a - 2 - a$	$5 = 4x - 3x + 5$
$5 = a$	$0 = x$

Now, although 5 is a root of the first equation, 0 is not a root of the second equation. [Substitute '0' for 'x' in the second equation. Do you get a true statement?] How do we explain the fact that our usual method of solving gave a root in the first case but did not give a root in the second?

We shall be in a better position to answer this question if we look more closely at what happens when we transform an equation as we did in the two examples. Consider the equation:

$$(1) \quad 3x = 21.$$

Suppose we transform this equation by writing a '$\cdot x$' on both sides. We get:

$$(2) \quad 3x \cdot x = 21 \cdot x.$$

Both 0 and 7 are roots of (2) because $3 \cdot 0 \cdot 0 = 21 \cdot 0$ and $3 \cdot 7 \cdot 7 = 21 \cdot 7$. Equation (1), on the other hand, has just the root 7. So, (1) and (2) are not equivalent. [However, the solution set of (1) is a subset of the solution set of (2). [See Part F on page 3-31.]]

Take another example. Transform the equation:

$$(1') \qquad xx = 2x$$

by writing a '$\cdot \frac{1}{x}$' on both sides. This gives us:

$$(2') \qquad xx \cdot \frac{1}{x} = 2x \cdot \frac{1}{x}.$$

(1′) has the roots 0 and 2, but (2′) has only the root 2. So, (1′) and (2′) are not equivalent. [However, the solution set of (2′) is a subset of the solution set of (1′).]

From these two examples we see that when we transform an equation by multiplication, we may not get an equivalent equation. In fact, we may "pick up" roots or "lose" them. The first occurs only if the "multiplier" has 0 as a value. The second occurs only if there are values of the pronumeral for which the multiplier has no value ['$\frac{1}{x}$' has no value which corresponds with the value 0 of 'x']. So, in transforming an equation by multiplication, if we restrict the set of values of the pronumeral to those for which the multiplier has nonzero values, we can be sure that those numbers in the restricted set of values of the pronumeral which satisfy the given equation will satisfy the derived equation, and vice versa.

Let's consider some more examples. Take the equation:

$$\frac{x - 1}{x + 1} = 2x - \frac{x + 3}{x + 1}.$$

We would like to transform this equation into one which has the same roots but which does not contain fractions. Previous exercises suggest that a first step is to transform by multiplying by '$x + 1$'.

$$(x + 1)\left(\frac{x - 1}{x + 1}\right) = \left(2x - \frac{x + 3}{x + 1}\right)(x + 1).$$

Now, since '$x + 1$' has the value 0 only for the value -1 of 'x' and, in fact, has nonzero values for all values of 'x' different from -1, we can be sure that this equation and the given one are equivalent with respect to the set of real numbers different from -1. That is, we can be sure that they are satisfied by the same members of this restricted set. To indicate this fact we usually write an '$[x \neq -1]$'

at the right of the second equation. Here is how we might write the steps in solving the equation.

$$\frac{x - 1}{x + 1} = 2x - \frac{x + 3}{x + 1}$$

$$(x + 1)(\frac{x - 1}{x + 1}) = (2x - \frac{x + 3}{x + 1})(x + 1), \quad [x \neq -1]$$

$$x - 1 = 2x(x + 1) - (x + 3)$$

$$= 2xx + 2x - x - 3$$

$$= 2xx + x - 3$$

$$(x - 1) - (x - 1) = (2xx + x - 3) - (x - 1)$$

$$0 = 2xx - 2$$

The roots of this last equation are 1 and −1. Of these, only 1 belongs to the restricted set of values of 'x'. So, we know that 1 is a root of the given equation, and that the given equation has no other roots in the restricted set. If the given equation has roots other than 1, they must be outside the restricted set. The only such number is −1, and we discover by substitution that −1 is not a root of the given equation. So, the solution set of the given equation is {1}.

Now, consider the equation:

$$3(x + 1) = (x + 2)(x + 1).$$

We might transform this equation into a simpler one by multiplying by '$\frac{1}{x + 1}$', first getting:

$$(\frac{1}{x + 1})[3(x + 1)] = [(x + 2)(x + 1)](\frac{1}{x + 1}),$$

and then simplifying both sides to get:

$$3 = x + 2.$$

Since '$\frac{1}{x + 1}$' has values [nonzero ones, at that] for all values of 'x' except −1, we can be sure that the given equation and the next one are equivalent with respect to the set of real numbers different from −1. [As before, we would show this by writing an '[x ≠ −1]' next to the second equation.] The only root of the third equation is 1. So,

since 1 belongs to the restricted set, we know that 1 is a root of the given equation. The only other numbers which could be roots of the given equation are numbers outside the restricted set. There is only one such number. It is -1. By substitution, we find that -1 is a root of the given equation. So, its solution set is $\{1, -1\}$.

*

Sample 5. $13 - \frac{2}{x+2} = \frac{4x+6}{x+2}$

Solution. $13 - \frac{2}{x+2} = \frac{4x+6}{x+2}$

$$(x+2)(13 - \frac{2}{x+2}) = (\frac{4x+6}{x+2})(x+2), \quad [x \neq -2]$$

$$(x+2)13 - 2 = 4x + 6$$

$$13x + 26 - 2 = 4x + 6$$

$$9x = -18$$

$$x = -2$$

The given equation has no roots in the restricted set. Obviously, -2 is not a root. So, the given equation has no roots at all.

*

[Consider the equation:

$$2x - \frac{1}{x-7} = 14 - \frac{1}{x-7}.$$

We write a '$+\frac{1}{x-7}$' on both sides to get:

$$2x - \frac{1}{x-7} + \frac{1}{x-7} = 14 - \frac{1}{x-7} + \frac{1}{x-7},$$

and then simplify to get:

$$2x = 14.$$

The root of this last equation is 7, but 7 is not a root of the given equation. What happened?]

*

<u>Sample 6</u>. $3 + \frac{7}{x - 5} = \frac{2x + 1}{x - 5}$

<u>Solution</u>. $3 + \frac{7}{x - 5} = \frac{2x + 1}{x - 5}$

$$(x - 5)\left(3 + \frac{7}{x - 5}\right) = \left(\frac{2x + 1}{x - 5}\right)(x - 5), \quad [x \neq 5]$$

$$3(x - 5) + 7 = 2x + 1$$

$$3x - 15 + 7 = 2x + 1$$

$$x = 9$$

The root is 9.

<u>Check</u>. $3 + \frac{7}{9 - 5} = \frac{2 \cdot 9 + 1}{9 - 5}$?

$$3 + \frac{7}{4} \quad \Big| \quad \frac{19}{4}$$

$$\frac{19}{4} = \frac{19}{4} \checkmark$$

<u>Sample 7</u>. $\frac{3x - 4}{2x} - \frac{x + 1}{3x} + \frac{x + 2}{5x} = \frac{2}{5}$

<u>Solution</u>. $\frac{3x - 4}{2x} - \frac{x + 1}{3x} + \frac{x + 2}{5x} = \frac{2}{5}$

$$30x\left(\frac{3x - 4}{2x} - \frac{x + 1}{3x} + \frac{x + 2}{5x}\right) = \left(\frac{2}{5}\right)30x, \quad [x \neq 0]$$

$$15(3x - 4) - 10(x + 1) + 6(x + 2) = 12x$$

$$45x - 60 - 10x - 10 + 6x + 12 = 12x$$

$$41x - 58 = 12x$$

$$29x = 58$$

$$x = 2$$

The root is 2.

<u>Check</u>. $\frac{3 \cdot 2 - 4}{2 \cdot 2} - \frac{2 + 1}{3 \cdot 2} + \frac{2 + 2}{5 \cdot 2} = \frac{2}{5}$?

$$\frac{2}{4} - \frac{3}{6} + \frac{4}{10} \quad \Big| \quad \frac{2}{5}$$

$$\frac{2}{5} = \frac{2}{5} \checkmark$$

<u>Sample 8</u>. $\frac{6}{5b} = \frac{2}{b + 4}$

<u>Solution</u>. $\frac{6}{5b} = \frac{2}{b + 4}$

$5b(b + 4)\left(\frac{6}{5b}\right) = \left(\frac{2}{b + 4}\right) 5b(b + 4)$, $[b \neq 0 \text{ and } b \neq -4]$

$$(b + 4)6 = (2)5b$$
$$6b + 24 = 10b$$
$$24 = 4b$$
$$6 = b$$

The root is 6.

<u>Check</u>.

$$\frac{6}{5 \cdot 6} = \frac{2}{6 + 4} \ ?$$
$$\frac{1}{5} = \frac{1}{5} \ \checkmark$$

104. $\frac{2}{b} = 3$

105. $\frac{3}{x} = -4$

106. $\frac{2}{3d} = \frac{1}{9}$

107. $\frac{d}{d - 3} = 2$

108. $\frac{2}{x + 3} = \frac{5}{x}$

109. $\frac{y}{y + 5} = \frac{1}{2}$

110. $\frac{5}{x} - \frac{1}{2} = 2$

111. $\frac{4}{k} + \frac{3}{2k} = \frac{11}{6}$

112. $\frac{x}{3} + 4 = \frac{x}{2} - \frac{x}{6}$

113. $\frac{x}{7} + \frac{x}{3} = \frac{x}{21}$

114. $\frac{3}{a} - 2 = \frac{2}{a} + 3$

115. $\frac{4}{x} + \frac{5}{2x} = 3 - \frac{1}{x}$

116. $\frac{5}{2y} + \frac{1}{3y} = \frac{1}{y} + 2$

117. $\frac{7}{k} + \frac{1}{5} = \frac{2}{15k}$

118. $\frac{8}{x} - 9 = \frac{17}{x}$

119. $\frac{3}{a} - 4 = \frac{7}{a} + 8$

120. $\frac{1 + y}{y} - 5 = \frac{1 - 3y}{y}$

121. $\frac{6}{5x} + 2 = \frac{8}{x}$

122. $\frac{7}{2b} - 6 = 9 - \frac{5}{4b}$

123. $\frac{3x + 8}{2x + 5} - \frac{x + 3}{2x + 5} = 20$

(continued on next page)

124. $\frac{x}{2} - \frac{x}{4} + \frac{2 - 3x}{6} = 0$

125. $\frac{m + 2}{3} = m + .5$

126. $\frac{m + 3}{2} = \frac{m + 7}{5}$

127. $\frac{4a + 1}{3} = \frac{5 - a}{6}$

128. $\frac{3x - 5}{x} = 2$

129. $\frac{5}{y + 3} = \frac{2}{y}$

130. $\frac{c - 1}{c - 2} = \frac{3}{2}$

131. $\frac{3}{2} = \frac{a}{a + 2}$

132. $\frac{k - 2}{k + 3} = \frac{3}{8}$

133. $\frac{m - 7}{m + 2} = \frac{1}{4}$

134. $\frac{x + 2}{2} + \frac{3x}{5} + \frac{x + 1}{4} = 16$

135. $\frac{500}{5t} = \frac{500}{t} - 8$

136. $\frac{b + 11}{6} + \frac{b - 10}{3} = 1$

137. $\frac{180}{\frac{3}{2}x} = \frac{180}{x} - 2$

138. $x + \frac{x + 1}{2} + \frac{4x}{5} = 58$

139. $\frac{x}{4} + \frac{x + 2}{2} = 10$

140. $\frac{44}{11 - x} = \frac{4x}{11 - x} - \frac{19}{3}$

141. $\frac{5}{2}\left(\frac{1}{x - 1}\right) + \frac{2}{3} = \frac{39x - 3x}{2(x - 1)}$

142. $\frac{7}{x - 2} = \frac{9}{x}$

143. $\frac{2}{x + 4} = \frac{1}{x - 3}$

144. $\frac{3}{2x + 1} = \frac{8}{4x + 3}$

145. $\frac{4}{2 - x} = \frac{6}{3x - 5}$

146. $\frac{5}{x - 1} = \frac{5}{1 - x}$

147. $\frac{2}{x - 3} = \frac{2}{x - 3}$

148. $x + \frac{3x}{x - 3} = \frac{9}{x - 3}$

149. $x - \frac{16}{x - 4} = \frac{-4x}{x - 4}$

150. $2x + \frac{7x}{x - 6} = \frac{72 - 5x}{x - 6}$

151. $\frac{2xx}{2x - 1} - 3x = 1 + \frac{x}{2x - 1}$

152. $8(x - 7) = (x - 7)(3x + 2)$

153. $(9x - 5)(x - 3) = (2x + 5)(5 - 9x)$

[More exercises are in Part G, Supplementary Exercises.]

3.06 Transforming a formula.--Sally Prentiss, a freshman at Zabranchburg High, is planning a summer trip to Europe with her parents. She is concerned about the kind of clothes she should take with her. So, she writes to a friend who lives in France and asks questions about the weather in various cities which they expect to visit. Her friend sends her the following list of average July temperatures in these cities.

Madrid	23.2°
Rome	24.5°
Athens	27.4°
Istanbul	23.6°
Vienna	18.8°
Copenhagen	16.5°
Oslo	17.0°
Glasgow	14.5°
Amsterdam	17.2°
Paris	18.5°

Sally is slightly surprised at these temperatures. Does it really get that cold in Europe in July? Then she remembers that they use a different kind of thermometer in Europe, something called 'a centigrade thermometer', and that people in the United States use a Fahrenheit thermometer. She even remembers that in science she learned a formula about the thermometers. She hunts up her notebook and finds this formula:

$$C = \frac{5}{9}(F - 32).$$

She decides that she can use this formula to figure out the Fahrenheit temperatures in these cities. She starts with Madrid, substituting '23.2' for 'C' to get the equation:

$$23.2 = \frac{5}{9}(F - 32).$$

Next, she solves this equation.

$$9 \times 23.2 = 9 \times \frac{5}{9}(F - 32)$$

$$208.8 = 5(F - 32)$$

$$208.8 = 5F - 160$$

$$208.8 + 160 = 5F - 160 + 160$$

$$368.8 = 5F$$

$$\frac{1}{5} \times 368.8 = 5F \times \frac{1}{5}$$

$$73.76 = F$$

So, the Madrid temperature in July is almost 74° Fahrenheit.

Now, how about Rome? Substitute again in the formula $C = \frac{5}{9}(F - 32)$:

$$24.5 = \frac{5}{9}(F - 32)$$

$$9 \times 24.5 = 9 \times \frac{5}{9}(F - 32)$$

$$220.5 = 5(F - 32)$$

$$220.5 = 5F - 160$$

$$220.5 + 160 = 5F - 160 + 160$$

$$380.5 = 5F$$

$$\frac{1}{5} \times 380.5 = 5F \times \frac{1}{5}$$

$$76.1 = F$$

The July temperature in Rome is about 76° Fahrenheit.

By now, Sally is annoyed at the prospect of having to solve so many equations. There must be an easier way! She sees that she followed exactly the same steps in solving the equations for Madrid and for Rome. And, she realizes that she would follow the same steps each time she tried to compute the Fahrenheit temperatures for the rest of the cities. So, there must be a pattern. She takes

the original formula:

$$C = \frac{5}{9}(F - 32)$$

and treats the 'C' as if it were a numeral.

$$\begin{aligned} 9 \times C &= 9 \times \frac{5}{9}(F - 32) \\ 9C &= 5(F - 32) \\ 9C &= 5F - 160 \\ 9C + 160 &= 5F - 160 + 160 \\ 9C + 160 &= 5F \\ \frac{1}{5}(9C + 160) &= (5F)\frac{1}{5} \\ \frac{1}{5}(9C + 160) &= F \end{aligned}$$

Now, if she substitutes for 'C' in this last equation, she can find the Fahrenheit temperature very quickly. To be sure that this will work, she checks the Madrid temperature again.

$$\begin{aligned} F &= \frac{1}{5}(9 \cdot 23.2 + 160) \\ &= \frac{1}{5}(208.8 + 160) \\ &= \frac{1}{5}(368.8) \\ &= 73.76. \end{aligned}$$

Sure enough, this is what she got the first time! There seem to be fewer steps, although the actual amount of computing is the same. She notices that she could cut down the amount of computing if she simplified the formula:

$$F = \frac{1}{5}(9C + 160)$$

to:

$$F = 1.8C + 32.$$

Compare the formulas:

$$(1) \quad C = \frac{5}{9}(F - 32)$$

and:

$$(2) \quad F = 1.8C + 32.$$

Do you see that when you substitute for 'C' in both formulas a numeral for the same number, the equations you get are equivalent? Also, do you get equivalent equations when you substitute a numeral for 'F' in both formulas? Formulas such as (1) and (2) are often called equivalent formulas. Either of two equivalent formulas can be transformed into the other by using the methods you learned in the preceding section.

You can see that knowing how to transform a formula makes it possible for you to have many formulas available without having to memorize them or even have them all written down. For example, Sally had just the formula '$C = \frac{5}{9}(F - 32)$' written in her notebook. This formula is most useful in computing centigrade temperatures when you are given Fahrenheit temperatures. The job Sally wanted to do required computing Fahrenheit temperatures from centigrade temperatures. A formula for this purpose would start:

$$F = \ldots$$

And, Sally could have derived such a formula immediately by transforming '$C = \frac{5}{9}(F - 32)$'.

Here is a formula for computing the perimeter of a rectangle:

$$P = 2(\ell + w).$$

If you know the measures of the length and width of a rectangle, you can use this formula to compute the perimeter.

But, suppose you know the perimeter and the width of a rectangle; is there a formula which you can use to compute the measure of the length of this rectangle? The answer is 'yes', and we can derive such

a formula by transforming '$P = 2(\ell + w)$' into a formula which starts '$\ell = \ldots$'.

$$P = 2(\ell + w)$$
$$P = 2\ell + 2w$$
$$P - 2w = 2\ell$$
$$\frac{P - 2w}{2} = \ell$$
$$\ell = \frac{P - 2w}{2}$$

This process is sometimes called 'solving an equation for one pronumeral in terms of the other pronumerals'. In the example just given, we solved the equation '$P = 2(\ell + w)$' for 'ℓ' in terms of 'w' and 'P'.

EXERCISES

A. Solve each of the following equations for the pronumeral indicated.

Sample 1. $P = a + b + c$; b

Solution. [We wish to transform the given equation into one whose left side is 'b' and whose right side is a pronumeral expression in which 'b' does not occur.]

$$P = a + b + c$$
$$P = b + (a + c)$$
$$P - (a + c) = b + (a + c) - (a + c)$$
$$b = P - a - c$$

1. $P = 2a + b$; b
2. $P = 2a + b$; a
3. $P = 4s$; s
4. $C = \pi d$; d
5. $P = 3x + 3y$; x
6. $P = 2a + 3b$; b
7. $x = 3y - z$; y
8. $x = 2y + z$; z

Sample 2. $k = 5m - n$; n

Solution. $k = 5m - n$

$$k - 5m = 5m - n - 5m$$
$$k - 5m = -n$$
$$-1(k - 5m) = -1(-n)$$
$$n = -k + 5m \quad [\text{or: } n = 5m - k]$$

9. $A = 5b - c$; c

10. $K = 2g - 3h$; h

11. $x = 2v - 3u + w$; v

12. $y = 2p + q - r$; r

13. $P = 2(a + b) - c$; c

14. $P = 3(x - y) - 5z$; y

<u>Sample 3</u>. $A = hb$; b

<u>Solution</u>. $A = hb$

$$A \cdot \frac{1}{h} = hb \cdot \frac{1}{h}, \quad [h \neq 0]$$

$$b = \frac{A}{h}$$

<u>Sample 4</u>. $i = prt$; r

<u>Solution</u>. $i = prt$

$$i = (pt)r$$

$$r = \frac{i}{pt}, \quad [pt \neq 0]$$

<u>Sample 5</u>. $A = p + prt$; p

<u>Solution</u>. $A = p + prt$

$$A = p(1 + rt)$$

$$p = \frac{A}{1 + rt}, \quad [rt \neq -1]$$

[Why would you be wrong if you gave as the answer:

$p = A - prt$?]

15. $c = np$; n

16. $b = \frac{p}{r}$; p

17. $A = \frac{1}{2}bh$; b

18. $r = \frac{E}{I}$; E

19. $x = y + yz$; z

20. $x = y + yz$; y

21. $\ell = a + (n - 1)d$; d

22. $\ell = a + (n - 1)d$; n

23. $A = \frac{1}{2}h(b_1 + b_2)$; h

24. $A = \frac{1}{2}h(b_1 + b_2)$; b_1

[More exercises are in <u>Part H</u>, <u>Supplementary Exercises</u>.]

B. Solve each of these equations for 'y' to obtain an equation of the form: $y = \_\_\_x + \ldots$.

Sample 1. $3x - 2y = 7$

Solution.

$$3x - 2y = 7$$
$$-2y = 7 - 3x$$
$$y = \frac{7 - 3x}{-2}$$
$$y = \frac{3}{2}x + -\frac{7}{2}$$

1. $2x + y = 9$
2. $5x - y = 15$
3. $x - y + 7 = 0$
4. $3x + 7y = 21$
5. $x - 7y = 28$
6. $2y - 3x = 18$
7. $4x + 2 - 6y = 9 - 3y - 5x$
8. $7x + 6y - 3 = 8y - 2x + 4$
9. $4(x - 5) + 5(y - 3) = 15$
10. $3(x + y - 2) + 7(y - x + 3) = 0$

Sample 2. $\frac{1}{x} + \frac{1}{y} = 5$

Solution.

Method I.

$$\frac{1}{x} + \frac{1}{y} = 5$$
$$xy\left(\frac{1}{x} + \frac{1}{y}\right) = 5xy, \quad [x \neq 0 \neq y]$$
$$y + x = 5xy$$
$$y + x - y = 5xy - y$$
$$x = y(5x - 1)$$
$$y = \frac{x}{5x - 1}, \quad [x \neq \frac{1}{5}]$$

Method II.

$$\frac{1}{x} + \frac{1}{y} = 5$$
$$\frac{1}{y} = 5 - \frac{1}{x}, \quad [x \neq 0 \neq y]$$
$$\frac{1}{y} = \frac{5x - 1}{x}$$
$$xy\left(\frac{1}{y}\right) = \left(\frac{5x - 1}{x}\right)xy$$
$$x = (5x - 1)y$$
$$y = \frac{x}{5x - 1}, \quad [x \neq \frac{1}{5}]$$

11. $\frac{3}{4x} + \frac{1}{2y} = \frac{5}{8}$
12. $3 = \frac{x}{7} - \frac{y}{3}$
13. $\frac{1}{p} + \frac{1}{q} = \frac{1}{y}$
14. $\frac{1}{p} + \frac{1}{y} = \frac{1}{q}$
15. $\frac{1}{y} + \frac{1}{a} = 6$
16. $\frac{2}{3n} - \frac{3}{5y} = \frac{1}{15}$
17. $\frac{x}{y} = \frac{7}{1 + y}$
18. $\frac{5}{a + y} = \frac{6}{a - y}$

[More exercises are in Part I, Supplementary Exercises.]

EXPLORATION EXERCISES

Complete with the simplest expression you can to make true sentences.

1. The sum of 5 and 9 is ______.

2. The sum of 3 and the number which is 4 more than 3 is ______.

3. The sum of 7 and the number which is 5 less than 7 is ______.

4. The number which is 2 more than 10 is ______.

5. The number which exceeds 5 by 3 is ______.

6. The number which exceeds 3 by 5 is ______.

7. Mary is 14 and Mike is twice as old as Mary; in 7 years Mike will be ______ years old and Mary will be ______ years old.

8. The sum of three consecutive integers, of which the first is $^{+}5$, is ______. [An integer is a real number whose absolute value is a whole number of arithmetic. Examples: -6, 0, 2, and 100.]

9. The integer between -6 and -8 is ______.

10. The difference of 12 from 15 is ______.

11. The difference of 15 from 12 is ______.

12. The difference of 12 from the number which is 3 more than 12 is ______.

13. The number by which 73 exceeds 62 is ______.

14. If Teresa weighs 101 pounds after gaining $2\frac{1}{2}$ pounds during March, she weighed ______ pounds at the beginning of March.

15. Mr. Petrini bought a home for $12, 350 and sold it the next year for ______ which was $750 less than he had paid for it.

16. If Pete is 12 years old now, he was ______ years old 5 years ago.

17. If Ned was born 5 years ago and his sister was born 7 years ago, Ned is ______ years older than his sister.

18. The product of 6 and 5 is ______.

19. The product of 6 and the number which is 3 more than 6 is ______.

20. The product of 4 and the number which is the product of 3 and 4 is ______.

21. The number which is 10% of 30 is ______.

22. The product of 150 and $\frac{1}{2}$ of 20 is ______.

23. The quotient of 70 by 2 is ______.

24. The number which is 3 more than the quotient of 15 by 10 is ______.

25. 75 is ______ per cent of 60.

26. 60 is ______ per cent of 75.

27. The number which exceeds 30 by 10% [of 30] is ______.

28. The number which exceeds 60 by 10% [of 60] is ______.

29. The number which exceeds 70 by 10% is ______.

30. The number which is 20% greater than 50 is ______.

31. The number which is 30% less than 90 is ______.

32. The sum of 50% of 70 and 60% of 30 is ______.

(continued on next page)

33. Joe has 36 pigeons; if Abe has 3 more than half this number of pigeons then Abe has ______ pigeons.

34. Mr. Abercrombie earns ______ dollars a year which is $720 more than twice the earnings of Mr. Sussman who earns $3100 yearly.

35. Phil added 60 stamps to his Switzerland collection; if this was $\frac{1}{2}$ as many as he already had, then altogether he has ______ Swiss stamps.

36. If unlined paper costs 15 cents a pad and lined paper costs 7 cents more per pad than unlined paper, the cost of a dozen pads of lined paper is ______ cents.

37. If a paper boy earns one cent for each paper delivered, and if he delivers papers to 35 customers each day except Sunday, then he earns ______ dollars per week.

38. If Amos buys pencils three for a dime, and sells them to his classmates at 5 cents each, then his profit on two dozen pencils is ______ cents.

39. If the cost price of an article is 25 dollars, and the margin is 20% of the cost price, the selling price is ______ dollars.

40. A man buys a house for $10,000 and sells it for 5% more than it cost; but, he pays 5% commission on the selling price. So, he makes ______ dollars.

41. The difference of 7 from the number which is 7 times as large as 6 is ______.

42. Carla is 10 years old now, and 3 years ago Dick was twice as old as Carla was then; Dick is ______ years old now.

43. If Butchie is now 5 years old and if Barbie's age is 3 years less than twice Butchie's age, Barbie is now ______ years old.

44. If white bread costs 3 cents less per loaf than whole wheat bread, six loaves of whole wheat bread cost ______ cents more than the same number of loaves of white bread.

45. If Rudolph is 7 years old, and Rupert is 12 years older than Rhoda, then Rudolph is ______ years younger than Rupert.

46. If the length of each side of a regular pentagon is 7 inches more than twice the length of each side of an equilateral triangle, and if the perimeter of the equilateral triangle is $10\frac{1}{2}$, then the perimeter of the pentagon is ______.

47. A rectangle is ______ inches wide if its perimeter is 20 and its length is 7 inches.

48. Martin has 3 dimes and 2 more quarters than dimes [and no other money]; so, he has ______ cents.

49. If the number of dimes in a sack of nickels and dimes is 3 more than twice the number of nickels, and if the sack contains 20 nickels, then all the coins in the sack are worth ______ cents.

50. Six quarters, six nickels and six dimes are together worth ______ cents.

51. In a sum of money consisting of just dimes and nickels there are 13 nickels. If the sum amounts to $1.00 then there are ______ dimes.

52. 10 pounds of coffee at 85 cents per pound and 7 pounds of coffee at 97 cents per pound will cost ______ cents.

53. If a coffee blend is made using 30 pounds of a 90-cents-a-pound grade and 40 pounds of an 85-cents-a-pound grade, then one pound of this blend is worth ______ cents.

54. A grocer makes 70 pounds of a coffee blend to sell at 85 cents per pound by mixing 30 pounds of a 90-cents-a-pound grade with ______ pounds of an 87-cents-a-pound grade.

(continued on next page)

55. The simple interest per year on \$700 invested at an annual rate of 4% is ______ dollars.

56. If a total of \$1400 is invested, with \$1000 at 3% and the rest at 4%, the annual return on the total investment is ______.

57. The annual income on \$3000 of which \$1000 is invested at 4.5% and the rest at 5.5% is ______ dollars.

58. There are ______ pints in 3 quarts.

59. There are ______ quarts in 7 pints.

60. Two gallons and three quarts together make ______ pints.

61. Mrs. Swenson needs ______ gallons of tomato juice to fill 5 pint jars and twice as many quart jars.

62. There are ______ feet in 48 inches.

63. There are 48 feet in ______ inches.

64. If a man can do a certain job in 12 minutes, then by working at the same rate he can do ______ of the job in 1 minute, ______ of the job in 2 minutes, ______ of the job in 9 minutes and ______ of the job in 12 minutes.

65. If a man can do a certain piece of work in 10 minutes, then he can do ______ part of the job in 6 minutes.

66. If Art can do a certain job in 3 hours, and Joe can do the same job in 4 hours, then in 1 hour Art can do ______ of the job and Joe can do ______ of the job and, so, in one hour together they can do ______ of the job.

67. If Bill can do a certain job in 2 hours and Bob can do the same job in half the time, then in 3 hours they can do ______% of the job.

68. You can walk ______ miles in 3 hours at the average rate of 4 miles per hour.

69. You can walk 7 miles in ______ hours at the average rate of 3 miles per hour.

70. You can walk 4 miles in 3 hours at the average rate of ______ miles per hour.

71. If Kathy's average rate of walking is 1 mile per hour more than Marilynn's average rate, and if Marilynn walks 5 miles in 2 hours, then Kathy walks 5 miles in ______ hours.

72. If Bill's wages are ______ dollars an hour, he receives $60 for 40 hours of work.

73. If Martha receives $8 more than Richard for working 4 hours less, Richard receives ______ dollars an hour.

74. If 5 quarts of an alcohol solution contain 4 quarts of water and 1 quart of alcohol, then 10 quarts of this same solution will contain ______ quarts of alcohol.

75. If an alcohol solution is 25% alcohol [and the rest water] then 8 quarts of this solution contain ______ quarts of water.

76. If you add 3 quarts of a 30% alcohol solution to 5 quarts of a 20% alcohol solution, you get 8 quarts of solution containing ______ quarts of alcohol.

77. If you add 2 quarts of pure alcohol to 4 quarts of a 25% alcohol solution, then the mixture contains a total of ______ quarts of liquid, of which ______ per cent is alcohol.

78. If you have 3 pounds of a candy mixture, 50% of which is peppermint candy, and you want to make a mixture which contains 80% peppermint candy, you can add ______ pounds of peppermint candy to the original mixture.

(continued on next page)

79. If 28 ounces of shelled peanuts are combined with 2 pounds of a nut mixture which contains 25% shelled peanuts, then _____ per cent of the new mixture is shelled peanuts.

80. Five gallons of 15%-alcohol are poured into 7 gallons of 12%-alcohol. The resulting solution contains ______ gallons of 3%-alcohol.

3.07 Solving problems.--The very best way of becoming a good problem solver is to solve lots of problems. In this section you will get lots of practice in solving problems. There is no one method which is best for solving all problems. There may be several very good ways of solving the same problem, as well as several poor ones. A "good" method of solving a problem is usually a method which takes less computing and which gives you ideas for solving other problems of the same kind. Sometimes you will find it helpful to take an easy problem and solve it in two or three different ways in order to get practice with these different methods, so that you will have a variety of tools available when you attempt more difficult problems.

Many of the problems you will practice on in this section are twentieth century versions of problems mathematics students have been doing for thousands of years. Most of them are really puzzles, but a few are of a practical nature. In either case, the important thing for you is that you read carefully and think clearly. And the more problems you work, the easier it will be to do both of these things.

Example 1.

A jar of coins contains 3 times as many dimes as nickels and twice as many quarters as nickels. The total value of the quarters, dimes, and nickels in the jar is $21.25. How many nickels are there in the jar?

One way to attack this problem is to imagine yourself putting coins into a jar and following the conditions of the problem. Can you picture yourself doing this?

Suppose you decide that you ought to put 10 nickels into the jar. The problem tells you that the jar is to contain 3 times as many dimes as nickels, and twice as many quarters as nickels. So, when you finish putting in the coins, the jar will contain

$$\begin{array}{r} 10 \text{ nickels} \\ 3 \times 10 \text{ dimes} \\ 2 \times 10 \text{ quarters.} \end{array}$$

Is the value of these coins \$21.25?

$$\begin{array}{r} 10 \text{ nickels are worth } 5 \times 10 \text{ cents,} \\ 3 \times 10 \text{ dimes are worth } 10 \times (3 \times 10) \text{ cents,} \\ 2 \times 10 \text{ quarters are worth } 25 \times (2 \times 10) \text{ cents,} \end{array}$$

and, altogether the coins are worth

$$5 \times 10 + 10 \times (3 \times 10) + 25 \times (2 \times 10) \text{ cents.}$$

You have solved the problem correctly <u>if</u> this many cents is 2125 cents. And, we find this out by seeing whether the following sentence is true:

$$\begin{array}{ccccc|c} 5 \times 10 & + & 10 \times (3 \times 10) & + & 25 \times (2 \times 10) = & 2125. \\ 50 & + & 300 & + & 500 & 2125 \\ & & & & 850 = & 2125. \end{array}$$

This sentence is not true, so 10 is <u>not</u> the right number of nickels.

Try again. This time put in, say, 19 nickels. Then the jar will contain

$$\begin{array}{r} \boxed{19} \text{ nickels,} \\ 3\ \boxed{19} \text{ dimes,} \\ 2\ \boxed{19} \text{ quarters,} \end{array}$$

and these coins would be worth

$$5\ \boxed{19} + 10 \cdot 3\ \boxed{19} + 25 \cdot 2\ \boxed{19} \text{ cents.}$$

Let's check to see whether this is right.

$$5\ \boxed{19} + 10 \cdot 3\ \boxed{19} + 25 \cdot 2\ \boxed{19} = 2125?$$

$$\begin{array}{r|l} 95 \quad + \quad 570 \quad + \quad 950 & 2125 \\ 1615 = & 2125\ ? \end{array}$$

No, wrong again!

Instead of trying another number of nickels, let's take a look at what has been done so far. For the number 10, you checked to see if the following was a true sentence:

(1) $$5\ \boxed{10} + 10 \cdot 3\ \boxed{10} + 25 \cdot 2\ \boxed{10} = 2125,$$

and, for the number 19, you tested the following sentence:

(2) $$5\ \boxed{19} + 10 \cdot 3\ \boxed{19} + 25 \cdot 2\ \boxed{19} = 2125.$$

It turned out that both (1) and (2) are false. How could you get a true sentence with the same pattern as (1) and (2)? Just by putting in the '$\square$'s numerals for the <u>correct</u> number of nickels. In other words, the correct number of nickels is a root of the equation:

$$5\ \square + 10 \cdot 3\ \square + 25 \cdot 2\ \square = 2125.$$

Solve this equation [you can use 'x's instead of '$\square$'s if you prefer], and see if the root is the correct number of nickels.

<u>Example 2.</u>

A committee sold 120 tickets for a school play. Some of the tickets were sold to adults at $.70 each; the remaining tickets were sold to students at $.50 each. A total of $70.40 was collected from ticket sales. How many adult tickets were sold?

Imagine yourself selling these 120 tickets. Suppose you sell $\boxed{40}$ adult ones. Then, how many do you sell to students? $120 - \boxed{40}$. From the adults you collect $70 \times \boxed{40}$ cents, and from the students you collect $50 \times (120 - \boxed{40})$ cents. Altogether, then, you collect

$$70 \times \boxed{40} + 50 \times (120 - \boxed{40}) \text{ cents.}$$

And, if 40 adult tickets <u>were</u> sold, the following sentence should be true.

$$70 \times \boxed{40} + 50 \times (120 - \boxed{40}) = 7040.$$

Instead of taking the time to check this sentence [it's probably false, anyway], we study the pattern of the sentence. Notice that the right answer to the problem would convert the open sentence:

$$(*) \quad 70 \square + 50(120 - \square) = 7040$$

into a true one. So, let's solve this equation, but using 'x's instead of '$\square$'s. [It's easier to write 'x's than to write '$\square$'s.]

$$\begin{aligned} 70x + 50(120 - x) &= 7040 \\ 70x + 6000 - 50x &= 7040 \\ 20x + 6000 &= 7040 \\ 20x &= 1040 \\ x &= 52 \end{aligned}$$

So, 52 is the solution of (*). We could check our computations by substituting in (*). This would tell us only whether we had solved (*) correctly. But, it is more important to check whether we solved the <u>ticket problem</u> correctly. And this we can tell by going back to the original statement.

How much money would 52 adult tickets bring in? How many student tickets were sold, and how much would they bring? How much is this all together? Is it $70.40?

$$\begin{array}{r} 52 \\ \times\ 70 \\ \hline 3640 \end{array} \qquad 120 - 52 = \begin{array}{r} 68 \\ \times\ 50 \\ \hline 3400 \end{array} \qquad \begin{array}{r} 3640 \\ 3400 \\ \hline 7040 \end{array} \checkmark$$

EXERCISES

A. Solve these problems.

1. Agnes bought 140 stamps for $4.95. Some were 3-cent stamps and the rest were 4-cent stamps. How many 3-cent stamps did she buy?

2. Mary is 3 years older than Bill. Ten years ago Mary was three times as old as Bill. How old is Bill now?

3. Taking $\frac{1}{5}$ of a certain number gives the same result as subtracting the number from 27. What is the number?

4. A rectangle is twice as long as it is wide. The width of a second rectangle is 3 units more than the width of the first, and the length of the first is half the length of the second. The second rectangle has a perimeter which is 8 less than twice the perimeter of the first. Find the width of the first.

5. John has a handful of dimes and nickels totaling $3.55. He has 7 more dimes than nickels. How many nickels does he have?

6. Delivered milk costs 30 cents a quart. This price is 20% higher than last year's price. What did delivered milk cost last year?

7. Taking 50% of a certain number is the same as adding 7 to that number. What is the number?

8. Three more than a certain number is six less than twice the number. What is the number?

9. A square and an equilateral triangle have equal perimeters. A side of the triangle is two inches longer than a side of the square. How long is a side of the square?

10. A salesman sold a number of pairs of shoes at $8 a pair, and 5 more than that number of pairs at $6 a pair. He received $184 for all the shoes sold. How many pairs did he sell at each price?

B. Complete with the simplest expressions you can to make true sentences.

1. For each x, the sum of x and the number -7 more than x is ______.

2. For each x, the sum of x and the number which is 2 less than the product of 3 and x is ______.

3. For each x, the number which is -5 times as large as x is ______.

4. For each x, the difference of x from a number 6 times as large as x is ______.

5. For each y, the number which is 2 greater than y is ______.

6. For each z, the number which is 10% of z is ______.

7. For each k, 15% of the number which is 7 times as large as k is ______.

8. For each p, the number which is 120% greater than p is ______.

9. For each r, the number which is 80% less than r is ______.

10. For each s, for each t, the sum of 50% of s and 60% of t is ______.

11. For each x, the product of $\frac{1}{2}x$ and 150 is ______.

12. For each z, 9 less than z is ______.

13. For each v, the product of 3 and $-v$ exceeds the sum of 2 and v by ______.

14. For each $x \neq 0$, the quotient of 8 by the product of 2 and x is ______.

15. For each number x of arithmetic, if Bill is x years old now, he will be ______ years old 6 years from now.

(continued on next page)

16. For each number y of arithmetic, if Carl is y years old now and Andy is twice as old as Carl then Andy will be ______ years old 7 years from now.

17. For each number of arithmetic $z > 2$, if Mary is now z years old and Bill's age is 4 years less than twice Mary's age, Bill is now ______ years old.

18. For each number x of arithmetic, if the difference of Jim's age from Andy's age is x years then the difference of Jim's age from Andy's age 4 years ago was ______ years.

19. For each number x of arithmetic, if unlined paper costs x cents a pad, and a pad of lined paper costs 5 cents more than a pad of unlined paper, the cost of 6 pads of lined paper and 5 pads of unlined paper is ______ cents.

20. For each number k of arithmetic, there are ______ pints in k quarts.

21. For each number m of arithmetic, m quarts and twice that many gallons together contain ______ pints.

22. For each number t of arithmetic, there are ______ feet in t inches.

23. For each number x of arithmetic, x yards, 4 times as many feet, and 7 times as many inches (as yards) together make ______ inches.

24. For each whole number m of arithmetic, there are ______ cents in m nickels.

25. For each whole number p of arithmetic, for each number t of arithmetic, there are ______ cents in a total of p nickels and t dimes.

26. For each whole number d of arithmetic, there are ______ dollars in d dimes.

27. For each whole number p of arithmetic, if John buys pencils at the rate of 3 pencils for 12 cents and sells them to his classmates at 5 cents each, his profit on the sale of 3p pencils is ______ cents.

28. For each number x of arithmetic, if the cost price of an article is x dollars and the margin is 25% of the cost price, then the selling price is ______ dollars.

29. For each number w of arithmetic, if the width of a rectangle is w units and the length is twice the width then the perimeter is ______.

30. For each number x of arithmetic, if a shorter side of a rectangle is x units long and a side of a square has the same length as this shorter side, then the perimeter of the square is ______.

31. For each number of arithmetic $k > 3$, if a longer side of a rectangle is k units, and a shorter side of this rectangle is 3 units less than a longer side, then the perimeter of the rectangle is ______.

32. For each x, the product of 7 by x multiplied by the sum of 3 and x is ______.

33. For each y, the sum of $\frac{1}{5}$ of y and the product of 2 by y is ______.

34. For each number N of arithmetic, if Mr. Ronk earns N dollars a year and Mr. Dunlap earns $520 more than two thirds of what Mr. Ronk earns, then Mr. Dunlap earns ______ dollars a year.

35. For each number h of arithmetic, you can walk ______ miles in h hours at the average rate of 4 miles per hour.

36. For each number r of arithmetic, you can travel ______ miles in 3 hours at the average rate of r miles per hour.

(continued on next page)

37. For each number r of arithmetic, for each number h of arithmetic, you can travel ______ miles in h hours at the average rate of r miles per hour.

38. For each number x of arithmetic, it takes ______ hours for a freight train to travel x miles if its average rate is 30 miles per hour.

39. For each number s of arithmetic, you must walk at an average rate of ______ miles per hour to travel s miles in 2 hours.

40. For each number x of arithmetic, the annual income (interest) on x dollars invested at 3% is ______ dollars.

41. For each number of arithmetic $y \leq 1400$, the annual income on $(1400 - y)$ dollars invested at 4.5% is ______ dollars.

42. For each number of arithmetic $k \leq 2000$, the total annual income on $2000 of which k dollars are invested at 3% and the rest at 4%, is ______ dollars.

43. For each number x of arithmetic, if the length of each of the two sides of equal length of an isosceles triangle is 2 inches more than the length of the base, and if the base is x inches long, then the perimeter is ______.

44. For each number t of arithmetic, if the circumference of a circle is 33t, a diameter measures ______.

45. For each number x of arithmetic, if an equilateral triangle has perimeter x then a square whose side is 4 units longer than a side of this triangle will have perimeter ______.

46. For each number of arithmetic $x > 0$, if Abe can do a certain job in x hours, he can do ______ of the job in 1 hour.

47. For each number x of arithmetic, if Bob can mow a lawn in 3 hours then he can mow ______ of the lawn in 1 hour, and ______ of the lawn in x hours.

48. For each number x of arithmetic, if Bob can mow a lawn in 3 hours and Tom can mow this lawn in 2 hours then together, they can mow ______ of the lawn in x hours.

49. For each number of arithmetic $x \geq 3$, if an inlet pipe can fill a tank in 3 hours and an outlet pipe can empty the tank in x hours, then, when both pipes are turned on [starting with an empty tank], ______ of the tank is filled at the end of 1 hour.

50. For each nonzero number x of arithmetic, if Raymond hops $\frac{3}{4}$ as fast as Harold and if Harold hops x feet per second, then Raymond takes ______ seconds to hop 15 feet.

51. For each whole number n of arithmetic, a pile of nickels, dimes and quarters which contains n nickels, twice as many quarters, and 10 more dimes than quarters is worth ______ cents.

52. For each number x of arithmetic, if the measure of the width of a rectangle is $\frac{2}{3}$ the perimeter x, then the length measures ______.

53. For each whole number of arithmetic $x > 30$, if 15 more than one half of the total number x of passengers in a bus get off at the first stop, and one third of the remaining passengers get off at the second, there are ______ passengers left.

54. For each number t of arithmetic, if a person can climb a mountain in t hours and descend 5 times as fast, the total time required for the trip up and down [no resting at the top] is ______ hours.

55. For each number x of arithmetic, if x pounds of nuts at 32 cents per pound are mixed with 4 more than twice as many pounds of nuts at 37 cents per pound, the resulting mixture is worth ______ cents per pound.

(continued on next page)

56. For each number x of arithmetic, if x pounds of nuts at 85 cents per pound are mixed with 15 pounds of nuts at 70 cents per pound, the resulting mixture contains ______ pounds worth 90 cents per pound.

57. For each number of arithmetic $x \leq 27$, the total cost of x pounds of coffee at 93 cents per pound and $(27 - x)$ pounds of coffee at 89 cents per pound is ______ cents.

58. For each number n of arithmetic, if n is a whole number, ______ is the next larger whole number.

59. For each number of arithmetic $n \geq 3$, if n is an odd whole number, ______ is the largest odd whole number smaller than n.

60. For each number m of arithmetic, if m is a whole number, the sum of the next two consecutive whole numbers is ______.

61. For each number x of arithmetic, x gallons of a 30% alcohol solution contain ______ gallons of alcohol.

62. For each number x of arithmetic, x gallons of a 20% alcohol solution contain ______ gallons of water.

63. For each number x of arithmetic, if 4 gallons of a 25% alcohol solution are added to x gallons of a 30% alcohol solution, the new mixture contains ______ gallons of alcohol.

64. For each number y of arithmetic, if y pints of a 3% iodine solution are added to 2 pints of a 7% iodine solution, the new mixture is a ______ per cent iodine solution.

65. For each number x of arithmetic, if x ounces of shelled peanuts are added to 2 pounds of a nut mixture which contains 30% shelled peanuts, the new mixture contains ______ pounds of shelled peanuts.

66. For each number x of arithmetic, if x gallons of a 30% alcohol solution are mixed with twice as many gallons of a 60% alcohol solution, the result is a mixture containing ______ gallons of alcohol.

67. For each number x of arithmetic, if 10 ounces of an x% gold alloy are combined with 3 ounces of pure gold, the new alloy is _______ per cent gold.

68. For each number of arithmetic $t \neq 0$, if it takes Benjamin t hours to walk 7 miles and if Theodore can walk $\frac{1}{2}$ mile per hour faster than Benjamin, then it takes Theodore _______ hours to walk half as far as Benjamin.

69. For each whole number of arithmetic $x \neq 0$, if x people share equally in the cost of a \$17 picnic, a picnic for 7 more than twice this number of people should cost _______ dollars.

[More exercises are in <u>Part J, Supplementary Exercises</u>.]

* * *

You have seen how to solve some problems by picking some number as a possible answer, going through the steps necessary to check this possible answer [but not simplifying], and arriving at a pattern. This pattern is expressed easily by an equation, and a root of the equation leads to the solution of the problem.

You may also have found in Part A that you could write the equation immediately without going through the process of checking possible answers. Of course, it is faster if you can write down the equation immediately, and you should practice doing this in the next set of problems. But, you can always use the procedure of checking a <u>possible</u> answer if this helps you get the "feel" of the problem.

<u>Example 3</u>.

If you increase a certain number by 17, you get the same result as if you had subtracted $\frac{1}{2}$ the number from 5. What is this number?

<u>Solution</u>. We know that, for each number x, if you increase x by 17, you get $x + 17$, and if you subtract $\frac{1}{2}x$ from 5, you get $5 - \frac{1}{2}x$. So, we want to find a number x such that

$$(1) \qquad x + 17 = 5 - \frac{1}{2}x.$$

If there <u>is</u> a number which meets the conditions of this problem,

it is a root of (1). So, we solve (1):

$$x + 17 = 5 - \frac{1}{2}x$$

$$2x + 34 = 10 - x$$

$$3x = -24$$

$$x = -8$$

Instead of checking to see whether the root of (1) is -8, we check to see whether -8 fits the conditions of the problem [Why do this?].

Check.

-8 increased by 17 is 9;
the difference of $\frac{1}{2} \cdot -8$ from 5 is 9. ✓

Answer. The number in question is -8.

Example 4.

John usually takes 40 minutes to ride his bicycle from home to school. When he is pressed for time, he can increase his average speed by 6 miles per hour and save 16 minutes. How far does John live from school?

Solution. For each number x of arithmetic, if John lives x miles from school and it takes him $\frac{2}{3}$ of an hour [40 minutes] at his usual rate to make the trip from home to school, his usual rate is $\frac{x}{2/3}$ [or: $\frac{3x}{2}$] miles per hour. If he increases this rate by 6 miles per hour, the new rate is $\frac{3x}{2} + 6$ miles per hour. At this new rate the time required for the trip is 16 minutes less than the usual time of 40 minutes. That is, the new time is 24 minutes, or $\frac{2}{5}$ of an hour. So, we are looking for a number x of arithmetic such that

$$\left(\frac{3x}{2} + 6\right)\frac{2}{5} = x.$$

Let's solve this equation.

$$\left(\frac{3x}{2} + 6\right)\frac{2}{5} = x$$

$$\frac{3x + 12}{2} \cdot \frac{2}{5} = x$$

$$\frac{3x + 12}{5} = x$$

$$\frac{3x + 12}{5} \cdot 5 = x5$$

$$3x + 12 = 5x$$

$$12 = 2x$$

$$6 = x$$

Check. If John lives 6 miles from school and it takes 40 minutes to make the trip from home to school, his usual rate is $6 \div \frac{2}{3}$ miles per hour; that is, 9 miles per hour. Now, if he increases his usual rate by 6 miles per hour, his new rate will be 15 miles per hour. Will this new rate make it possible for him to get to school in 16 minutes less time [i. e., in 24 minutes], as the problem stated? Well, if he lives 6 miles from school, and travels at a rate of 15 miles per hour, it will take him $6 \div 15$ hours to get there. $6 \div 15$ is $\frac{2}{5}$ hours, or 24 minutes--which is 16 minutes less than 40 minutes!

Answer. John lives 6 miles from school.

Example 5.

How many quarts of a 30% alcohol solution should be added to 8 quarts of a 40% alcohol solution to make a new solution which is 38% alcohol?

Solution. Suppose you add x quarts of the 30% alcohol solution to the 8 quarts of the 40% alcohol solution. Since what you add contains $.3x$ quarts of alcohol, and since the original 8 quarts of solution contain $.4(8)$ quarts of alcohol, the new solution contains $[.3x + .4(8)]$ quarts of alcohol. But, the new solution contains a total of $(x + 8)$ quarts of liquid. So, we are looking for a number x of arithmetic such that

$$.3x + .4(8) = .38(x + 8).$$

We solve this equation.

$$\begin{aligned} 100[.3x + .4(8)] &= [.38(x + 8)]100 \\ 30x + 320 &= 38x + 304 \\ 16 &= 8x \\ x &= 2 \end{aligned}$$

Check. 2 quarts of a 30% alcohol solution contain .6 quarts of alcohol. 8 quarts of a 40% alcohol solution contain 3.2 quarts of alcohol. So, the new mixture of 10 quarts of solution contains 3.8 quarts of alcohol, and 3.8 is 38% of 10.

Answer. 2 quarts of a 30% alcohol solution should be added.

Example 6.

Mr. Alders invests a total of $3800 in two enterprises, one giving an income of 3% and the other an income of 5%. If the total income from these investments is $166, how much is invested in each enterprise?

Solution. Suppose he invests x dollars at 3%. Then he invests $(3800 - x)$ dollars at 5%. The income from these investments is

$$.03x + .05(3800 - x) \text{ dollars.}$$

So, we are looking for a number x of arithmetic such that

$$.03x + .05(3800 - x) = 166.$$

[Finish the solution and check.]

Example 7.

If Albert can mow a lawn in 2 hours and Bill can mow this lawn in 3 hours, how long will it take them to mow the lawn if they work together?

Solution. Method I

You know that Albert will mow $\frac{1}{2}$ of the lawn in one hour, and Bill will mow $\frac{1}{3}$ of it in one hour, if they work at steady rates. Suppose they work together for x hours. Then, together, they would mow $\frac{1}{x}$ part of the lawn in 1 hour. So, we are looking for a number x of arithmetic, such that

$$\frac{1}{2} + \frac{1}{3} = \frac{1}{x}.$$

[Solve this equation and check.]

Method II

Suppose it takes x hours to mow the lawn if both boys work together. Then Albert would mow $\frac{x}{2}$ part of the lawn during this time, and Bill would mow $\frac{x}{3}$ part of it at the same time, and they would be finished. So, we need to find a number x of arithmetic such that

$$\frac{x}{2} + \frac{x}{3} = 1.$$

[Solve this equation and check.]

* * *

C. Solve these problems.

1. One pint of an alcohol solution contains 15% alcohol. How much pure alcohol [100% alcohol solution] must be added to make a solution which contains 35% alcohol?

2. Jim picked a number, tripled it, added 4 to the result, divided the sum by 8, and got 5. What number did he pick?

3. Edward is two years older than Charles. Eleven years ago Edward was twice as old as Charles. How old is each boy now?

(continued on next page)

4. A business man has 7 minutes to catch a train at a station which is 8 miles from his home. His taxi covers half of this distance traveling at an average speed of 30 miles per hour. What should be the average speed of the taxi during the second half of the trip to enable the man to catch the train?

5. A confectioner is making a mixture of almonds and cashews. The cashews are worth \$.90 a pound and the almonds are worth \$.75 a pound. How many pounds of each kind of nut should be used to make 30 pounds of a mixture worth \$.81 per pound?

6. Two boys start around a 1300-foot track, running in opposite directions. If one boy runs 6 feet more per second than the other, and they meet in 24 seconds, what is the rate of the faster boy?

7. Two cyclists start at the same time and from the same place and travel in opposite directions. In twenty minutes they are 11 miles apart. The faster cyclist travels at an average speed which is 3 miles per hour more than the average speed of the slower cyclist. What is the average speed of each cyclist?

8. A freight train and a passenger train on parallel tracks are 7 miles apart at 1:00 p.m. and are traveling in opposite directions. The passenger train's average speed is 35 miles per hour more than the average speed of the freight train. If they maintain their average speeds and are 45 miles apart at 1:24 p.m., what is the average speed of the freight train?

9. Bill can mow a lawn in 35 minutes and his brother can do the same job in 40 minutes. If they were to work together, how long would they take to mow the lawn?

10. A tank has two inlet pipes. One pipe by itself can fill the tank in 17 minutes; the other pipe by itself can fill the tank in 21 minutes. How long will it take to fill the tank if both pipes are opened?

11. There are 783 pupils in Zabranchburg High School. If the ratio of girls to boys is 5 to 4, how many boys are there in the school? [If there are 4x boys, there are 5x girls.]

12. A man has a total of \$3000 earning interest, some at 5% and the remainder at 6%. The amount of annual interest on both investments is \$155. How much is invested at each rate?

13. A man who can row 5 miles an hour in still water rows up a stream for 3 hours and then rows back to his starting point in 2 hours. At what rate does the stream flow?

14. Howard can read at a rate which is 4 pages an hour more than one and a half times Paul's rate. If Howard were to decrease his rate by 50% and if Paul were to increase his by 20%, each would read a book of 397 pages in 12 hours. What is Paul's usual reading rate?

15. A man has \$3.50 in dimes and quarters. He has 17 coins in all. How many coins of each denomination does he have?

16. Divide \$155 among A, B, C, and D so that A and B together receive \$40, C receives twice as much as A, and D receives three times as much as B.

17. If −6 is added to half a certain number, the result is 15. What is the number?

18. Jack wants a sweater that costs \$.15 more than 3 times the amount of money he now has. If the sweater costs \$4.50, how much money does Jack have now?

19. Herbert is walking up a long flight of steps. He climbs 6 less than half the total number, then he climbs 4 more than a third of the number remaining. He rests for a while, and then climbs 3 less than a fourth of what still remains. There are 48 steps left to reach the top. How many steps are there in the flight?

(continued on next page)

20. Noodles, Bismark, and Clem are three dachshund puppies. Noodles is one hour more than half as old as Bismark, and 3 hours older than Clem. Four hours ago Bismark's age was $4\frac{2}{3}$ Clem's age. How old is each puppy?

21. Two grades of coffee were accidentally mixed, and thus produced a new grade. To meet the cost of advertising this new grade, the coffee distributor had to pay 7% of the gross income he had expected to receive from the sale of the original two grades [11.7 tons for $17,550 and 8.3 tons for $14,940]. How much [to the nearest cent] per pound must he charge for the new grade to meet the advertising costs, and to give him his originally expected income?

22. How many pounds of coffee at 75 cents per pound should be mixed with 337 pounds of coffee at 90 cents per pound to produce 1000 pounds of mixture worth 80 cents per pound?

23. How many pounds of coffee at 75 cents per pound should be mixed with 337 pounds at 90 cents per pound to produce a mixture worth (a) 50 cents per pound, (b) 80 cents per pound, (c) $1.00 per pound?

24. Andrew has twice as much money as Scott. If Andrew were to lend Scott a quarter then both boys would have the same amount of money. How much money does each boy have?

25. (a) Two bees working together, can gather nectar from 100 hollyhock blossoms in 30 minutes. Assuming that each bee works the standard eight-hour day, five days a week, how many blossoms do these bees gather nectar from in a summer season of fifteen weeks?

    (b) In working on a batch of 100 blossoms, one of the bees stops after 18 minutes [just to smell the flowers], and it takes the other bee 20 minutes to finish the batch. How long would it take the diligent bee to gather nectar from 100 blossoms if she worked all by herself?

[More exercises are in <u>Part K</u>, <u>Supplementary Exercises</u>.]

EXPLORATION EXERCISES

Can you do these problems mentally?

$13 \times 18 = ?$ $\qquad$ $24 \times 26 = ?$

$17 \times 15 = ?$ $\qquad$ $42 \times 47 = ?$

Study the following examples. They may suggest a short cut.

Example 1. $\quad 13 \times 18 = ?$

$$\begin{aligned}(10 + 3) \times (10 + 8) &= (10 + 3)10 + (10 + 3)8 \\ &= 10 \cdot 10 + 3 \cdot 10 + 10 \cdot 8 + 3 \cdot 8 \\ &= 100 + (3 + 8)10 + 24 \\ &= 100 + 11 \cdot 10 + 24 \\ &= 100 + 110 + 24 \\ &= 210 + 24 \\ &= 234 .\end{aligned}$$

Example 2. $\quad 24 \times 26 = ?$

$$\begin{aligned}(20 + 4)(20 + 6) &= (20 + 4)20 + (20 + 4)6 \\ &= 20 \cdot 20 + 4 \cdot 20 + 20 \cdot 6 + 4 \cdot 6 \\ &= 400 + (4 + 6)20 + 24 \\ &= 400 + 10 \cdot 20 + 24 \\ &= 600 + 24 \\ &= 624 .\end{aligned}$$

Example 3. $\quad 17 \times 15 = ?$

$$\begin{aligned}(10 + 7)(10 + 5) &= 100 + (7 + 5)10 + 35 \\ &= 100 + 12 \cdot 10 + 35 \\ &= 100 + 120 + 35 \\ &= 220 + 35 \\ &= 255 .\end{aligned}$$

Example 4. $\quad 42 \times 47 = ?$

$$\begin{aligned}42 \times 47 &= 1600 + 9 \cdot 40 + 14 \\ &= 1960 + 14 \\ &= 1974 .\end{aligned}$$

A. Simplify mentally.

1. 12×17
2. 22×24
3. 18×11
4. 22×29
5. 16×18
6. 23×27
7. 32×34
8. 43×47
9. 55×55
10. 48×42
11. 62×67
12. 83×88
13. 88×81
14. 94×92
15. 74×75

B. Tell which is larger and by how much.

1. 23×25 or 22×26
2. 39×31 or 38×32
3. 43×47 or 45×45
4. 68×62 or 67×63
5. 72×76 or 71×77
6. 33×35 or 36×32
7. 84×87 or 89×82
8. 91×99 or 96×94
9. 76×76 or 78×74
10. 87×88 or 86×89
11. 372×374 or 371×375
12. 891×894 or 893×892
13. 5873×5876 or 5874×5875
14. 92585×92583 or 92586×92582
15. 92588×92582 or 92584×92584
16. 9462593×9462598 or 9462597×9462594
17. $(50 + 2)(50 + 4)$ or $(50 + 3)(50 + 3)$
18. $(60 + 8)(60 + 3)$ or $(60 + 9)(60 + 2)$
19. $(67 + 8)(67 + 3)$ or $(67 + 9)(67 + 2)$
20. $(983 + 4)(983 + 5)$ or $(983 + 1)(983 + 8)$
21. $(983 + 40)(983 + 10)$ or $(983 + 30)(983 + 20)$
22. $(60052 + 99)(60052 + 1)$ or $(60052 + 27)(60052 + 83)$
23. $(-5 + 7)(-5 + 8)$ or $(-5 + 9)(-5 + 6)$
24. $(-27 + 1)(-27 + 5)$ or $(-27 + 2)(-27 + 4)$

*

True or false?

25. $(72 + 3)(72 + 7) > (72 + 2)(72 + 8)$.

26. For each x, $(x + 3)(x + 7) > (x + 2)(x + 8)$.

27. For each x, $(x + 9)(x + 8) > (x + 5)(x + 12)$.

28. For each k, $(k + 1)(k + 18) > (k + 9)(k + 10)$.

*

Complete to true sentences.

29. For each x, $(x + 3)(x + 7) - (x + 2)(x + 8) =$ __________.

30. For each x, $(x + 9)(x + 8) - (x + 5)(x + 12) =$ __________.

31. For each k, $(k + 1)(k + 18) - (k + 9)(k + 10) =$ __________.

☆32. For each x, $(x + 10)(x + 2) - (x + 5)(x + 4) =$ __________.

"EXPANDING" PRONUMERAL EXPRESSIONS

Let's look again at Exercise 32 of Part B above. The expressions

'$(x + 10)(x + 2)$' and '$(x + 5)(x + 4)$'

can be simplified by using principles for real numbers.

$(x + 10)(x + 2)$	$(x + 5)(x + 4)$
$(x + 10)x + (x + 10)2$	$(x + 5)x + (x + 5)4$
$xx + 10x + 2x + 20$	$xx + 5x + 4x + 20$
$xx + (10 + 2)x + 20$	$xx + (5 + 4)x + 20$
$xx + 12x + 20$	$xx + 9x + 20$

So, for each x,

$$\begin{aligned} &(x + 10)(x + 2) - (x + 5)(x + 4) \\ &= (xx + 12x + 20) - (xx + 9x + 20) \\ &= 3x. \end{aligned}$$

[Did you get this answer in Exercise 32?]

EXERCISES

A. Transform each of the following expressions into an equivalent one which does not contain grouping symbols. [This procedure is sometimes called expanding.]

1. $(x + 5)(x + 3)$ [Answer: $xx + 8x + 15$]

2.	$(x + 2)(x + 7)$	3.	$(x + 6)(x + 2)$	4.	$(x + 1)(x + 7)$
5.	$(x + 8)(x + 9)$	6.	$(x + 3)(x + 5)$	7.	$(x + 2)(x + 2)$
8.	$(x + 7)(x + 7)$	9.	$(x + 11)(x + 9)$	10.	$(x + 1)(x + 1)$
11.	$(y + 3)(y + 7)$	12.	$(y + 4)(y + 5)$	13.	$(a + 1)(a + 15)$

Sample 1. $(y - 3)(y + 7)$

Solution. $(y - 3)(y + 7)$
$= (y + -3)(y + 7)$
$= yy + (-3 + 7)y + -3 \cdot 7$
$= yy + 4y - 21.$

14.	$(x - 3)(x + 5)$	15.	$(x + 9)(x - 2)$	16.	$(z - 8)(z + 12)$
17.	$(a - 11)(a + 15)$	18.	$(a + 6)(a + 7)$	19.	$(a - 10)(a + 9)$
20.	$(b + 4)(b - 3)$	21.	$(b + 6)(b - 6)$	22.	$(b - 11)(b + 11)$
23.	$(b + 6)(b - 15)$	24.	$(m + 2)(m - 3)$	25.	$(m - 6)(m + 20)$
26.	$(x - 8)(x - 2)$	27.	$(x - 7)(x - 6)$	28.	$(x - 3)(x - 11)$
29.	$(x - 3)(x - 3)$	30.	$(x + 5)(x - 7)$	31.	$(x - 2)(x + 19)$
32.	$(m - 4)(m - 7)$	33.	$(m + 8)(m - 8)$	34.	$(m - 5)(m - 50)$

Sample 2. $(3x + 7)(2x + 5)$

Solution. $(3x + 7)(2x + 5)$
$= (3x + 7)2x + (3x + 7)5$
$= 3x \cdot 2x + 7 \cdot 2x + 3x \cdot 5 + 35$
$= 6xx + (7 \cdot 2 + 3 \cdot 5)x + 35$
$= 6xx + (14 + 15)x + 35$
$= 6xx + 29x + 35.$

Sample 3. $(2y - 5)(3y + 4)$

Solution. $(2y - 5)(3y + 4)$

$= 6yy + (-15 + 8)y - 20$

$= 6yy - \quad 7y \quad - 20.$

35. $(4y + 9)(3y + 2)$	36. $(2x + 5)(3x + 2)$	37. $(7x + 4)(2x + 5)$
38. $(6x - 7)(2x + 9)$	39. $(9x - 2)(3x + 1)$	40. $(2g + 5)(8g - 7)$
41. $(a - 7)(a - 3)$	42. $(a - 0)(a - 7)$	43. $(a - 1)(a - 2)$
44. $(2a + 3)(8a + 1)$	45. $(6x + 5)(x + 5)$	46. $(3a + 8)(2a + 11)$
47. $(2x - 5)(5x - 2)$	48. $(8y - 3)(7y - 4)$	49. $(9s - 5)(3s - 7)$
50. $(a + 4)(a - 17)$	51. $(x - 21)(x + 20)$	52. $(x - 80)(x + 80)$
53. $(a + 7)(7a + 1)$	54. $(b + 4)(3b + 8)$	55. $(9b + 1)(3b + 5)$
56. $(5 + x)(7 + x)$	57. $(3 - x)(8 + x)$	58. $(7 - x)(7 + x)$
59. $(5u - 3)(8u + 9)$	60. $(10x - 2)(10x + 2)$	61. $(5 + 3x)(7 - 2x)$

B. Complete to make true sentences.

1. For each x, $xx + 5x + 6 = (x + 3)(\quad)$.
2. For each n, $nn + 8n + 15 = (\quad)(n + 5)$.
3. For each y, $yy + 7y + 12 = (y + 4)(\quad)$.
4. For each z, $zz + 2z - 48 = (\quad)(z - 6)$.
5. For each g, $gg - 16g + 63 = (g - 9)(\quad)$.
6. For each a, $aa - 9 = (a - 3)(\quad)$.
7. For each b, $bb + 10b + 25 = (b + 5)(\quad)$.
8. For each s, $ss - 12s + 36 = (\quad)(s - 6)$.
9. For each x, $xx - 7x - 18 = (x - 9)(\quad)$.
10. For each c, $cc - 11c + 18 = (\quad)(c - 2)$.
11. For each y, $30yy + 43y + 4 = (3y + 4)(\quad)$.
12. For each b, $15bb - 2b - 8 = (\quad)(5b - 4)$.
13. For each m, $20mm + 17m - 10 = (5m - 2)(\quad)$.
14. For each x, $xx - 25 = (\quad)(x + 5)$.

* * *

People often abbreviate expressions like '(xx)' and '(yy)' to 'x^2' and 'y^2'. [Read 'x^2' as 'x squared' or as 'the square of x'.] So, for example, $6^2 = 36$, $3^2 = 9$, $(.01)^2 = .0001$, and $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$. The raised numeral is called <u>an exponent symbol</u>, or, for short, an <u>exponent</u>.

* * *

<u>C</u>. True or false?

1. $(6 \times 3)^2 = 6^2 \times 3^2$
2. $(6 + 3)^2 = 6^2 + 3^2$
3. $(6 \div 3)^2 = 6^2 \div 3^2$
4. $(6 - 3)^2 = 6^2 - 3^2$
5. $(5 + 2)^2 = (5 + 2)(5 + 2)$
6. $(4 + 3)^2 = 4^2 + 2(4)(3) + 3^2$
7. $\left(\frac{12}{2}\right)^2 = \frac{12^2}{2^2}$
8. $\left(\frac{5 + 9}{2}\right)^2 = \frac{5^2 + 9^2}{2^2}$
9. $\left(\frac{3 \times 4}{2}\right)^2 = \frac{3^2 \times 4^2}{2^2}$
10. $\left(\frac{14 - 4}{2}\right)^2 = \frac{14^2 - 2(14)(4) + 4^2}{2^2}$
11. $(2 \cdot 7)(3 \cdot 7) = 6 \cdot 7^2$
12. $9 \cdot 5^2 = 9 \cdot 9 \cdot 5 \cdot 5$
13. $(-5)^2 = 5^2$
14. $-5^2 = -25$

<u>D</u>. Write an equivalent expression (as simple an expression as you can) which does not contain an exponent symbol.

1. 3^2
2. $(-3)^2$
3. -3^2
4. $-(3^2)$
5. $(2 \times 3)^2$
6. $(2 + 3)^2$
7. $(2 - 3)^2$
8. $(2 \div 3)^2$
9. $(.001)^2$
10. $(-.001)^2$
11. $(6 - 6)^2$
12. $(6 + 6)^2$
13. $\left(\frac{4 \times 3}{5}\right)^2$
14. $\left(\frac{5 + 2}{3}\right)^2$
15. $\left(\frac{6 - 4}{3}\right)^2$

<u>E</u>. Expand.

1. $(x + 3)(x + 8)$
2. $(y - 4)(y + 9)$
3. $(y - 3)(y + 17)$
4. $(a - 4)(a + 12)$
5. $(a - 2)(a + 3)$
6. $(2x - 3)(5x + 7)$

<u>Sample 1</u>. $(x + 3)^2$

<u>Solution</u>. $(x + 3)^2 = (x + 3)(x + 3)$

$= xx + 6x + 9$

$= x^2 + 6x + 9.$

<u>Sample 2</u>. $(3x + 5)^2$

<u>Solution</u>. $(3x + 5)^2 = (3x + 5)(3x + 5)$

$= (3x + 5)3x + (3x + 5)5$

$= (3x)^2 + 5 \cdot 3x + 3x \cdot 5 + 25$

$= 9x^2 + (5 + 5)3x + 25$

$= 9x^2 + 10 \cdot 3x + 25$

$= 9x^2 + 30x + 25.$

7. $(x + 1)^2$
8. $(a + 5)^2$
9. $(k + 2)^2$
10. $(m - 1)^2$
11. $(n - 4)^2$
12. $(y - 11)^2$
13. $(3x + 7)^2$
14. $(2y + 5)^2$
15. $(3z + 3)^2$
16. $(4y - 1)^2$
17. $(3k - 4)^2$
18. $(7n - 3)^2$
19. $(2 - 4j)^2$
20. $(6 - 8r)^2$
21. $(3x - 10)^2$
22. $(b - 11)(b + 13)$
23. $(6x - 5)(2x + 3)$
24. $(7x + 3)(x - 2)$
25. $(6a - 5)(2a + 7)$
26. $(5b - 4)(3b - 2)$
27. $(6r + 7)(2r - 3)$
28. $(t + 5)^2$
29. $(r + 7)^2$
30. $(z + 9)^2$
31. $(m - 3)^2$
32. $(m + 3)^2$
33. $(a - .5)^2$
34. $(7m + 2)^2$
35. $(2a - 3b)^2$
36. $(5 + 9s)^2$
37. $(8x + 1)(3 + 2x)$
38. $(5 - 3y)(7 - 2y)$
39. $(6 + 5y)(2 - 9y)$
40. $(-x + 3)(x - 5)$
41. $(-5r - 3)(-3r - 4)$
42. $(-6j - 5)(-8 + 2j)$
43. $(r - \frac{1}{2})^2$
44. $(s + \frac{1}{3})^2$
45. $(t + \frac{1}{2})^2$
46. $(2a + 3b)^2$
47. $(4 + 3p)^2$
48. $(3x_1 - 5x_2)^2$
49. $(5x - 2)(5x + 2)$
50. $(8y + 3)(8y - 3)$
51. $(3z - 1)(3z - 1)$

<u>Sample 3.</u> $a(2b - 3)^2$

<u>Solution.</u>
$$\begin{aligned} a(2b - 3)^2 &= a(2b - 3)(2b - 3) \\ &= a(4b^2 - 12b + 9) \\ &= 4ab^2 - 12ab + 9a. \end{aligned}$$

<u>Sample 4.</u> $x(3y - 2)(y + 5)$

<u>Solution.</u>
$$\begin{aligned} & x(3y - 2)(y + 5) \\ &= x(3y^2 + 13y - 10) \\ &= 3xy^2 + 13xy - 10x. \end{aligned}$$

52. $c(a + b)^2$ 53. $x(2y - 5)^2$ 54. $d(a - b)^2$

55. $a(3x + y)(3x - y)$ 56. $n(2x - 5y)(3x + 5y)$

57. $r(x_1 - x_2)^2$ 58. $(5m - 2n)(2n + 5m)$

59. $(3a + 4b)(3a - 4b)$ 60. $b(3 + a)(5 - a)$

61. $(x + \bigcirc)(x + \square)$ 62. $(x + \triangle)^2$

63. $(⬡\, x + \square)^2$ 64. $(\square\, x + ⬡)(\bigcirc x + \triangle)$

[More exercises are in <u>Part L</u>, <u>Supplementary Exercises</u>.]

FACTORING PRONUMERAL EXPRESSIONS

The process of transforming an expression such as:

$$x^2 + 6x + 8$$

into the equivalent one:

$$(x + 4)(x + 2)$$

is called <u>factoring</u>. Some other examples of factoring are transforming

'12' into '$3 \cdot 4$'
'100' into '$10 \cdot 10$'
'100' into '$20 \cdot 5$'
'60' into '$3 \cdot 20$' [or: '$3 \cdot 4 \cdot 5$']
'80a' into '$4 \cdot 20a$'
'$x^2 + 3x$' into '$x(x + 3)$'
'$x^2y + xy^2$' into '$xy(x + y)$'
'$24a^2 - 60a$' into '$a(24a - 60)$'
'$24a^2 - 60a$' into '$4a(6a - 15)$'
'$24a^2 - 60a$' into '$12a(2a - 5)$'.

EXERCISES

A. Factor.

1. 12 2. 144 3. 35 4. 51

5. $33x$ 6. $27xy^2$ 7. $24y$ 8. $60yz$

Sample. $30a + 6$

Solution. $6(5a + 1)$
[Check the result of factoring by seeing whether you can transform '$6(5a + 1)$' into '$30a + 6$'.]

9. $7x + 28$ 10. $8y - 40$ 11. $21x + 70$

12. $3x + 12y$ 13. $16a - 20b$ 14. $5m - 10p + 25q$

15. $xy + 3x$ 16. $5xy - 10y$ 17. $2ax - 6ay + 10az$

18. $a^2b + b^2a$ 19. $3at + 9a^2$ 20. $-2a^2b + 8a^2bc$

B. Complete to make true sentences.

1. For each x, $xx + 7x + 12 = (x + 4)(\quad)$.
2. For each x, $x^2 + 11x + 18 = (\quad)(x + 2)$.
3. For each y, $y^2 + 10y + 9 = (y + 9)(\quad)$.
4. For each y, $y^2 + 7y - 170 = (\quad)(y + -10)$.
5. For each a, $a^2 - 16 = (a - 4)(\quad)$.
6. For each a, $a^2 - 49 = (a + 7)(\quad)$.
7. For each t, $t^2 + 6t + 9 = (\quad)(t + 3)$.
8. For each t, $t^2 - 4t + 4 = (t - 2)(\quad)$.
9. For each m, $m^2 - 20m + 100 = (\quad)(m - 10)$.
10. For each z, $z^2 + 4z - 21 = (z + 7)(\quad)$.
11. For each z, $6z^2 + 23z + 20 = (2z + 5)(\quad)$.
12. For each x, $10x^2 + 33x + 27 = (\quad)(5x + 9)$.
13. For each x, $12x^2 - 4x - 21 = (6x + 7)(\quad)$.
14. For each t, $30t^2 - 91t - 30 = (\quad)(3t - 10)$.
15. For each n, for each a, $na^2 - 5an + 6n = n(\qquad)$
$= n(a - 3)(\quad)$.

C. Factor.

Sample 1. $x^2 + 7x - 8$

Solution. We suspect that factoring '$x^2 + 7x - 8$' will lead to an expression of the form:

$$(x + \triangle)(x + \text{⬡}).$$

Now, if we expand this last expression, we get:

$$x^2 + (\triangle + \text{⬡})x + \triangle \times \text{⬡}.$$

Compare this with the expression to be factored:

$$x^2 + (\quad 7 \quad)x + (\quad -8 \quad).$$

What we want to find are numbers $\triangle$ and ⬡ such that

$$\triangle \times \text{⬡} = -8 \qquad \text{and} \qquad \triangle + \text{⬡} = 7.$$

Do you see that such numbers are 8 and -1? So, factoring '$x^2 + 7x - 8$' gives us '$(x + -1)(x + 8)$', or:

$$(x - 1)(x + 8)$$

[Check by expanding '$(x - 1)(x + 8)$'. Do you get '$x^2 + 7x - 8$'?]

Sample 2. $n^2 + 10n + 25$

Solution.
$$(n + \square)(n + \bigcirc)$$
$$= n^2 + (\square + \bigcirc)n + \square \times \bigcirc.$$

What we want to find are numbers $\square$ and $\bigcirc$ such that

$$\square \times \bigcirc = 25 \qquad \text{and} \qquad \square + \bigcirc = 10.$$

$$5 \times 5 = 25 \qquad \text{and} \qquad 5 + 5 = 10.$$

So, factoring '$n^2 + 10n + 25$' gives:

$$(n + 5)(n + 5)$$

or, for short:

$$(n + 5)^2.$$

1. $y^2 + 2y - 35$
2. $y^2 + 6y - 27$
3. $x^2 - 2x - 99$
4. $x^2 - 13x + 12$
5. $r^2 + 6r + 9$
6. $p^2 - 22p + 121$
7. $y^2 + 4y - 12$
8. $n^2 - 16n + 64$
9. $x^2 - 9x + 20$
10. $a^2 + 10a + 16$
11. $y^2 + 20y + 100$
12. $d^2 - 12d + 36$

Sample 3. $6x^2 + 23x + 20$

Solution. Factoring will lead to an expression of the form:

$$(\square x + \bigcirc)(\triangle x + \hexagon).$$

If we expand this last expression we get:

$$(\square \times \triangle)x^2 + (\square \times \hexagon + \bigcirc \times \triangle)x + \bigcirc \times \hexagon.$$

Compare this with the given expression:

$$(\quad 6 \quad)x^2 + (\qquad 23 \qquad)x + \quad 20 \;.$$

We try various factorings for '6' and for '20'.

$\square \times \triangle = 6$	$\bigcirc \times \hexagon = 20$	$\square \times \hexagon + \bigcirc \times \triangle = 23$
6×1	10×2	$6 \times 2 + 10 \times 1 \neq 23$
6×1	2×10	$6 \times 10 + 2 \times 1 \neq 23$
6×1	5×4	$6 \times 4 + 5 \times 1 \neq 23$
6×1	4×5	$6 \times 5 + 4 \times 1 \neq 23$
3×2	5×4	$3 \times 4 + 5 \times 2 \neq 23$
3×2	4×5	$3 \times 5 + 4 \times 2 = 23$

So, factoring '$6x^2 + 23x + 20$' leads to:

$$(3x + 4)(2x + 5).$$

Check this by expanding.

(continued on next page)

13. $3a^2 + 7a + 2$
14. $2a^2 - 5a + 1$
15. $5p^2 + 34p - 7$
16. $25m^2 + 10m + 1$
17. $9r^2 - 6r + 1$
18. $11t^2 - 32t - 3$
19. $d^2 + 6d + 5$
20. $b^2 + 4b - 45$
21. $7a^2 - 11a - 6$
22. $r^2 + 5r + 6$
23. $10r^2 + 39r + 14$
24. $6x^2 + x - 1$
25. $12m^2 - 2m - 2$
26. $15p^2 + 29p + 8$
27. $15p^2 + 43p + 8$
28. $d^2 - 12d + 36$
29. $q^2 + 26q + 169$
30. $c^2 + 7c - 6$
31. $y^2 - 24y + 144$
32. $49x^2 + 42x + 9$
33. $n^2 - 25$
34. $n^2 - 9$
35. $x^2 + 19x + 90$
36. $d^2 + 10d + 21$
37. $a^2 - 10a + 21$
38. $m^2 + 3m - 28$
39. $49 - t^2$
40. $a^2 - b^2$
41. $14x^2 + 39x + 10$
42. $24t^2 + 52t + 20$
43. $36r^2 - 25$
44. $9 - 16s^2$
45. $b^2 + 6b - 16$
46. $c^2 - 15c - 16$
47. $18t^2 + 15t + 3$
48. $21k^2 - 5k - 6$
49. $9x^2 - 25y^2$
50. $9x^2 - 30xy + 25y^2$

<u>Sample 4</u>. $3a^2n - 12n$

<u>Solution</u>. $3a^2n - 12n$

$= 3n(a^2 - 4)$

$= 3n(a - 2)(a + 2)$.

[Check this by expanding.]

Sample 5. $12a^2b - 4ab - 40b$

Solution. $12a^2b - 4ab - 40b$

$= 2b(6a^2 - 2a - 20)$

$= 2b(3a + 5)(2a - 4).$

[Check this by expanding.]

51. $4a^2 - 12ab$
52. $x^2 + 5x$
53. $6n^2 + 18$
54. $2r^2 - 18$
55. $x^2y + 16xy + 64y$
56. $2y^2 + 30y + 72$
57. $by^2 + 21by + 110b$
58. $m^2 + 14m - 51$
59. $N^2 + 3N - 10$
60. $P^2 - 9PQ + 18Q^2$
61. $20m^2n - 9mn - 18n$
62. $6x^2z + xyz - y^2z$
63. $45y^2 - 15yx - 6x^2$
64. $y^2 + 22y + 40$
65. $9xy^2 - 25x$
66. $18xy^2 - 50xz^2$
67. $y^2 + 13y + 40$
68. $y^2 + 14y + 40$
69. $3r^2 - 36r + 105$
70. $at^2 - 3at - 28a$
71. $36 + 15r + r^2$
72. $42 - 13u + u^2$
73. $49u^2 - 14uv + v^2$
74. $100t^2 - r^2$
75. $12z^2 - 16zx - 3x^2$
76. $10x_1^2 - 7x_1x_2 - 12x_2^2$

[More exercises are in Part L, Supplementary Exercises.]

D. Solve these equations.

1. $(x - 3)(x - 5) = 0$
2. $(x + 4)(x - 7) = 0$
3. $(y + 3)(y + 7) = 0$
4. $x(x - 5) = 0$
5. $(2x - 5)(x + 8) = 0$
6. $(3y + 5)(2y - 7) = 0$
7. $x^2 - 36 = 0$
8. $x^2 - 5x + 6 = 0$

3.08 Quadratic equations.--Let's try to solve the following problem.

> Ed is 3 years older than Mary,
> and 5 years ago the product of
> their ages was 180. How old
> are they now?

Suppose that Mary is now x years old. Then Ed is now $(x + 3)$ years old. Five years ago, Mary was $(x - 5)$ years old and Ed was $(x + 3 - 5)$ [or: $(x - 2)$] years old. Hence, we are looking for a number x such that

$$(x - 5)(x - 2) = 180,$$

or,

$$(1) \qquad x^2 - 7x + 10 = 180.$$

So, we try to solve equation (1). The exercises in Part D on page 3-95 give us a clue to how this can be done.

Transform (1) into an equation which has one side '0':

$$x^2 - 7x + 10 - 180 = 180 - 180.$$

$$(2) \qquad x^2 - 7x - 170 = 0.$$

Then, factor the left side of (2) to get:

$$(3) \qquad (x - 17)(x + 10) = 0.$$

Now, we are looking for a number x such that the product of $x - 17$ by $x + 10$ is 0. The 0-product theorem tells us that if the product of a number by a number is 0 then one of these numbers is 0. Also, the principle for multiplying by 0 [and the commutative principle for multiplication] tells us that the product of a number by a number is 0 if one of the numbers is 0. So, equation (3) is equivalent to the sentence:

$$(4) \qquad x - 17 = 0 \quad \text{or} \quad x + 10 = 0.$$

And, (4) is equivalent to the sentence:

$$(5) \qquad x = {}^{+}17 \quad \text{or} \quad x = {}^{-}10.$$

The numbers which satisfy (5) are $^{+}17$ and $^{-}10$.

Do you see that sentence (5) is equivalent to equation (3), that

is, that they are satisfied by the same numbers? So, equation (1) and sentence (5) are equivalent. Hence, the solutions of (1) are $^{+}17$ and $^{-}10$.

But the number we are looking for in the age problem is the number of years in Mary's present age. This is a number of arithmetic. And it must satisfy equation (1). Since $^{+}17$ and $^{-}10$ are the only real numbers which satisfy (1), it must be the case that 17 is the only number of arithmetic which satisfies (1) [Why?].

So, Mary is now 17 years old. [Check. Ed is 20 if Mary is 17. Five years ago they were 15 and 12, and the product of 15 and 12 is 180.]

* * *

Notice that solving this problem involved solving the equation:

$$x^2 - 7x - 170 = 0.$$

This equation is called a quadratic equation. Other examples of quadratic equations are:

$$3x^2 + 11x - 4 = 0, \qquad x^2 - 3x = 0,$$

$$x^2 + 6x + 9 = 0, \qquad 8x^2 - 392 = 0.$$

In fact, any equation you get from the open sentence:

$$\square\, x^2 + \triangle\, x + \text{⬡} = 0$$

by substituting numerals for the frames [but not substituting a name of 0 for '$\square$'] is called a quadratic equation in 'x'. [Sometimes, quadratic equations like these are said to be in standard form, and other equations which can be transformed into quadratic equations in standard form are also called 'quadratic equations'. For example, we can call the equation:

$$x^2 - 7x + 10 = 180$$

a quadratic equation' because we can transform it into a quadratic equation in standard form:

$$x^2 - 7x - 170 = 0.]$$

As you have seen, one way of trying to solve a quadratic equation

is to transform it to one of standard form and then try to factor the left side. Consider this example:

$$(1) \qquad 2x(x + 6) = 5 - x(x + 2).$$

We expand to get:

$$(2) \qquad 2x^2 + 12x = 5 - x^2 - 2x.$$

Next, transform (2) to get a quadratic equation in standard form:

$$(3) \quad [2x^2 + 12x] + -5 + x^2 + 2x = [5 - x^2 - 2x] + -5 + x^2 + 2x.$$

Simplify the sides to get:

$$(4) \qquad 3x^2 + 14x - 5 = 0.$$

Now, factor the left side:

$$(5) \qquad (3x - 1)(x + 5) = 0.$$

Notice that equation (1) and equation (5) are equivalent.

To solve (5) is to find numbers x such that the product of the number $3x - 1$ and the number $x + 5$ is 0. But, we know that, given a first number and a second number, if one of these numbers is 0 then their product is 0, and if their product is 0 then one of them is 0. So, equation (5) is equivalent to the sentence:

$$(6) \qquad 3x - 1 = 0 \quad \text{or} \quad x + 5 = 0,$$

which is equivalent to the simpler sentence:

$$(7) \qquad x = \frac{1}{3} \quad \text{or} \quad x = -5.$$

Since the numbers which satisfy sentence (7) are just the numbers $\frac{1}{3}$ and -5, it follows from the fact that (7) and (1) are equivalent that the roots of (1) are $\frac{1}{3}$ and -5.

We check to make sure our computations are correct.

$$\begin{array}{rcl} 2x(x + 6) & = & 5 - x(x + 2) \\ 2 \cdot \frac{1}{3}(\frac{1}{3} + 6) & = & 5 - \frac{1}{3}(\frac{1}{3} + 2)\ ? \\ \frac{2}{3} \cdot \frac{19}{3} & \vert & 5 - \frac{7}{9} \\ \frac{38}{9} & = & \frac{38}{9}\ \checkmark \end{array}$$

$$\begin{array}{rcl} 2x(x + 6) & = & 5 - x(x + 2) \\ 2 \cdot -5(-5 + 6) & = & 5 - -5(-5 + 2)\ ? \\ -10 \cdot 1 & \vert & 5 - 15 \\ -10 & = & -10\ \checkmark \end{array}$$

EXERCISES

Solve these equations.

Sample 1. $x^2 - 3x - 40 = 0$

Solution. [The quadratic equation is in standard form. So, we try to factor the left side.]

$$x^2 - 3x - 40 = 0$$
$$(x - 8)(x + 5) = 0$$

$$x - 8 = 0 \quad \text{or} \quad x + 5 = 0$$
$$x = 8 \quad \text{or} \quad x = -5$$

The roots are 8 and -5.

Check.

$$8^2 - 3 \cdot 8 - 40 = 0 \ ? \qquad (-5)^2 - 3 \cdot -5 - 40 = 0 \ ?$$
$$64 - 24 - 40 \mid 0 \qquad 25 - -15 - 40 \mid 0$$
$$0 = 0 \checkmark \qquad 0 = 0 \checkmark$$

1. $x^2 - 3x + 2 = 0$
2. $x^2 + 7x + 6 = 0$
3. $x^2 - 1 = 0$
4. $x^2 + 10x + 25 = 0$
5. $2x^2 + 5x - 12 = 0$
6. $3x^2 + 4x - 15 = 0$
7. $x^2 + 5x = 14$
8. $9 - 17x = 2x^2$
9. $x^2 = 49$
10. $x^2 = 8x$
11. $6x^2 - x - 2 = 0$
12. $8y^2 = 21 - 2y$
13. $42 + 5z - 2z^2 = 0$
14. $3x^2 - 90 + 17x = 0$
15. $6x^2 = 6(2x - 1) + x$
16. $y(5y + 1) = 3(5y + 1)$
17. $28x = 12x^2$
18. $2(x^2 + 6x + 8) = 0$
19. $2x^2 + 12x + 16 = 0$ [Hint: See Exercise 18.]
20. $7y^2 - 49y + 84 = 0$
21. $5x^2 = 10x - 5$
22. $\frac{1}{2}x^2 + 4x + 6 = 0$
23. $\frac{1}{3}x^2 + x - \frac{10}{3} = 0$

[More exercises are in Part N, Supplementary Exercises.]

3.09 Solving inequations. --You have used basic principles and theorems in solving equations. Particularly useful ones are these.

The addition transformation principle

(a) $\forall_x \forall_y \forall_z$ if $x = y$ then $x + z = y + z$,

(b) $\forall_x \forall_y \forall_z$ if $x + z = y + z$ then $x = y$.

The two parts of this principle are often combined into the single sentence:

$$\forall_x \forall_y \forall_z \ (x + z = y + z \text{ if and only if } x = y).$$

The multiplication transformation principle

$$\forall_x \forall_y \forall_{z \neq 0} \ (xz = yz \text{ if and only if } x = y).$$

The factoring transformation principle

$$\forall_x \forall_y \ [(x = 0 \text{ or } y = 0) \text{ if and only if } xy = 0].$$

You might suspect that there are similar principles which are useful for solving inequations. Let's try solving an inequation by transforming, as we did in solving equations, and see how we come out.

(1) $2b + 3 > 9$

(2) $2b + 3 + -3 > 9 + -3$

(3) $2b > 6$

(4) $2b \cdot \frac{1}{2} > 6 \cdot \frac{1}{2}$

(5) $b > 3$

So, it appears that the solution set of (1) is $\{x\colon\ x > 3\}$. Do you believe that it is? In order to justify the procedure, we need to show that (1) and (2) are equivalent sentences, that (2) and (3) are equivalent, etc.

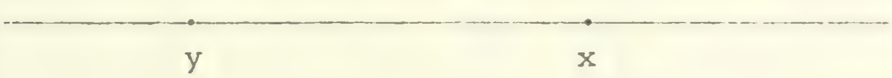
(2) and (3) are equivalent sentences because '2b + 3 + −3' and '2b' are equivalent expressions and so are '9 + −3' and '6'. Similarly, (4) and (5) are equivalent sentences.

Are (1) and (2) equivalent? That is, is the solution set of (1) a subset of the solution set of (2), and conversely? Given a first number [2b + 3] and a second number [9] such that the first number is greater than the second, if you add a third number [−3] to each, will the sum of the first and third be greater than the sum of the second and third? And, conversely, suppose the sum of the first and third is greater than the sum of the second and the third. Does it follow that the first is greater than the second? In other words, is it the case that

(i) $\forall_x \forall_y \forall_z$ if $x > y$ then $x + z > y + z$,

and (ii) $\forall_x \forall_y \forall_z$ if $x + z > y + z$ then $x > y$?

It is easy to see that (i) is the case. Think of it geometrically.



If $x > y$ then the graph of x is to the right of the graph of y. Now, suppose you add the same number z to each of x and y. You can think of adding z as jumping the graphs of x and y either to the right [if z is positive] or to the left [if z is negative]. In either case, the graph of $x + z$ is to the right of the graph of $y + z$.

Similarly, if the graph of $x + z$ is to the right of the graph of $y + z$ then the graph of x is to the right of the graph of y. So, (ii) is the case.

Generalizations (i) and (ii) together give us

<u>The addition transformation principle for inequations.</u>

$$\forall_x \forall_y \forall_z \; (x + z > y + z \text{ if and only if } x > y).$$

[Since '$x > y$' means the same thing as '$y < x$', and '$x + z > y + z$' means the same thing as '$y + z < x + z$', this principle tells us that

$$\forall_x \forall_y \forall_z \; (y + z < x + z \text{ if and only if } y < x).$$

So, the addition transformation principle for inequations is applicable to "<-tions" as well as to ">-tions". Also, you can combine this principle with the one for equations to get a principle which is applicable to "≤-tions" and to "≥-tions".]

Now, consider sentences (3) and (4):

(3) $2b > 6,$

(4) $2b \cdot \frac{1}{2} > 6 \cdot \frac{1}{2}.$

One is tempted to say that these are equivalent by virtue of the generalization:

(*) $\forall_x \forall_y \forall_{z \neq 0}$ ($xz > yz$ if and only if $x > y$).

Let's try to justify this generalization as we did the one for addition.

If we multiply a number by 1/2, we get a number whose graph is halfway between the graph of 0 and the graph of the original number. So, the result of multiplying by 1/2 is to jump the graphs of numbers toward the graph of 0 but to leave the graphs in the same order. Multiplying by 2 jumps the graphs away from the graph of 0 but still leaves them in the same order. Multiplying by 1 does nothing. [What would multiplying by 0 do?] Does this convince you that (*) is the case? It shouldn't. Consider what multiplying by a negative number does to the order of the graphs.

So, here is the generalization which tells us that (3) and (4) are equivalent:

(a) $\forall_x \forall_y \forall_{z > 0}$ ($xz > yz$ if and only if $x > y$).

Now, suppose we want to solve the inequation:

$$-3x > 12.$$

If we try to apply (*), we get after simplifying:

$$x > -4$$

which is <u>not</u> equivalent to the given inequation. But, the simple inequation which is equivalent to the given one is:

$$x < -4.$$

If you experiment with multiplying numbers by a negative number and observing what happens to the order of their graphs, you will no doubt arrive at this generalization:

(b) $\forall_x \forall_y \forall_{z < 0}$ ($xz < yz$ if and only if $x > y$).

Together, (a) and (b) constitute <u>the multiplication transformation principle for inequations</u>.

There is also an analogue for the factoring transformation principle. Since you are well acquainted with the facts on which it is based, we merely state it, and ask you to convince yourself of it by experimenting.

The factoring transformation principle for inequations

(a) $\forall_x \forall_y \forall_z$ (xy > 0 if and only if [(x > 0 and y > 0) or (x < 0 and y < 0)]).

(b) $\forall_x \forall_y \forall_z$ (xy < 0 if and only if [(x > 0 and y < 0) or (x < 0 and y > 0)]).

Let us now use the inequation transformation principles.

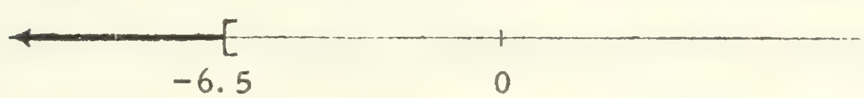
Example 1. Find the solution set and draw the graph of:

$$5x - 4 > 7x + 9.$$

Solution.

$$5x - 4 > 7x + 9$$
$$5x - 4 + 4 > 7x + 9 + 4$$
$$5x > 7x + 13$$
$$5x + -7x > 7x + 13 + -7x$$
$$-2x > 13$$
$$-2x \cdot -\frac{1}{2} < 13 \cdot -\frac{1}{2}$$
$$x < -6.5$$

The solution set is $\{x: x < -6.5\}$. Here is the graph.



* * *

Now, before reading any further, you should make up some inequations like the one given in Example 1, solve them, and graph. [Throughout your study of mathematics, you will find it helpful to make up your own examples to illustrate new ideas and techniques.]

* * *

<u>Example 2</u>. Find the solution set and draw the graph of:

$$x^2 + 4 > 5(2 - x).$$

<u>Solution</u>. [This appears to be a quadratic inequation, so we'll transform to standard form.]

$$x^2 + 4 > 5(2 - x)$$

$$x^2 + 4 > 10 - 5x$$

$$x^2 + 5x - 6 > 0$$

$$(x + 6)(x - 1) > 0$$

$$(x + 6 > 0 \text{ and } x - 1 > 0) \text{ or } (x + 6 < 0 \text{ and } x - 1 < 0)$$

$$\left.\begin{array}{ccc} (x > -6 \text{ and } x > 1) & \text{or} & (x < -6 \text{ and } x < 1) \\ x > 1 & \text{or} & x < -6 \end{array}\right\} \text{[Why?]}$$

So, the solution set is $\{x\colon\ x > 1 \text{ or } x < -6\}$. Here is the graph.

<u>Example 3</u>. Find the solution set and draw the graph of:

$$7 \leq \frac{5x - 2}{4} < 17.$$

<u>Solution</u>. [This sentence is a conjunction of the two inequations:

$$7 \leq \frac{5x - 2}{4}, \quad \text{and:} \quad \frac{5x - 2}{4} < 17.$$

If a number satisfies the given sentence then it satisfies <u>both</u> inequations, and conversely.]

Since solving the given sentence amounts to solving two inequations simultaneously, we proceed as follows.

$$7 \leq \frac{5x - 2}{4} < 17$$

$$7 \leq \frac{5x - 2}{4} \quad \text{and} \quad \frac{5x - 2}{4} < 17$$

$$28 \leq 5x - 2 \quad \text{and} \quad 5x - 2 < 68$$

$$30 \leq 5x \quad \text{and} \quad 5x < 70$$

$$6 \leq x \quad \text{and} \quad x < 14$$

$$6 \leq x < 14$$

So, the solution set is $\{x\colon\ 6 \leq x < 14\}$. Here is the graph.

[You could save some extra writing by writing the steps this way:

$$7 \leq \frac{5x - 2}{4} < 17$$

$$28 \leq 5x - 2 < 68$$

etc.]

<u>Example 4</u>. Find the solution set and draw the graph of:

$$|3x - 5| < 13.$$

<u>Solution</u>. The sentence '$|3x - 5| < 13$' is equivalent to:

$$^-13 < 3x - 5 < {}^+13.$$

So, we could proceed as in Example 3. Finish the problem.

EXERCISES

<u>A</u>. Solve each of the following inequations and draw its graph. [To solve an inequation means to give the simplest description of its solution set, as illustrated in the preceding examples.]

1. $3x + 4 > 14 - 7x$
2. $2x + 5 > x - 7$
3. $2x + 6 < 4 - 6x$
4. $\frac{x}{2} - 1 > 3 - x$

(continued on next page)

5. $\frac{2y - 5}{2} > \frac{3y + 4}{3}$

6. $2x^2 - 9x - 35 > 0$

7. $x^2 + 2(x - 2) < x + 8$

8. $3(x + 4) > 9 + 3x$

9. $2x - 3 > 3x + 5$

10. $3x + 5 < 2x - 3$

11. $6.6 - 1.5z \geq 3 - z$

12. $8p + 9 \leq 21 - 7p$

13. $5 < 2x - 7 < 13$

14. $-8 \leq 12 - 2x \leq 4$

15. $10 \leq \frac{3x + 11}{2} \leq 13$

16. $\frac{14}{3} < \frac{1}{2}(5 - x) + \frac{2}{3} < \frac{19}{6}$

17. $|2x + 5| \leq 23$

18. $|1 - 5y| \leq 0.5$

☆19. $\frac{1}{x} > 2$

☆20. $\frac{3x - 5}{x} > 2$

☆21. $(x - 2)(x + 3)(x - 4) > 0$

☆22. $(x - 1)^2 > x - 1$

☆23. $\frac{(x + 2)(x - 5)}{(x - 4)} < 0$

☆24. $|x^2 - 5x| < 6$

☆B. 1. Use '$\forall_x \forall_y \forall_z$ if $x > y$ then $x + z > y + z$' together with the basic principles and theorems to prove:

$$\forall_x \forall_y \forall_z \text{ if } x + z > y + z \text{ then } x > y.$$

2. Take as a basic principle the generalization:

(G) $\forall_x \forall_y$ ($x > y$ if and only if $x - y$ is positive),

and use it to prove:

$$\forall_x \forall_y \forall_z \ (x + z > y + z \text{ if and only if } x > y).$$

*

[In order to prove more theorems about $>$, you would need other basic principles about positive numbers in addition to (G). The following four are sufficient:

(P_1) $\forall_x$ if $x \neq 0$ then x is positive or $-x$ is positive,

(P_2) $\forall_x$ not both x and $-x$ are positive,

(P_3) $\forall_x \forall_y$ if x and y are positive then $x + y$ is positive,

(P_4) $\forall_x \forall_y$ if x and y are positive then xy is positive.

From these you can, for example, derive the transformation principles for inequations.]

EXPLORATION EXERCISES

A. Graph each of the following sentences.

Sample. $2x + 30 > 70$

Solution.

1. $5y + 9 > 4$
2. $3x - 2 < 7$
3. $2 < x < 5$
4. $3 < y + 2 < 7$
5. $9 \leq x - 1 \leq 12$
6. $-11 \leq 2y + 1 \leq 17$
7. $10 \leq x < 11$
8. $9.5 \leq y < 10.5$
9. $3.5 \leq x < 3.6$
10. $12.3 \leq 3x < 12.6$
11. $4.25 \leq x < 4.35$
12. $4.257 \leq x < 4.258$
13. $|x - 2| = 0$
14. $|x - 2| \leq 0$
15. $|x - 3| < 1$
16. $|x - 3| < 0.1$
17. $|x - 3| \leq 0.1$
18. $|x - 6.1| < 0.05$

B. True or false?

1. $\{x: 3 \leq x < 4\} \subseteq \{x: 3 \leq x < 5\}$
2. $\{x: 3.5 \leq x < 4.5\} \subseteq \{x: 3 \leq x < 4\}$
3. $\{x: 6.26 < x < 6.28\} \subseteq \{x: 6.25 \leq x < 6.35\}$
4. $\{x: 5.31 < x < 5.39\} \subseteq \{x: 5.3 \leq x < 5.4\}$
5. $\{x: 7.3984 < x < 7.3992\} \subseteq \{x: 7.3985 \leq x < 7.3995\}$
6. $\{x: |x - 3| < 1\} \subseteq \{x: |x - 3| \leq 1\}$
7. $\{x: 4.5 \leq x < 5.5\} \subseteq \{x: |x - 5| < 0.5\}$
8. $\{x: |x - 6.3| < 0.05\} \subseteq \{x: 6.25 \leq x < 6.35\}$
9. $\{x: 5.477 < x < 5.478\} \subseteq \{x: 5.47 < x < 5.48\}$
10. $\{x: 1.414 < x < 1.415\} \subseteq \{x: 4.24 \leq 3x < 4.25\}$
11. $\{x: \frac{1}{3} < x < \frac{2}{3}\} \subseteq \{x: 0.33 < x < 0.67\}$
12. $\{x: \frac{1}{7} < x < \frac{2}{7}\} \subseteq \{x: 0.1429 < x < 0.286\}$
13. $\{x: 0.11 < x < 0.12\} \subseteq \{x: \frac{1}{9} < x < \frac{1}{8}\}$

3.10 Square roots.--Consider the equation:

$$x^2 - 25 = 0.$$

To solve this equation is to find all numbers each of which when squared gives 25. Since '$x^2 - 25 = 0$' is equivalent to '$(x - 5)(x + 5) = 0$', it follows that the solution set of the given equation is $\{5, -5\}$. Now, consider the equation:

$$x^2 - 30 = 0.$$

What is its solution set? If there is a number whose square is 30, this number belongs to the solution set. Also, its opposite belongs to the solution set. So, if we knew the positive numbers which satisfy the equation then we would know the solution set of the equation. This solution set consists just of the positive numbers which satisfy the equation together with their opposites. Let us first ask how many positive numbers satisfy the equation. Can there be two? Must there be at least one?

The first question ["Can there be two positive roots?"] is easy to answer; let's answer it. Suppose m and n are positive numbers whose squares are 30. Then $m^2 = n^2$. So, $m^2 - n^2 = 0$. Hence, $(m - n)(m + n) = 0$. Thus, $m = n$ or $m = -n$. But, since m and n are positive, m cannot be the opposite of n. So, $m = n$. There can't be two positive roots of '$x^2 - 30 = 0$'.

Now, let's consider the second question: Must there be a positive number whose square is 30? Let's try to find one. Since $30 > 25$, and since $5^2 = 25$, it is reasonable to expect that if there is a positive number whose square is 30, the number is greater than 5. Also, since $36 > 30$, and since $6^2 = 36$, if there is a positive number whose square is 30, this number is less than 6. Let's consider some numbers between 5 and 6 along with their squares.

$$(5.1)^2 = 26.01$$
$$(5.2)^2 = 27.04$$
$$(5.3)^2 = 28.09$$
$$(5.4)^2 = 29.16$$
$$(5.5)^2 = 30.25$$

You can see that if there is a positive number whose square is 30, this

number must be between 5.4 and 5.5 [Why?]. [If there is a positive number whose square is 30, then the only other number whose square is 30 is a negative number. What can you say about this negative number?]

Here is another list.

$$(5.41)^2 = 29.2681$$
$$(5.42)^2 = 29.3764$$
$$(5.43)^2 = 29.4849$$
$$(5.44)^2 = 29.5936$$
$$(5.45)^2 = 29.7025$$
$$(5.46)^2 = 29.8116$$
$$(5.47)^2 = 29.9209$$
$$(5.48)^2 = 30.0304$$

Do you see that if there is a positive number whose square is 30, this number is between 5.47 and 5.48? After more tedious multiplication, we find that

$$(5.477)^2 = 29.997529$$

and that $$(5.478)^2 = 30.008484.$$

We leave to you the job of showing that the number we are seeking [if there is such a number] is between 5.4772 and 5.4773.

It appears that we can find positive numbers whose squares are <u>as close to</u> 30 as we wish. [For most applications of mathematics this is enough.] However, nothing we have done helps to show that there <u>is</u> a positive number whose square is 30. To prove that there is such a positive number is a difficult task and requires additional basic principles about real numbers. You may study such a proof in a later course. It is enough for present purposes that you accept the statement that there is such a number. The usual name for this number is:

$$\sqrt{30}.$$

['$\sqrt{30}$' is read as 'the principal square root of 30', or as '<u>the</u> square root of 30'. The '$\sqrt{}$' is called 'a radical sign'.] The negative number whose square is 30 is

$$-\sqrt{30}.$$

['$-\sqrt{30}$' is read as 'the negative square root of 30'.]

So, the solutions of the equation:

$$x^2 - 30 = 0$$

are $\sqrt{30}$ and $-\sqrt{30}$. The check is easy. Since $\sqrt{30}$ is the positive number whose square is 30, $(\sqrt{30})^2$ is 30, and $30 - 30 = 0$. And, since $(-\sqrt{30})^2 = (\sqrt{30})^2$ [Why?], $(-\sqrt{30})^2$ is 30, and $30 - 30 = 0$.

EXERCISES

A. Find the solution sets of these equations.

1. $x^2 - 17 = 0$ [Answer: $\{\sqrt{17}, -\sqrt{17}\}$.]

2. $x^2 - 38 = 0$
3. $y^2 - 91 = 0$
4. $y^2 = 71$
5. $0 = z^2$
6. $x^2 = 81$
7. $50 - a^2 = 0$

B. True or false?

Sample. $\sqrt{600} = 10\sqrt{6}$

Solution. We are trying to determine whether '$\sqrt{600}$' and '$10\sqrt{6}$' are names for the same number. We know that '$\sqrt{600}$' is a name for the positive number whose square is 600. So, we must determine whether $10\sqrt{6}$ is the positive number whose square is 600. Since 10 and $\sqrt{6}$ are positive numbers, and since the product of two positive numbers is positive, it follows that $10\sqrt{6}$ is a positive number. Also,

$$\begin{aligned}(10\sqrt{6})^2 &= (10\sqrt{6})(10\sqrt{6}) \\ &= 10^2 \cdot (\sqrt{6})^2 \\ &= 100 \cdot 6 \\ &= 600.\end{aligned}$$

So, the sentence '$\sqrt{600} = 10\sqrt{6}$' is true. [Is the sentence '$-\sqrt{600} = 10\sqrt{6}$' true?]

1. $\sqrt{5000} = 10\sqrt{50}$
2. $\sqrt{2000} = 100\sqrt{2}$
3. $\sqrt{8} = 2\sqrt{2}$
4. $\sqrt{9} = 3$

5. $\sqrt{16} = -4$

6. $\sqrt{50} = \sqrt{25}\cdot\sqrt{2}$

7. $\sqrt{4\cdot 9} = \sqrt{4}\cdot\sqrt{9}$

8. $\sqrt{4+9} = \sqrt{4} + \sqrt{9}$

9. $\sqrt{75} = 5\sqrt{3}$

10. $\sqrt{7500} = 50\sqrt{3}$

11. $\sqrt{.02} = .1\sqrt{2}$

12. $\sqrt{.002} = .01\sqrt{2}$

13. $\sqrt{89} \times \sqrt{89} = 89$

14. $89 \div \sqrt{89} = \sqrt{89}$

C. Simplify.

Sample. $5\sqrt{2} + 3\sqrt{2} + 7$

Solution. $5\sqrt{2} + 3\sqrt{2} + 7$

$= (5+3)\sqrt{2} + 7$

$= 8\sqrt{2} + 7.$

1. $6\sqrt{3} + 2\sqrt{3}$
2. $9\sqrt{5} + \sqrt{5}$
3. $7\sqrt{7} - 2\sqrt{7}$
4. $4\sqrt{2} + 3 + 7\sqrt{2}$
5. $3\sqrt{5} + 2\sqrt{5} + 8$
6. $5\sqrt{3} - 10\sqrt{2} + 5\sqrt{3}$
7. $5(3 + 2\sqrt{2}) + 7\sqrt{2}$
8. $4(2\sqrt{2} + \sqrt{3}) - 5(3\sqrt{2} - \sqrt{3})$

D. Expand.

Sample. $(\sqrt{5} + 2)(\sqrt{5} + 3)$

Solution. Use the short cuts you developed earlier in the unit.

$(\sqrt{5} + 2)(\sqrt{5} + 3)$

$= (\sqrt{5})^2 + (2+3)\sqrt{5} + 6$

$= 5 + 5\sqrt{5} + 6$

$= 11 + 5\sqrt{5}.$

1. $(\sqrt{3} + 4)(\sqrt{3} + 3)$
2. $(\sqrt{2} + 5)(\sqrt{2} - 7)$
3. $(\sqrt{13} - 2)(\sqrt{13} + 4)$
4. $(3\sqrt{17} - 2)(4\sqrt{17} + 5)$
5. $(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})$
6. $(\sqrt{5} - \sqrt{7})(\sqrt{5} + \sqrt{7})$
7. $(\sqrt{5} + 4)(\sqrt{5} + 4)$
8. $(\sqrt{5} + 4)^2$
9. $(\sqrt{7} + 1)^2$
10. $(\sqrt{10} - 3)^2$
11. $(2 + \sqrt{7})^2$
12. $(x + \sqrt{y})(x - \sqrt{y})$
13. $(3a - 2\sqrt{b})(4a + 5\sqrt{b})$
14. $(\sqrt{x} - y)^2$
15. $(2\sqrt{a} + 3b)^2$

DEGREES OF APPROXIMATION

In some of the applications of mathematics it is important to be able to compute an approximation to the square root of a number. For example, suppose you wanted a carpenter to make a square table top, 30 square feet in area. The carpenter would need to know the number of feet in each side of the square. You could tell him that the table top should be $\sqrt{30}$ feet on each side, but this information would be of little help to him since the carpenter's rule is not marked with numerals like '$\sqrt{30}$'. It would be more helpful for him to know that the sides should be approximately 5.5 feet long, or 5.48 feet long, or 5.477 feet long. Which approximation to $\sqrt{30}$ he would use would depend upon how accurate a job he planned to do.

So, what we want to consider now are ways of computing approximations to square roots of numbers. Before doing this, we need to discuss different degrees of approximations to positive numbers. For example, we need to agree on what is meant by statements such as:

0.33	is the approximation to	1/3	correct to 2 decimal places,
0.88	is the approximation to	7/8	correct to the nearest 0.01,
0.55	is the approximation to	5/9	correct to 2 decimal places,
0.56	is the approximation to	5/9	correct to the nearest 0.01,
2	is the approximation to	7/3	correct to the units place,
5.4	is the approximation to	$\sqrt{30}$	correct to 1 decimal place,
5.4	is the approximation to	$\sqrt{30}$	correct to the nearest 0.1,
9.258	is the approximation to	9.25875	correct to 3 decimal places,
9.259	is the approximation to	9.25875	correct to the nearest 0.001.

A careful reading of the foregoing examples may have shown you what we mean by such phrases as 'correct to the nearest 0.01' and 'correct to 3 decimal places'. But, let's be explicit.

> For each positive number x, there is just one integer y such that
>
> $$y \leq x < y + 1.$$
>
> This number y is the approximation to x <u>correct to the units place</u>.

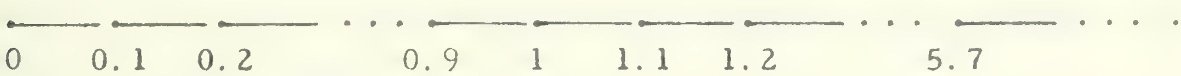
An easy way to think of this is to picture $\overrightarrow{0, 1}$ as follows.

Each positive real number belongs to just one of these half-open intervals, that is, to just one of the sets

$$\{x\colon\ 0 \le x < 1\}, \quad \{x\colon\ 1 \le x < 2\}, \quad \{x\colon\ 2 \le x < 3\}, \ \ldots\ .$$

And, the approximation to a positive number correct to the units place is the left end point of that set to which it belongs. What is the approximation correct to the units place for $12\frac{3}{4}$? For 6.01? 17.999? 3/4? π? 19?

In order to say what we mean by 'correct to 1 decimal place' [or: 'correct to the tenths place'], it is convenient to introduce the term '<u>tenth-integer</u>'. A tenth-integer is a real number whose product by 10 is an integer. For example, 3.1, $^{-}2.4$, and $^{+}6$ are tenth-integers. Now, picture $\overrightarrow{0, 1}$ as follows.



Each positive real number belongs to just one of the sets

$$\{x\colon\ 0 \le x < 0.1\}, \ \{x\colon\ 0.1 \le x < 0.2\}, \ \ldots, \ \{x\colon\ 8.9 \le x < 9\}, \ \ldots\ .$$

And, the approximation to a positive number correct to 1 decimal place is that tenth-integer which is the left end point of the set to which the positive number belongs. So, for each positive number x,

> the approximation to x <u>correct to 1 decimal place</u>
> is the tenth-integer y such that
>
> $$y \le x < y + 0.1.$$

What is the approximation correct to 1 decimal place for 1.19? For 51/4? 1.003? 1/3? 2/3? π? 8.1? 9?

What is a hundredth-integer? A thousandth-integer? A millionth-integer? Tell what is meant by 'the approximation to a positive number correct to 2 decimal places'. Now, answer the questions at the top of the next page.

(1) What is the approximation to 78.9381 correct to 3 decimal places? Correct to 2 decimal places? Correct to the units place? Correct to 4 decimal places? Correct to 5 decimal places?

(2) What is the approximation to $\frac{1}{8}$ correct to 1 decimal place? 2 decimal places? 3 decimal places? 10 decimal places?

(3) Suppose a number is between 0.2 and 0.3. What is the approximation to this number correct to 1 decimal place? Can you tell the approximation to this number correct to 2 decimal places?

(4) 6.7 is the approximation to 6.73 correct to 1 decimal place. It is also the approximation to 6.7598 correct to 1 decimal place. Describe the set of positive numbers for each of which 6.7 is the approximation correct to 1 decimal place.

(5) Describe the set of positive numbers for each of which 6.7 is the approximation correct to 2 decimal places.

(6) On page 3-109 you discovered that

$$5.4772 < \sqrt{30} < 5.4773.$$

From this tell the approximation to $\sqrt{30}$ which is correct to the units place. Correct to 1 decimal place. To 2 decimal places. To 3 decimal places. Can you tell from this the approximation to $\sqrt{30}$ correct to 4 decimal places? To 5 decimal places?

You have seen, for example, that because

$$3 \leq 3\frac{2}{3} < 4,$$

the approximation to $3\frac{2}{3}$ correct to the units place is the integer 3. But, there is another integer which is <u>closer</u> to $3\frac{2}{3}$ than 3 is. This is the integer 4. We say that 4 is the approximation to $3\frac{2}{3}$ <u>correct to the nearest unit</u>.

Let us now make precise the notions expressed by 'correct to the nearest unit', 'correct to the nearest 0.1', 'correct to the nearest 0.0001', etc.

Picture $\overrightarrow{0, 1}$ as follows.

Each positive number belongs to just one of the sets

$$\{x\colon\ 0 \le x < 0.5\},\quad \{x\colon\ 0.5 \le x < 1.5\},\quad \{x\colon\ 1.5 \le x < 2.5\},\ \ldots\ .$$

The approximation to a positive number correct to the nearest unit is the integer which belongs to that set to which the positive number belongs. [In fact, except for numbers in $\overline{0,\ 0.5}$, the approximation correct to the nearest unit is the midpoint of the set.] So, for each positive number x,

> the approximation to x <u>correct to the nearest unit</u>
> is the integer y such that

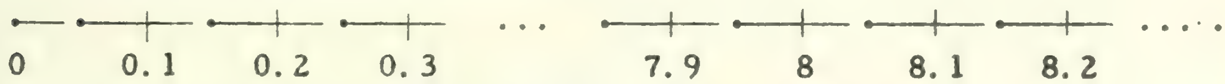
$$y - 0.5 \le x < y + 0.5.$$

Thus, for example, since

$$4 - 0.5 \le 3\frac{2}{3} < 4 + 0.5,$$

we say that 4 is the approximation to $3\frac{2}{3}$ correct to the nearest unit. What is the approximation correct to the nearest unit for 6.3? For 2.73? 5.5? 17? 1/7? 4/7? π?

Now, picture $\overrightarrow{0, 1}$ as follows.



Each positive number belongs to just one of the sets

$$\{x\colon\ 0 \le x < 0.05\},\ \{x\colon\ 0.05 \le x < 0.15\},\ \ldots,\ \{x\colon\ 12.95 \le x < 13.05\},\ \ldots\ .$$

For each positive number x, the approximation to x <u>correct to the nearest 0.1</u> is the tenth-integer y such that

$$y - 0.05 \le x < y + 0.05.$$

We know that $3\frac{2}{3}$ is between 3.66 and 3.67. From this it follows that

$$3.7 - 0.05 \le 3\frac{2}{3} < 3.7 + 0.05.$$

So, 3.7 is the approximation to $3\frac{2}{3}$ correct to the nearest 0.1. What is the approximation correct to the nearest 0.1 for 2.81? For 2.87? 2.85? 2.97? π? 17? 1/7? 6/7? 8.6543?

Tell what is meant by 'the approximation to a positive number correct to the nearest 0.01'. By 'the approximation to a positive number correct to the nearest 0.001'.

(1) What is the approximation to 78.9381 correct to the nearest 0.001? Correct to the nearest 0.01? Correct to the nearest unit? Correct to the nearest 0.0001?

(2) What is the approximation to $\frac{1}{8}$ correct to the nearest 0.1? To the nearest 0.01? To the nearest 0.001? To the nearest 0.0001? To the nearest 0.000001?

(3) Suppose a number is between 0.15 and 0.25. What is the approximation to this number correct to the nearest 0.1? Can you tell the approximation to this number correct to the nearest 0.01? Correct to 1 decimal place?

(4) 6.7 is the approximation to 6.73 correct to the nearest 0.1. It is also the approximation to 6.6839 correct to the nearest 0.1. Describe the set of positive numbers for each of which 6.7 is the approximation correct to the nearest 0.1. Describe the set of positive numbers for each of which 6.7 is <u>both</u> the approximation correct to the nearest 0.1 and the approximation correct to 1 decimal place.

(5) Describe the set of positive integers for each of which 6.7 is the approximation correct to the nearest 0.01.

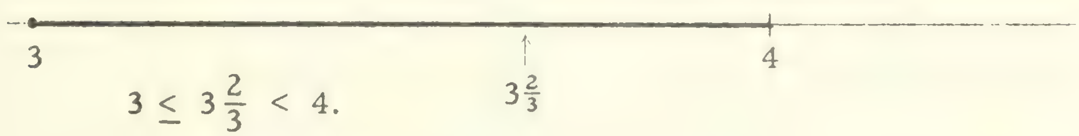
(6) On page 3-109 you discovered that

$$5.4772 < \sqrt{30} < 5.4773.$$

From this tell the approximation to $\sqrt{30}$ which is correct to the nearest unit. Correct to the nearest 0.1. To the nearest 0.01. To the nearest 0.001. Can you tell from this the approximation to $\sqrt{30}$ correct to the nearest 0.0001?

Let us summarize this discussion of degrees of approximation by considering various approximations to $3\frac{2}{3}$.

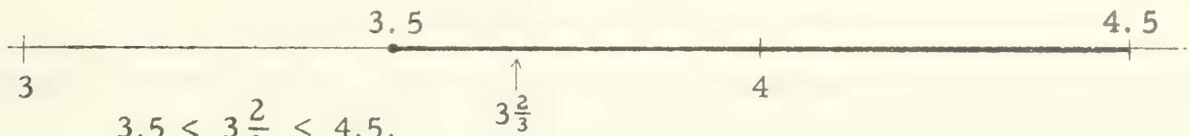
(a)



$3 \leq 3\frac{2}{3} < 4.$

So, 3 is the approximation to $3\frac{2}{3}$ correct to the units place.

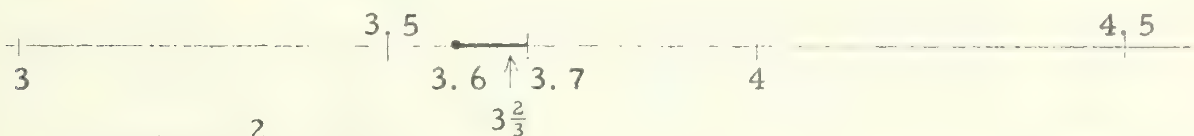
(b)



$3.5 \leq 3\frac{2}{3} < 4.5.$

So, 4 is the approximation to $3\frac{2}{3}$ correct to the nearest unit.

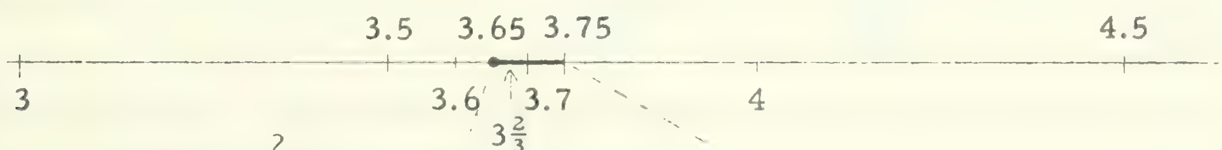
(c)



$3.6 \leq 3\frac{2}{3} < 3.7.$

So, 3.6 is the approximation to $3\frac{2}{3}$ correct to 1 decimal place.

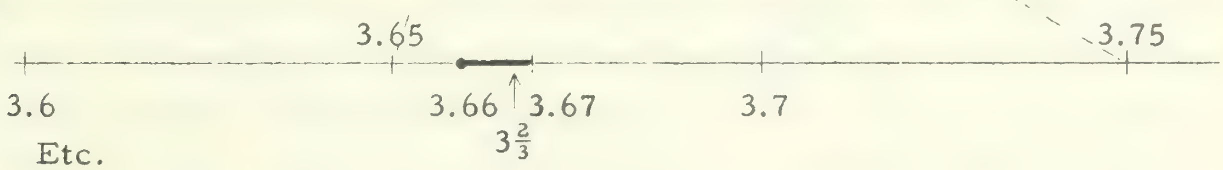
(d)



$3.65 \leq 3\frac{2}{3} < 3.75.$

So, 3.7 is the approximation to $3\frac{2}{3}$ correct to the nearest 0.1.

(e)



Etc.

[Approximations to negative real numbers are found by obtaining approximations to their opposites, and taking <u>their</u> opposites. For example, the approximation to $-3\frac{2}{3}$ correct to the nearest 0.1 is -3.7. Also, the approximation to $-3\frac{2}{3}$ correct to 1 decimal place is -3.6.]

ESTIMATES OF ERRORS

When you know that a number y is the approximation to a number x correct, say, to the nearest 0.01, you know that

$$y - \frac{0.01}{2} \leq x < y + \frac{0.01}{2} .$$

That is, you know that

$$-0.005 \leq x - y < 0.005.$$

So, for each x for which y is the approximation correct to the nearest 0.01,

$$|x - y| \leq 0.005.$$

We say that the approximation y <u>is in error by at most</u> 0.005. [If we know that x is not a hundredth-integer then the approximation to x which is correct to the nearest 0.01 <u>is in error by less than</u> 0.005.]

In general, the approximation to a positive number which is correct to the nearest unit, or 0.1, or 0.01, or 0.001, etc., is in error by at most $\frac{1}{2}$, or $\frac{0.1}{2}$, or $\frac{0.01}{2}$, or $\frac{0.001}{2}$, etc.

When you know that a number y is the approximation to a number x correct, say, to 2 decimal places, you know that $y \leq x < y + 0.01$. That is, you know that $0 \leq x - y < 0.01$. So, all you know about the <u>size</u> of the error is that $|x - y| < 0.01$. [Of course, you also know that an approximation which is correct to a certain number of decimal places is never greater than the number for which it is an approximation.]

In general, the approximation to a positive number which is correct to the units place, or to 1 decimal place, or to 2 decimal places, or to 3 decimal places, etc., is in error by less than 1, or 0.1, or 0.01, or 0.001, etc.

When you know <u>either</u> that a number is an approximation correct, say, to 2 decimal places <u>or</u> that it is an approximation correct to the nearest 0.01, in either case you know that the number being approximated is in a half-open interval of length 0.01. In the second case you know that the error is at most 0.005 while in the first case your only knowledge of the size of the error is that it is less than 0.01. But, this is compensated for by additional information about the direction of the error.

EXERCISES

A. Here are several statements about $\sqrt{112}$.

(a) $10.5 < \sqrt{112} < 10.6$

(b) $10.58 < \sqrt{112} < 10.59$

(c) $10.583 < \sqrt{112} < 10.584$

(d) $10.583 < \sqrt{112} < 10.5831$

(e) $10.583 < \sqrt{112} < 10.58301$

(f) $10.583005 < \sqrt{112} < 10.583006$

1. Use one number line picture [page 3-117] to show (a) and (b).

2. Use one number line picture to illustrate both (e) and (f).

3. What is the approximation to $\sqrt{112}$ correct to 2 decimal places?

4. What is [the approximation to] $\sqrt{112}$ correct to the nearest unit?

5. What is $\sqrt{112}$ correct to the nearest 0.001?

6. What is $\sqrt{112}$ correct to the nearest 0.0001?

7. $\sqrt{112}$ = _______, correct to the nearest hundredth?

8. $\sqrt{112}$ = _______, correct to 6 decimal places?

9. $\sqrt{112}$ = _______, correct to the nearest 0.000001?

10. True or false?

(a) $|\sqrt{112} - 10.583| < 0.001$ (b) $|\sqrt{112} - 10.583| < 0.0005$

(c) $|\sqrt{112} - 10.584| < 0.001$ (d) $|\sqrt{112} - 10.584| < 0.0005$

11. Which of the six statements can be used to derive all the rest?

B. The approximation to $\sqrt{39}$ correct to 7 decimal places is 6.2449979. That is, $\sqrt{39} = 6.2449979$, correct to 7 decimal places.

1. What is the approximation to $\sqrt{39}$ correct to one decimal place? Two decimal places? Three decimal places? Four decimal places? Five decimal places?

2. What is $\sqrt{39}$, correct to the nearest 0.1? 0.01? 0.001? 0.0001? Hundred thousandth? 0.000001?

3. Give one number which differs from $\sqrt{39}$ by less than 0.0000001. Give another number which differs from $\sqrt{39}$ by less than 0.0000001. Give still another number.

C. 1. Given only the information that

$$12.124 < \sqrt{147} < 12.131,$$

which of the following can you be sure of?

(a) $\sqrt{147} = 12$, correct to the nearest unit.

(b) $\sqrt{147} = 12.1$, correct to 1 decimal place.

(c) $\sqrt{147} = 12.1$, correct to the nearest 0.1.

(d) $\sqrt{147} = 12.12$, correct to 2 decimal places.

(e) $\sqrt{147} = 12.12$, correct to the nearest 0.01.

(f) $|\sqrt{147} - 12.1| < 0.1$ (g) $|\sqrt{147} - 12.1| < 0.05$

(h) $|\sqrt{147} - 12.1| < 0.01$ (i) $|\sqrt{147} - 12.1| < 0.005$

2. Given that $9.4 < \sqrt{89} < 9.47$. Use this information to justify as many statements like those in the preceding exercise as you can.

3. Repeat Exercise 2 using the facts that $13.227 < \sqrt{175} < 13.23$.

4. Repeat Exercise 2 using the facts that $9.4339 < \sqrt{89} < 9.4341$.

5. Each of the following sentences gives you information concerning the square root of some number. In each case <u>where you have sufficient information</u> give the approximation to the square root correct to 3 decimal places and the approximation correct to the nearest 0.01.

(a) $3.741 < \sqrt{14} < 3.742$.

(b) $6.928202 \leq \sqrt{48} < 6.928206$

(c) $15.554 < \sqrt{242} < 15.565$

(d) $|\sqrt{26} - 5.099| < 0.0005$

(e) $17.143 < \sqrt{274} < 17.15$

(f) $|\sqrt{37} - 6.082763| < 0.0000005$

(g) $1.7724 < \sqrt{\pi} < 1.7725$

(h) $11.180 < \sqrt{125} < 11.185$

COMPUTING APPROXIMATIONS TO SQUARE ROOTS

One method of computing approximations to the square root of a number was illustrated earlier when we were searching for the positive number whose square is 30. By doing a great deal of squaring we found that

$$5 < \sqrt{30} < 6,$$
$$5.4 < \sqrt{30} < 5.5,$$
$$5.47 < \sqrt{30} < 5.48,$$
$$5.477 < \sqrt{30} < 5.478.$$

Clearly, we could continue the squaring procedure and "close in" on $\sqrt{30}$ as closely as we might want to.

There is another method of closing in which requires less work. Let's illustrate it. Suppose we want to find the approximation to $\sqrt{89}$ which is correct to the nearest hundredth. The first thing we do is to make a guess. We guess, say, that $\sqrt{89}$ is approximately 9. Then, we divide 89 by 9, and get approximately 9.9. If 9 <u>were</u> the square root of 89, the quotient would be 9 [Why?]. Since the quotient is larger than 9, we know that $\sqrt{89}$ is between 9 and $89 \div 9$. Since $89 \div 9 < 9.9$, $\sqrt{89}$ is between 9 and 9.9. So, let's use a number between 9 and 9.9 as our next guess. One of the numbers between 9 and 9.9 is their average which is 9.4, correct to 1 decimal place.

Now, let's divide 89 by 9.4.

$$\begin{array}{r}
9.46 \\
9.4\overline{)\,89.000} \\
\underline{84\,6} \\
4\,40 \\
\underline{3\,76} \\
640 \\
\underline{564}
\end{array}$$

The quotient of 89 by 9.4 is 9.46, correct to 2 decimal places. So, $9.4 < \sqrt{89} < 9.47$. [Do you see that this last statement tells us that $\sqrt{89}$ is 9.4, correct to 1 decimal place?] Take the average of 9.4 and 9.47.

$$\frac{9.4 + 9.47}{2} = \frac{18.87}{2} = 9.43, \text{ correct to 2 decimal places.}$$

Use 9.43 as the next guess.

Divide 89 by 9.43

```
         9.43
9.43 ) 89.0000
       8487
        4130
        3772
         3580
         2829
```

We are now sure that $9.43 < \sqrt{89} < 9.44$. So, 9.43 is the approximation to $\sqrt{89}$ correct to 2 decimal places. But, is it the approximation correct to the nearest 0.01? Might this approximation be 9.44? To decide, just square 9.435. $(9.435)^2 = 89.019225 > 89$. So, 9.43 is the approximation to $\sqrt{89}$ correct to the nearest 0.01.

Let's try another example. Find the approximation to $\sqrt{175}$ which is correct to the nearest 0.001. First, we make a guess [it can be a wild one]. Try 10. Now divide 175 by 10 to get 17.5, and average 17.5 with 10 to get approximately 13.7. 13.7 is the next guess. Divide 175 by 13.7.

```
          12.7
13.7 ) 175.00
       137
        380
        274
        1060
         959
```

Since $12.7 < \sqrt{175} < 13.7$, the average of 12.7 and 13.7 should be close to $\sqrt{175}$. This average is 13.2. Divide again.

```
          13.25
13.2 ) 175.000
       132
        430
        396
         340
         264
          760
          660
```

So, $\sqrt{175}$ is between 13.2 and 13.26. [What is the approximation to $\sqrt{175}$ correct to 1 decimal place?] Our next approximation is the average of 13.2 and 13.26, that is 13.23.

$$
\begin{array}{r}
13.227 \\
13.23\overline{)175.00000} \\
\underline{1323} \\
4270 \\
\underline{3969} \\
3010 \\
\underline{2646} \\
3640 \\
\underline{2646} \\
9940 \\
\underline{9261}
\end{array}
$$

This tells us that $13.227 < \sqrt{175} < 13.23$. So, $\sqrt{175}$ is 13.22, correct to 2 decimal places, and $\sqrt{175}$ is 13.23, correct to the nearest 0.01. Also, the approximation correct to the nearest 0.001 is either 13.227, 13.228, 13.229, or 13.23. We must average and divide again to get more information. $\dfrac{13.227 + 13.23}{2} = 13.228$, correct to 3 decimal places.

$$
\begin{array}{r}
13.2295 \\
13.228\overline{)175.0000000}
\end{array}
$$

Now, we know that $13.228 < \sqrt{175} < 13.2296$. We still can't tell whether 13.228, 13.229, or 13.23 is the approximation to $\sqrt{175}$ correct to the nearest 0.001. We could find out by squaring each and determining which square is closest to 175. Or, we can make our next guess by averaging 13.228 and 13.2296, and then dividing. Finish the problem.

EXERCISES

For each exercise, find the approximation correct to 3 decimal places, and the approximation correct to the nearest 0.01.

1. $\sqrt{59}$	2. $-\sqrt{29}$	3. $\sqrt{2}$	4. $\sqrt{8.73}$
5. $\sqrt{205}$	6. $\sqrt{3}$	7. $\sqrt{5839}$	8. $\sqrt{0.68}$
9. $\sqrt{171}$	10. $\sqrt{1.71}$	11. $\sqrt{17100}$	12. $\sqrt{1710000}$
13. $\sqrt{1710}$	14. $\sqrt{17.10}$	15. $\sqrt{0.1710}$	16. $\sqrt{171000}$

[More exercises are in Part O, Supplementary Exercises.]

☆MORE ON DIVIDING-AND-AVERAGING

The dividing-and-averaging procedure can be used with just a slight modification as a very efficient method for finding extremely accurate approximations to square roots.

The first step consists in finding an approximation whose error is less than 0.1. After this, one finds successively better approximations by dividing-and-averaging, carrying out the divisions so as to obtain twice as many decimal places in the quotient numeral as there are in the divisor numeral, and carrying out the averagings so as to obtain the approximation to the average which is correct to this doubled number of decimal places. Let's try this in finding approximations to $\sqrt{89}$.

First, we need an approximation to $\sqrt{89}$ which is in error by less than 0.1. We have already found that 9.4 is the approximation to $\sqrt{89}$ correct to 1 decimal place. [We knew this because $89 \div 9.4 = 9.4$, correct to 1 decimal place.] So, $|\sqrt{89} - 9.4| < 0.1$. Now, we continue the division of 89 by 9.4, carrying out the division to obtain two decimal places in the quotient numeral.

$$9.4\overline{)\,89.000}\quad\text{quotient } 9.46$$

Next, we average 9.4 and 9.46, keeping two decimal places in the numeral for the average.

$$\frac{9.2 + 9.46}{2} = 9.43, \text{ correct to 2 decimal places.}$$

This completes the second step, giving us 9.43 as the second approximation to $\sqrt{89}$.

Now, we divide-and-average again.

$$9.43\overline{)\,89.000000}\quad\text{quotient } 9.4379$$

$$\frac{9.43 + 9.4379}{2} = 9.4339, \text{ correct to 4 decimal places.}$$

So, our third approximation to $\sqrt{89}$ is 9.4339.

Divide-and-average again.

$$9.4339\overline{)\,89.000000000000}\quad\text{quotient } 9.43406226$$

$$\frac{9.4339 + 9.43406226}{2} = 9.43398113, \text{ correct to 8 places.}$$

Our fourth approximation to $\sqrt{89}$ is 9.43398113.

Once more.

$$\begin{array}{r} 9.43398113211320 76 \\ 9.43398113\overline{)89.0000000000000000000000000} \end{array}$$

The average is 9.4339811310566038, correct to 16 places.

Here is a list of the successive approximations to $\sqrt{89}$.

9.4

9.43

9.4339

9.43398113

9.4339811310566038

We already know that

$$|\sqrt{89} - 9.4| < 0.1. \qquad \text{[1 decimal place]}$$

The second division [89 ÷ 9.43] tells us that

$$9.43 < \sqrt{89} < 9.438.$$

So, $$|\sqrt{89} - 9.43| < 0.01. \qquad \text{[2 decimal places]}$$

The third division [89 ÷ 9.4339] tells us that

$$9.4339 < \sqrt{89} < 9.4341.$$

So, $$|\sqrt{89} - 9.4339| < 0.0002.$$

But, the fourth division [89 ÷ 9.43398113] tells us that

$$\sqrt{89} < 9.43399.$$

So, $$|\sqrt{89} - 9.4339| < 0.0001. \qquad \text{[4 decimal places]}$$

The fourth division also tells us that

$$9.43398113 < \sqrt{89} < 9.433981133.$$

So, $$|\sqrt{89} - 9.43398113| < 0.00000001. \qquad \text{[8 decimal places]}$$

Do you see that as far as we have checked them each approximation is "at least twice as accurate" as the preceding one? As a matter of fact, it can be proved that this always happens when your first approximation is in error by less than 0.1 and you "double the number of places" for the quotient numeral in each division. This being so, we can be certain that

$|\sqrt{89} - 9.4339811310566038| < 0.0000000000000001$ [16 decimal places]

Let's return to our third approximation, 9.4339, and the error estimate:

$$|\sqrt{89} - 9.4339| < 0.0001.$$

This estimate tells us that

$$9.4338 < \sqrt{89} < 9.434.$$

So, from it we can conclude that 9.434 is the approximation to $\sqrt{89}$ correct to the nearest 0.001. [From the division $89 \div 9.4339$ we see that $9.4339 < \sqrt{89}$. And, this tells us that 9.4339 is the approximation to $\sqrt{89}$ correct to 4 decimal places.]

The error estimate for the fourth approximation, 9.43398113:

$$|\sqrt{89} - 9.43398113| < 0.00000001$$

tells us that

$$9.43398112 < \sqrt{89} < 9.43398114.$$

So, from it we can conclude that 9.4339811 is the approximation to $\sqrt{89}$ correct to the nearest 0.0000001. [This also tells us that either 9.43398112 or 9.43398113 is the approximation to $\sqrt{89}$ correct to 8 decimal places. How can you tell which?]

What is the approximation to $\sqrt{89}$ correct to the nearest 0.000000000000001? What can you say about the approximation to $\sqrt{89}$ which is correct to 16 decimal places? How could you find out what this approximation is?

Use this procedure to find the approximation to $\sqrt{175}$ correct to the nearest 0.0000001, and the approximation to $\sqrt{175}$ correct to 8 decimal places.

EXERCISES

A. Find the approximations correct to the nearest 0.01.

Sample 1. $\sqrt{48}$

Solution. One way to do this is to use the divide-and-average method. Another method is to notice that $48 = 16 \times 3$ and that 16 is the square of an integer. Thus,

$$\sqrt{48} = \sqrt{16 \times 3} = \sqrt{16} \times \sqrt{3} = 4\sqrt{3}.$$

So, we can find approximations to $\sqrt{48}$ by multiplying 4 by approximations to $\sqrt{3}$. [But, the error in such an approximation to $\sqrt{48}$ will be 4 times the error in the approximation used for $\sqrt{3}$.] Now, suppose you know that the approximation to $\sqrt{3}$ correct to the nearest 0.000001 is 1.732051, that is,

$$1.7320505 \leq \sqrt{3} < 1.7320515.$$

From this we could get an estimate for $\sqrt{48}$:

$$4 \times 1.7320505 \leq 4 \times \sqrt{3} < 4 \times 1.7320515,$$

that is,

$$6.928202 \leq \sqrt{48} < 6.928206.$$

This tells us that 6.93 is the approximation to $\sqrt{48}$ correct to the nearest 0.01. But, we didn't need to use so accurate an approximation to $\sqrt{3}$ to find this out. Doing so caused us more computational labor than was necessary. [Of course, if you don't mind doing a bit of extra computing, this is a sure way of getting the answer. All you need do is multiply 1.732051 by 4, and round to the nearest 0.01.]

Another procedure is to notice that

$$1.732 < \sqrt{3} < 1.733.$$

So, multiplying by 4, gives us:

$$6.928 < \sqrt{48} < 6.932.$$

So, 6.93 is the approximation to $\sqrt{48}$ correct to the nearest 0.01. [Would noticing that $1.73 < \sqrt{3} < 1.74$ have helped?]

(continued on next page)

Suppose we wanted to find $\sqrt{48}$ correct to the nearest 0.001. We would need to use at least the information that

$$1.7320 < \sqrt{3} < 1.7321.$$

Multiplying by 4, we find that:

$$6.9280 < \sqrt{48} < 6.9284.$$

So, 6.928 is the approximation to $\sqrt{48}$ correct to the nearest 0.001, [and it is also the approximation to $\sqrt{48}$ correct to 3 decimal places].

<u>Sample 2</u>. Find $\sqrt{242}$, correct to the nearest 0.01.

<u>Solution</u>. Since $242 = 121 \times 2$ and $121 = (11)^2$, it follows that $\sqrt{242} = 11\sqrt{2}$. Hence, we can find an approximation to $\sqrt{242}$ if we know an approximation to $\sqrt{2}$. [But, the error in such an approximation to $\sqrt{242}$ will be 11 times the error in the approximation to $\sqrt{2}$.] Suppose we know that

$\sqrt{2} = 1.41421356$, correct to the nearest 0.00000001.

This tells us, for example, that

$$1.414 < \sqrt{2} < 1.415.$$

So, multiplying by 11, we learn that

$$15.554 < \sqrt{242} < 15.565.$$

This tells only that either 15.55 or 15.56 is the approximation to $\sqrt{242}$ correct to the nearest 0.01. We must use a more accurate estimate for $\sqrt{2}$.

$$1.4142 < \sqrt{2} < 1.4143$$

Multiplying by 11, we get:

$$15.5562 < \sqrt{242} < 15.5573.$$

And, from this we see that 15.56 is the approximation to $\sqrt{242}$ correct to the nearest 0.01. [Also, we see that 15.55 is the approximation to $\sqrt{242}$ correct to 2 decimal places.]

*

[You will find it helpful to memorize the facts that

1.4142 is the approximation to $\sqrt{2}$ correct to the nearest 0.0001,
1.7321 is the approximation to $\sqrt{3}$ correct to the nearest 0.0001,
2.2361 is the approximation to $\sqrt{5}$ correct to the nearest 0.0001.]

*

Find approximations correct to the nearest 0.01.

1. $\sqrt{32}$ 2. $\sqrt{12}$ 3. $\sqrt{80}$ 4. $\sqrt{8}$ 5. $\sqrt{20}$ 6. $\sqrt{72}$

<u>Sample 3</u>. $\sqrt{12} + \sqrt{75}$

<u>Solution</u>. $\sqrt{12} + \sqrt{75} = 2\sqrt{3} + 5\sqrt{3} = 7\sqrt{3}$

Now, compute the approximation to $\sqrt{12} + \sqrt{75}$ correct to the nearest 0.01 by using an appropriate approximation to $\sqrt{3}$.

7. $\sqrt{8} + \sqrt{50}$ 8. $\sqrt{3} + \sqrt{27}$ 9. $3\sqrt{50} + \sqrt{72}$

<u>Sample 4</u>. $\sqrt{6} \times \sqrt{12}$

<u>Solution</u>. $\sqrt{6} \times \sqrt{12} = \sqrt{6} \times (\sqrt{6} \times \sqrt{2}) = 6\sqrt{2}$. [Complete the problem.]

10. $\sqrt{7} \times \sqrt{21}$ 11. $\sqrt{5} \times \sqrt{30}$ 12. $\sqrt{80} \div \sqrt{10}$

<u>B</u>. Simplify each of the following expressions.

<u>Sample</u>. $\sqrt{75}$

<u>Solution</u>. $\sqrt{75} = \sqrt{25 \times 3} = \sqrt{25} \times \sqrt{3} = 5\sqrt{3}$.

The expression '$5\sqrt{3}$' is considered simpler than '$\sqrt{75}$' because if one wanted to find an approximation to $\sqrt{75}$ and knew an approximation to $\sqrt{3}$, he could find the approximation to $\sqrt{75}$ by easy multiplication.

1. $\sqrt{50}$ 2. $\sqrt{98}$ 3. $\sqrt{147}$
4. $\sqrt{200}$ 5. $\sqrt{300}$ 6. $\sqrt{.03}$
7. $\sqrt{.02}$ 8. $\sqrt{.75}$ 9. $\sqrt{125}$
10. $\sqrt{75} + \sqrt{48}$ 11. $\sqrt{45} - 2\sqrt{5} + 3\sqrt{20}$

(continued on next page)

12. $\sqrt{7} \times \sqrt{14}$

13. $7\sqrt{3} \times 2\sqrt{12} \times 3\sqrt{9}$

14. $2\sqrt{48} - \sqrt{3}$

15. $\frac{1}{2}\sqrt{32} + \frac{1}{3}\sqrt{18} - \frac{1}{5}\sqrt{50}$

16. $\sqrt{200} \times \sqrt{\frac{1}{2}}$

17. $\sqrt{242} + \sqrt{162} - \sqrt{72}$

18. $\sqrt{57} \div \sqrt{19}$

19. $5\sqrt{18} \times 6\sqrt{8} \times 3\sqrt{25}$

20. $(\sqrt{50})^2 \times (3\sqrt{7})^2$

21. $\sqrt{20} + 2\sqrt{75} + \sqrt{45} - \sqrt{300}$

[More exercises are in Part P, Supplementary Exercises.]

C. Solve these equations. Give the roots, and also give approximations which are correct to the nearest 0.01.

Sample. $2x^2 - 5 = 0$

Solution.
$$2x^2 - 5 = 0$$
$$4x^2 - 10 = 0$$
$$(2x - \sqrt{10})(2x + \sqrt{10}) = 0$$
$$2x - \sqrt{10} = 0 \text{ or } 2x + \sqrt{10} = 0$$
$$2x = \sqrt{10} \text{ or } 2x = -\sqrt{10}$$
$$x = \frac{1}{2}\sqrt{10} \text{ or } x = -\frac{1}{2}\sqrt{10}$$

The "exact" roots are $\frac{1}{2}\sqrt{10}$ and $-\frac{1}{2}\sqrt{10}$.

Approximations to the roots are obtained by using an approximation to $\sqrt{10}$. By the dividing-and-averaging process we find that

$$3.16 < \sqrt{10} < 3.17.$$

So, $1.58 < \frac{1}{2}\sqrt{10} < 1.585$, and 1.58 is the approximation to $\sqrt{10}$ which is correct to the nearest 0.01 [as well as to 2 decimal places]. Hence, 1.58 and −1.58 are the approximations to the roots correct to the nearest 0.01.

Roots: $\frac{1}{2}\sqrt{10}$ and $-\frac{1}{2}\sqrt{10}$

Approximations: 1.58 and −1.58

1. $x^2 - 48 = 0$
2. $y^2 = 75$
3. $x^2 - 98 = 0$
4. $100y^2 = 8$
5. $25x^2 - 2 = 0$
6. $50x^2 - 81 = 0$
7. $64 = z^2$
8. $12 - k^2 = 0$
9. $x^2 - \frac{3}{2} = 0$
10. $5x - \sqrt{3} = 0$
11. $x^2 - 5.83 = 0$
12. $0.0082 = y^2$

PRINCIPAL SQUARE ROOT

What is the square root of 25? What is the negative square root of 25?

What is the square root of 73? What is the negative square root of 73?

Does the real number 0 have square roots? Does it have a positive square root? Does it have a negative square root?

Does the negative number, -4, have square roots? Does it have a positive square root? Does it have a negative square root?

Correct answers to these questions suggest the following generalizations.

(1) For each number $x > 0$,
there is just one positive number, $\sqrt{x}$, whose square is x, and there is just one negative number, $-\sqrt{x}$, whose square is x.

(2) 0 has just one square root, 0.

(3) Negative real numbers do not have real number square roots.

The nonnegative square root of a nonnegative real number is called <u>the principal square root</u>, or, simply, <u>the square root</u>.

What is the square of 3? What is the square root of the square of 3? What is the negative square root of the square of 3?

What is the square of -3? What is <u>the</u> square root of the square of -3? What is the negative square root of the square of -3?

Study each of the following true sentences:

(a) $\sqrt{(-3)^2} = \sqrt{(+3)^2} = +3$

(b) $\sqrt{(+5)^2} = \sqrt{(-5)^2} = +5$

(c) $-\sqrt{(-5)^2} = -\sqrt{(+5)^2} = -5$

(d) $\sqrt{(8 - 2)^2} = 8 - 2$

(e) $\sqrt{(2 - 8)^2} = 8 - 2$

(f) $\sqrt{(2 - 8)^2} = -(2 - 8)$

The examples at the bottom of the preceding page suggest the following generalization:

For each x,

$$\sqrt{x^2} = x \text{ if } x \geq 0, \text{ and } \sqrt{x^2} = -x \text{ if } x < 0.$$

Recalling our agreement to the effect that '$|x|$' is ambiguous and can be used as an abbreviation for '$^+|x|$', we can say that

$$\text{for each } x,\ \sqrt{x^2} = |x|.$$

Instances of this generalization are:

$\sqrt{7^2} = |7| = 7,$ $\sqrt{(-3)^2} = |-3| = 3,$

$\sqrt{(3-9)^2} = |3-9| = 6,$ $\sqrt{0^2} = |0| = 0.$

EXERCISES

A. Simplify.

Sample 1. $\sqrt{x^2 - 6x + 9}$

Solution. Since, for each x,

$$x^2 - 6x + 9 = (x-3)^2,$$

it follows that, for each x,

$$\sqrt{x^2 - 6x + 9} = \sqrt{(x-3)^2}.$$

And, by the generalization stated above,

$$\text{for each } x,\ \sqrt{(x-3)^2} = |x-3|.$$

Sample 2. $\sqrt{4y^2}$

Solution. $\sqrt{4y^2} = \sqrt{(2y)^2} = |2y|$

1. $\sqrt{16x^2}$
2. $\sqrt{64z^2}$
3. $-\sqrt{81a^2}$
4. $-\sqrt{25b^2}$
5. $\sqrt{36a^2b^2}$
6. $-\sqrt{49x^2y^2}$
7. $-\sqrt{100(x+3)^2}$
8. $\sqrt{k^2 + 10k + 25}$
9. $\sqrt{a^2 - 4a + 4}$
10. $\sqrt{x^2 - 18x + 81}$
11. $\sqrt{81 - 18x + x^2}$
12. $\sqrt{a^2 - 2ab + b^2}$
13. $\sqrt{x^2 + 2xy + y^2}$

14. $\sqrt{9x^2 - 12x + 4}$

15. $\sqrt{49x^2 - 70xy + 25y^2}$

16. $\sqrt{4p^2 - 12pq + 9q^2}$

17. $\sqrt{(x + 3)^2(x - 5)^2}$

18. $\sqrt{10x^2(x^2 + 8x + 16)}$

19. $\sqrt{x^2 + y^2}$

[More exercises are in Part Q, Supplementary Exercises.]

B. Solve these equations and inequations.

Sample. $x = 9 - \frac{8}{x}$

Solution.

$$x = 9 - \frac{8}{x}$$

$$x^2 = x(9 - \frac{8}{x}), \quad [x \neq 0]$$

$$x^2 = 9x - 8$$

$$x^2 - 9x + 8 = 0$$

$$(x - 1)(x - 8) = 0$$

$$x - 1 = 0 \quad \text{or} \quad x - 8 = 0$$

$$x = 1 \quad \text{or} \quad x = 8$$

The roots are 1 and 8.

Check.

$1 = 9 - \frac{8}{1}$?	$8 = 9 - \frac{8}{8}$?
$1 \mid 9 - 8$	$8 \mid 9 - 1$
$1 = 1$ ✓	$8 = 8$ ✓

1. $\frac{21}{x} - 4 = x$

2. $5 + \frac{25}{y} = 6y$

3. $1 + \frac{11}{x} + \frac{18}{x^2} = 0$

4. $\frac{6}{y^2} = \frac{25}{3y} + 1$

5. $x + \frac{x + 3}{x - 9} = \frac{12}{x - 9} - 4$

6. $1 + \frac{3}{y + 2} = \frac{y + 4}{y + 2} - y$

7. $(y + 3)^2 < 6(y + 15)$

8. $x(x + 4) > 3 + 3(9 + x)$

9. $\frac{2x + 3}{3x - 3} = \frac{x + 9}{2x - 2}$

10. $\frac{5y + 2}{3y + 4} = \frac{6y - 3}{8y - 5}$

[More exercises are in Part R, Supplementary Exercises.]

C. Solve these problems.

Sample. A salesman travels 224 miles at a rate which is 4 miles per hour faster than his usual rate. This saves him 1 hour. How many hours does he usually take to travel this distance?

Solution. Suppose that he usually takes x hours to travel this distance. Then, his usual rate is $\frac{224}{x}$ miles per hour. His faster rate is $(\frac{224}{x} + 4)$ miles per hour. So, his faster time is $\frac{224}{\frac{224}{x} + 4}$ hours.

Hence, we are looking for a number x such that

$$(*) \qquad \frac{224}{\frac{224}{x} + 4} = x - 1.$$

We solve equation (*).

$$\frac{224}{\frac{224 + 4x}{x}} = x - 1, \quad [x \neq 0,\ 224 + 4x \neq 0]$$

$$\frac{224x}{224 + 4x} = x - 1$$

$$(224 + 4x)\frac{224x}{224 + 4x} = (x - 1)(224 + 4x)$$

$$224x = (x - 1)(4x + 224)$$

$$224x = 4x^2 + 220x - 224$$

$$4x^2 - 4x - 224 = 0$$

$$\frac{1}{4}(4x^2 - 4x - 224) = 0 \cdot \frac{1}{4}$$

$$x^2 - x - 56 = 0$$

$$(x - 8)(x + 7) = 0$$

$$x = +8 \quad \text{or} \quad x = -7$$

Since we are looking for a number of arithmetic, we know that 8 is the only number of arithmetic which satisfies (*).

So, the salesman's usual time of travel is 8 hours.

Check. The usual rate is $\frac{224}{8}$ miles per hour, that is, 28 miles per hour. The faster rate is 32 miles per hour, and the time required at this faster rate is $\frac{224}{32}$ hours, that is, 7 hours, which is 1 hour less than the usual time.

1. A group of boys decided to build a clubhouse, and share the cost equally. They estimated that the total cost would be \$120. Since this made their shares more than they could afford, they invited two other boys to join them. This reduced the cost per boy by \$2. How many boys were there in the original group?

2. Robert can mow a lawn in 2 hours less time than Raymond, and together they take 2.4 hours. How long does it take each one separately?

3. On a fishing trip a man can go 9 miles upstream and return in a total of 4 hours. If the current flows at the rate of 3 miles per hour, what is the speed of the boat in still water?

4. Bill bought several second-hand bicycles for \$180 during his summer vacation and tried to sell them to his friends, all at the same price. He sold all but two of them, and collected a total of \$250. How many bicycles did he buy?

5. A tank has an inlet pipe and an outlet pipe. The inlet pipe can fill the tank in 1 hour less time than the outlet pipe can empty it. The inlet pipe is turned on to fill the tank. After 2 hours, someone discovers that the outlet pipe is open. He turns off the outlet pipe. If only 10% of the tank has been filled, how long does it take the inlet pipe to finish filling the tank?

(continued on next page)

6. One side of a rectangle is 4 inches longer than another side. If the smaller side were doubled [and the longer side not changed], you would get a rectangle whose area was 2.5 square inches more than 1.5 times the original area. Find the dimensions of the original rectangle.

7. Al is 5 years older than Bud. Twice the product of their ages 5 years ago is 100 more than the product of their present ages. Find the present ages of Al and Bud.

8. Find the number which when divided by 1 more than twice itself gives a result which is $\frac{3}{2}$ more than $\frac{1}{2}$ the original number.

9. A colony of ants come upon 10 ounces of bread crumbs scattered around after a picnic. The ant leader decides that if there were 500 more ants, each would have to carry $\frac{1}{100}$ of an ounce less. How many ants are there in the colony?

D. Solve for 'x'.

Sample. $x^2 - ax - 20a^2 = 0$

Solution.

$$x^2 - ax - 20a^2 = 0$$
$$(x - 5a)(x + 4a) = 0$$
$$x - 5a = 0 \quad \text{or} \quad x + 4a = 0$$
$$x = 5a \quad \text{or} \quad x = -4a$$

[This solution shows that, for each a,
$\{x: x^2 - ax - 20a^2 = 0\} = \{x: x = 5a \text{ or } x = -4a\}$.]

1. $x^2 - 9nx + 20n^2 = 0$

2. $x^2 + 7rx - 8r^2 = 0$

3. $x^2 + 2ax + a^2 = 0$

4. $x^2 - 6bx + 9b^2 = 0$

MISCELLANEOUS EXERCISES

A. Solve these equations and inequations.

1. $x + 8 = 17$
2. $3 + y = 7$
3. $5 - z = -6$
4. $2x + 4 = 10$
5. $9 = 8 + 3y$
6. $-2 = 4 - x$
7. $6 + x = 7 + x$
8. $6x = 7x$
9. $-x = 3$
10. $5 = -x - 3$
11. $3a + 4 + 2a = 18$
12. $7x - 3 - x = 15$
13. $2y - (4 - y) = 5$
14. $z - (3 - z) = 0$
15. $8x + 2(3 - 4x) = 9$
16. $5y + 2(7 - 2y) = 15$
17. $x(x - 3) = x(3 - x)$
18. $y(y + 1) + y(y - 3) = y(7 + 2y)$
19. $5r - 2 > 7 - 4r$
20. $6 - 5m < 7m + 18$
21. $7t - t = 3t - 8t$
22. $4(1 + x) = 7 - x$
23. $\frac{a}{5} - 3 = \frac{a}{2} - 6$
24. $\frac{x}{3} + \frac{x}{4} = 13 - \frac{x}{2}$
25. $8 - (3 - 2x) = 7(x + 1) + 1$
26. $4 - 2(x - 3) = 1 - (1 - x)$
27. $\frac{x + 3}{2} - \frac{x}{3} = \frac{2x - 3}{5}$
28. $\frac{3 - y}{11} + \frac{4 + y}{2} = 1 + \frac{y}{4}$
29. $\frac{5}{x - 7} = \frac{3}{x + 2}$
30. $\frac{5}{2z} + \frac{9}{4z - 1} = \frac{5}{z}$
31. $\frac{x - 3}{x + 5} = \frac{x - 7}{x - 8}$
32. $\frac{2y + 1}{3y - 2} = \frac{6y + 1}{9y + 2}$
33. $x^2 + 9x + 20 = 0$
34. $y^2 - 17y + 70 = 0$
35. $x + \frac{11}{2} = \frac{21}{2x}$
36. $\frac{1}{2}x(x + 1) = 56 - \frac{5x}{2}$
37. $4x + 5 > 2x - 7$
38. $5(x - 1) > 6x - 3(x - 2)$
39. $\frac{5 - 3x}{7} < \frac{x + 5}{9}$
40. $9 - 3(2x + 4) < \frac{x}{2} - \frac{3x + 1}{3}$
41. $x^2 - 1 > 3[(x - 7) - 6]$
42. $x^2 + 25 \geq 10x$

B. If both members of an equation are fractions, the equation is called a proportion. Here are examples of proportions:

$$\frac{3}{5} = \frac{6}{10}, \quad \frac{x}{7} = \frac{3}{14}, \quad \frac{5}{9} = \frac{3}{y}, \quad \frac{4}{3} = \frac{8}{17}.$$

Sample. Solve the proportion:

$$\frac{2}{x} = \frac{5}{7}.$$

Solution.

$$\frac{2}{x} = \frac{5}{7}$$

$$7x\left(\frac{2}{x}\right) = 7x\left(\frac{5}{7}\right), \quad [x \neq 0]$$

$$14 = 5x$$

The root is $\frac{14}{5}$.

Solve these proportions. [Be on the alert for short cuts.]

1. $\frac{3}{x} = \frac{8}{5}$
2. $\frac{7}{4} = \frac{2}{a}$
3. $\frac{6}{5} = \frac{4}{y}$
4. $\frac{2}{k} = \frac{3}{7}$
5. $\frac{9}{x} = \frac{5}{2}$
6. $\frac{8}{x} = \frac{9}{10}$
7. $\frac{7}{7} = \frac{12}{x}$
8. $\frac{y}{3} = \frac{7}{9}$
9. $\frac{6}{11} = \frac{x}{22}$
10. $\frac{3}{5} = \frac{7}{A}$
11. $\frac{4}{B} = \frac{9}{4}$
12. $\frac{8}{y} = \frac{15}{4}$
13. $\frac{6.1}{3.7} = \frac{2.5}{x}$
14. $\frac{38}{x} = \frac{17}{29}$
15. $\frac{6.02}{0.03} = \frac{5.19}{x}$
16. $\frac{3}{7} = \frac{6}{x - 2}$
17. $\frac{16}{x - 3} = \frac{4}{3}$
18. $\frac{6}{5} = \frac{x + 9}{15}$
19. $\frac{5}{y - 8} = \frac{7}{2}$
20. $\frac{2y + 3}{5} = \frac{9}{13}$
21. $\frac{5 - x}{7} = \frac{3}{8}$
22. $\frac{8.3}{x - 1} = \frac{9.7}{2.4}$
23. $\frac{5.8}{x + 2.4} = \frac{6.1}{8.9}$
24. $\frac{3.4 + x}{7.1} = \frac{2.2}{11.1}$
25. $\frac{3}{x} = \frac{x}{12}$
26. $\frac{48}{k} = \frac{k}{3}$
27. $\frac{x}{169} = \frac{1}{x}$
28. $\frac{7}{y} = \frac{y}{3}$
29. $\frac{3}{t} = \frac{t}{4}$
30. $\frac{27}{x - 4} = \frac{x - 4}{3}$
31. Solve each for 'x': $\frac{a}{b} = \frac{x}{c}, \quad \frac{a}{b} = \frac{c}{x}, \quad \frac{a}{x} = \frac{x}{c}$.

C. Solve these equations for the pronumeral indicated.

1. $K = 2x - y$; x
2. $t = 7s - 3v$; v
3. $m = t\sqrt{s}$; s
4. $2r = s\sqrt{1 - k}$; k
5. $3x + 5y - 17 = 0$; y
6. $2(x - 3) + 7(y - 4) + 8 = 0$; x
7. $\frac{1}{x} - \frac{1}{y} = \frac{1}{z}$; y
8. $\frac{a + b}{ab} = c$; a

D State the generalization involved in each of the following descriptions and prove it.

1. The square of the sum of a first number and a second number is the sum of the square of the first number, the square of the second number, and twice the product of the first and second numbers.

2. The product of the sum of a first number and second number by the difference of the second number from the first number is the difference of the square of the second from the square of the first.

3. The product of the sum of a first number and a second number by the sum of a third number and a fourth number is the sum of the products of these four numbers taken two at a time, one from each sum.

4. The square of the sum of a number and $\frac{1}{2}$ is $\frac{1}{4}$ added to the product of the number by 1 more than the number.

5. The operation of squaring is distributed over multiplication.

*

[In Exercises 6 and 7, use the fact that for each $x \geq 0$, $\sqrt{x}$ = the $z \geq 0$ such that $z^2 = x$.]

☆6. The product of the square root of a first nonnegative number by the square root of a second nonnegative number is the square root of the product of the first number by the second number.

☆7. The quotient of the square root of a first nonnegative number by the square root of a second positive number is the square root of the quotient of the first number by the second number.

E. Evaluate each of the following pronumeral expressions using the given values of the pronumerals. [Answers which involve square roots should be given in exact form and as approximations correct to the nearest 0.01.]

1. $\sqrt{b^2 + a^2}$; '7' for 'b', '24' for 'a'

2. $\sqrt{c^2 - a^2}$; '9' for 'c', '3' for 'a'

3. $4xy\sqrt{z - 5}$; '3' for 'x', '2' for 'y', '17' for 'z'

4. $\sqrt{s(s - a)(s - b)(s - c)}$; '30' for 's', '10' for 'a', '24' for 'b', '26' for 'c'

5. $4\pi r^2$; '$\frac{1}{2}\sqrt{3}$' for 'r'.

6. $(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})$; '6' for 'x', '5' for 'y',

7. $\sqrt{\frac{3V}{\pi h}}$; '528π' for 'V', '6' for 'h'

8. $\sqrt{\frac{2A}{b_1 + b_2}}$; '252' for 'A', '18' for 'b_1', '24' for 'b_2'

9. $\sqrt{2gh}$; '64' for 'g', '200' for 'h'

10. $\frac{2\pi r}{T - 2\pi r^2}$; '168π' for 'T', '$\sqrt{2}$' for 'r'

11. $\sqrt{\frac{4\pi^2 \ell}{g}}$; '16' for 'g', '4' for 'ℓ'

12. $\sqrt{\ell^2 + w^2 + h^2}$; '10' for 'ℓ', '7' for 'w', '5' for 'h'

13. $\sqrt{x^2 - 2xy + y^2}$; '−9873' for 'x', '127' for 'y'

F. Complete to make true sentences.

1. The sum of 7 and the number which is 2 less than 7 is ____.

2. The sum of 12 and the number which is $^{-}3$ more than 12 is ____.

3. The number which is 25 more than 12 is ____.

4. The number which exceeds 18 by 5 is ____.

5. The number which exceeds 6 by 17 is ____.

6. The sum of three consecutive whole numbers, of which the largest is 13, is ____.

7. The difference of 39 from 46 is ____.

8. The difference of 46 from 39 is ____.

9. The difference of 58 from the number which is 5 more than 58 is ____.

10. The number by which 132 exceeds 123 is ____.

11. Mr. Hughes sold a house for \$23,350, and bought another for ____ which was \$1,475 less than he had received for the house he sold.

12. If Alice is 16 years old now, she was ____ years old 7 years ago.

13. If Ruthie was born 3 years ago and her brother was born $5\frac{1}{2}$ years ago, Ruthie is ____ years older than her brother.

14. The product of 12 and the number which is 2 less than 12 is ____.

15. The number which is 20% of 40 is ____.

16. The product of 175 and $\frac{1}{3}$ of 30 is ____.

(continued on next page)

17. The number which is 2 more than the quotient of 20 by 8 is ____.

18. If Debra is 10 years old, she was ____ years old 3 years ago.

19. 96 is ____ per cent of 80.

20. 80 is ____ per cent of 96.

21. The number which exceeds 50 by 15% [of 50] is ____.

22. The number which exceeds 80 by 15% is ____.

23. The number which is 40% less than 60 is ____.

24. The sum of 25% of 60 and 30% of 70 is ____.

25. Mike has 42 marbles; if Larry has 5 more than one third of this number of marbles then Larry has ____ marbles.

26. Mary added 15 bells to her bell collection; if this was one third as many as she already had, then altogether she has ____ bells.

27. If paper cups for cold liquids cost 18 cents per dozen, and paper cups for hot liquids cost one half cent more per cup than those for cold liquids, then the cost of 3 dozen cups for hot liquids is ____ cents.

28. If a school club bought sacks of salted peanuts at the rate of 3 sacks for 8 cents, and sold them at the ball game for 5 cents each, then their profit on 10 dozen sacks of peanuts was ____ cents.

29. If the cost price of an article is 38 dollars, and the margin is 25% of the cost price, the selling price is ____ dollars.

30. A man bought a horse for $350 and sold it at public auction for 10% more than it cost, but he had to pay an auction fee of 10% of the selling price. So, he made ____ dollars.

31. The difference of 9 from the number which is 9 times as large as 4 is ____.

32. If Mel is now 16 years old and if Lew is 7 years less than one and one half times Mel's age, Lew is now ____ years old.

33. If red plums cost 6 cents more per pound than blue plums, 5 pounds of blue plums cost ____ cents less than the same number of pounds of red plums.

34. If the length of each side of a regular hexagon is 1 inch more than 3 times the length of each side of a square, and if the perimeter of the square is 10, then the perimeter of the hexagon is ____.

35. Pete has 5 quarters and 3 more half-dollars than quarters [and no other money]; hence, he has ____ cents.

36. If the number of quarters in a sack of quarters and dimes is 2 more than 3 times the number of dimes, and if the sack contains 11 dimes, then all the coins in the sack are worth ____ cents.

37. In a sum of money consisting of just quarters and half-dollars there are 7 quarters. If the sum amounts to $3.00 then there are ____ half-dollars.

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38. 3 pounds of apples at 12 cents per pound and 5 pounds of apples at 15 cents per pound will cost ____ cents.

39. If a nut mixture is made using 10 pounds of cashews at $1.05 per pound and 5.4 pounds of pecans at 90 cents per pound then one pound of this mixture is worth ____ cents.

40. A grocer prepares 50 pounds of mixed hard candy to sell at 39 cents per pound by mixing 20 pounds of a 30 cents-a-pound kind with ____ pounds of a 45 cents-a-pound kind.

41. If a total of $8000 is invested, with $3000 at 5% and the rest at $3\frac{1}{2}$%, the annual return on the total investment is ____ dollars.

42. There are ____ quarts in 2 gallons.

43. There are ____ gallons in 9 quarts.

44. Three quarts and three pints together make ____ cups.

45. Ann Parker has 9 half-pint jars, and 2 more than one third as many pint jars [as half-pints] which she wants to fill with strawberry jam; she will need ____ quarts of strawberry jam to do this.

46. There are ____ yards in 51 feet.

47. There are 51 yards in ____ feet.

48. If a girl can do a certain household task in 15 minutes, then she can do ____ of the job in 10 minutes.

49. If Mary can iron a certain number of sheets in 2 hours, and Gladys can iron the same number of sheets in $2\frac{1}{2}$ hours, then in 1 hour Mary can iron _____ of the sheets and Gladys can iron _____ of the sheets; so, in one hour, if both girls iron sheets, they can iron _____ of them.

50. If Stan walks at an average rate of 5 miles an hour, he can walk _____ miles in $2\frac{1}{2}$ hours.

51. If Mary walks at an average rate of 4 miles per hour, she can walk 6 miles in _____ hours.

52. Hitchcock walks 10 miles in 1 hour and 40 minutes. So, his average rate of walking is _____ miles per hour.

53. Gerald gets $79.80 for a 42 hour work week. So, his wages are _____ cents per hour.

54. 45 pounds of a certain type of fertilizer contains 20 pounds of nitrogen and 25 pounds of phosphoric acid; 135 pounds of this same kind of fertilizer will contain _____ pounds of phosphoric acid.

55. If an alloy contains 80% copper and the rest tin, then 5600 pounds of this alloy will contain _____ pounds of tin.

56. If 6 quarts of water are added to 12 quarts of an acid solution that contains 15% acid, the new solution will contain _____ quarts of water.

57. If you add 2cc. of a 10% argyrol solution to 3cc. of a 5% argyrol solution, you get 5 cc. of solution containing _____ cc. of argyrol.

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58. When 200 tickets had been sold for a school play, it was found that only 40% of them had been sold to adults; in order to have 60% of the total tickets sold be those sold to adults, it would be necessary to sell ____ more adult tickets.

59. Six gallons of a 10% salt solution are poured into eleven gallons of a 15% salt solution. The resulting solution contains ____ gallons of salt.

60. For each a, the product of 7 by the sum of a and 3 is ____.

61. For each b, the sum of b and the number which is $^{-}2$ times b is ____.

62. For each c, the difference of c from a number 12 times as large as c is ____.

63. For each d, the number which is 12% of d is ____.

64. For each e, the number which is 125% [of e] greater than e is ____.

65. For each f, the number which is 55% [of f] less than f is ____.

66. For each g, for each h, the sum of 33% of g and 71% of h is ____.

67. For each i $\neq 0$, the quotient of 36 by the product of 9 and i is ____.

68. For each number j of arithmetic, if Jim is j years old now, he will be ____ years old 5 years from now.

69. For each number k of arithmetic, if Karl is k years old now and Max is 3 times as old as Karl then Max will be ____ years old 9 years from now.

70. For each number L of arithmetic, if the difference of Helen's age from Laura's age is L years then the difference of Helen's age from Laura's age 3 years ago was ____ years.

71. For each number m of arithmetic, there are ____ half-pints in m quarts.

72. For each number n of arithmetic, there are ____ yards in n inches.

73. For each number P of arithmetic, P gallons, 3 times as many quarts, and 5 times as many pints (as gallons), together make ____ pints.

74. For each whole number p of arithmetic, for each number q of arithmetic, there are ____ cents in a total of p quarters and q half-dollars.

75. For each number r of arithmetic, if the selling price of an article is r dollars and the margin is 18% of the selling price, then the cost price is ____ dollars.

76. For each number s of arithmetic, if the length of a rectangle is s units and the width is 2 more than one half the length, then the perimeter is ____.

77. For each number t of arithmetic, if the base of an isosceles triangle is t units, and each of the two equal sides is 3 units more than twice the base, then the perimeter of this triangle is ____.

(continued on next page)

78. For each $u \neq -\frac{3}{2}$, the product of 3 by u divided by the sum of 6u and 9 is ____.

79. For each number v of arithmetic, you can drive ____ miles in v hours at the average rate of 60 miles per hour.

80. For each number w of arithmetic, you can travel ____ miles in 5 hours at the average rate of w miles per hour.

81. For each number x of arithmetic, for each number y of arithmetic, you can travel ____ miles in x hours at the average rate of y miles an hour.

82. For each number of arithmetic $z > 0$, it takes ____ hours for an airplane to travel z miles if its cruising speed is 325 miles per hour.

83. For each number d of arithmetic, the annual income on d dollars invested at $4\frac{1}{2}\%$ is ____ dollars.

84. For each number b of arithmetic, if the diameter of a circle measures b, the circumference is ____.

85. For each number c of arithmetic, if a regular hexagon has perimeter c then an equilateral triangle whose side is 2 units longer than a side of this hexagon will have perimeter ____.

86. For each number of arithmetic $d > 0$, if Bart can shovel the snow off the sidewalks at his home in d hours, he can clean ____ of the sidewalks in 1 hour.

87. For each number e of arithmetic, if Bart can clean the snow off the sidewalks in 2 hours, and Martin can do the same job in 3 hours, they can clean ____ of the sidewalks in e hours.

88. For each number of arithmetic $f > 0$, if Bill climbs $\frac{2}{3}$ as fast as Dale, and if Dale climbs f feet per minute, then Bill requires ____ minutes to climb 20 feet.

89. For each whole number g of arithmetic, a pile of pennies, dimes, and half-dollars contains g dimes, 2 more than 3 times as many pennies, and 5 more half-dollars than dimes; the pile of coins is worth ____ cents.

90. For each whole number of arithmetic $k > 1$, if h three-cent stamps and 5 less than 4 times as many four-cent stamps are purchased, the entire purchase is worth ____ cents.

91. For each number i of arithmetic, if i pounds of potatoes at 13 cents a pound are mixed with 3 pounds of potatoes at 15 cents a pound, the resulting mixture contains ____ pounds worth 23 cents a pound.

92. For each j, if j is an even integer, ____ is the next smaller even integer.

93. For each k, if k is an integer, the sum of the next larger integer and the next smaller integer is ____ .

94. For each number L of arithmetic, L gallons of milk containing 3% butterfat will contain ____ gallons of butterfat.

95. For each number m of arithmetic, if m gallons of a fruit juice and ginger ale mixture contains $33\frac{1}{3}\%$ ginger ale, the mixture contains ____ gallons of fruit juice.

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96. For each number n of arithmetic, if n ounces of black tea are added to 2 pounds of a blend which contains 40% black tea, the new blend contains ____ ounces of black tea.

97. For each number q of arithmetic, if 10 pounds of a flour mixture which contains q% whole wheat flour [and the rest white flour] are combined with 3 pounds of whole wheat flour, the new mixture is ____ per cent whole wheat flour.

98. For each number of arithmetic $r > 0$, if it takes Ann r hours to fly an airplane 200 miles, and Marion flies 10 miles per hour faster than Ann, then it takes Marion ____ hours to fly $\frac{2}{5}$ as far as Ann.

99. For each whole number of arithmetic $s > 0$, if s people share equally in the cost of a $15 Christmas gift, a gift paid for by 2 less than 3 times this number of people should cost ____ dollars [if each person contributes the same amount as before].

G. Solve these problems.

1. The length of a rectangle is 7 feet more than its width, and its perimeter is 36. Find the dimensions of the rectangle.

2. Ben has 615 stamps in his collection. He has 5 times as many U.S. stamps as French stamps, 60 more British stamps than French stamps, and 75 more stamps of other nations [not including France and Britain] than of the U.S. How many stamps of each kind does he have?

3. A coin box contains quarters, dimes, and nickels which are together worth $18.55. The number of dimes is 1 more than twice the number of quarters, and the number of nickels is 3 less than three times the number of quarters. How many coins of each kind are in the box?

4. The perimeter of a triangle is 62. The length of the longest side is double that of the shortest side, and the length of the third side is 3 inches less than that of the longest side. Find the length of each side.

5. Miss Mack had $22000 to invest, and she bought bonds yielding 4% interest with part of it. She invested twice as much in stocks as she had in bonds, and the stocks yielded 6% interest. Finally, she loaned the remainder of the $22000 on a mortgage which would pay 5% interest. The total annual interest received from these three investments was $1150. How much money had she invested at each rate?

6. Rhoda said, "I am thinking of a number. If you triple it, add 7, and divide this sum by 2, the result is 29." What number was Rhoda thinking of?

7. Joyce Ann is 4 years older than her brother John. Eight years ago, John was $\frac{2}{3}$ as old as Joyce Ann was then. How old is each now?

8. The length of a rectangle is 2 feet more than its width. If the measure of the width is increased by 2, and the measure of the length is increased by 3, the area of the new rectangle is twice the area of the original. What were the dimensions of the original rectangle?

9. An estate of $30,000 was divided in the ratio of 3:4:5 and given to three charitable organizations. How much money did each of the three receive?

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10. The number named by the denominator of a fraction is 8 more than the number named by the numerator of this fraction. If both the numerator-number and the denominator-number are increased by 9, the resulting fraction stands for $\frac{2}{3}$. What was the original fraction?

11. Dale can go into town from camp in one hour and fifteen minutes on his bicycle, but it takes him 3 hours for the trip if he walks. His rate of walking is 7 miles an hour less than his rate on the bicycle. Find Dale's rate of walking.

12. A farmer sold 60 dozen eggs to a market. For the extra large eggs he received 55 cents a dozen, and for the large ones, 46 cents a dozen. He received a check for $30.12. How many dozen of each size did he sell?

13. One number is 13 more than another. Twelve times the smaller number is 19 more than 5 times the larger. What are the numbers?

14. The product of two consecutive integers is equal to the sum of 26 and 10 times the larger number. Find the two integers.

15. A building contractor sold a house for $12, 325, which was 15% less than it cost him. How much did it cost him?

16. Mrs. Reich paid $8.35 for 100 Christmas cards. For some of them she paid 12 cents each; then she picked out some 10-cent cards, and took 3 more of them than she had of the 12-cent cards. The number of 5-cent cards bought was $1\frac{2}{3}$ times the number of 12-cent cards, and the rest of the cards in the purchase were $8\frac{1}{3}$ cents each. How many cards did she buy of each kind?

17. One number is 6 more than another, and their product is 891. What are the numbers?

18. An apple orchard contains 1015 trees planted in rows, with each row containing the same number of trees. The number of rows is 6 less than the number of trees in each row. How many trees are there in each row?

19. The treasurer of the Zabranchburg High Y-Teen Club needed to get money from the school bank to fill change boxes for a candy sale. She decided that she should get $1\frac{1}{2}$ times as many quarters as half-dollars, three times as many dimes as half-dollars, and 2 more nickels than dimes. Altogether, she wanted to have \$16.00 worth of coins. How many coins of each kind should she get?

20. In an isosceles triangle, the length of the base is 6 feet less than the length of one of the two equal-length sides. If the perimeter is 57, find the length of the base.

21. Twenty-four grams of water are mixed with 16 grams of acetic acid. What per cent of the resulting solution is acetic acid? If a certain amount of this solution is poured out, and it is replaced by an equal amount of a 30% solution of acetic acid and water, the new solution will contain 32% acetic acid. How many grams of the original solution were poured out, and how many grams of the 30% solution were added, to get this 32% solution?

22. A man invests \$1200 at a certain yearly rate of interest, and he invests \$1300 at a rate one half per cent higher than that paid on his \$1200 investment. His total annual income from these two investments is \$106.50. What are the two rates of interest?

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☆23. A money box contains nickels, dimes, quarters, and half-dollars, and nothing else. Altogether these coins are worth $6.30. There are twice as many nickels as half-dollars, and 8 times as many quarters as nickels. How many dimes are there in the box?

☆24. A man leaves his home at 1:30 p.m. and drives to Zilchville, 72 miles away, to transact some business. It takes him one hour and ten minutes to conclude his business deal, and then he takes fifteen minutes to have a cup of coffee before starting home. On the return trip, traffic is heavier than before, so his average speed is 6 miles per hour less than it was as he drove to Zilchville. If he arrived home at 5:45 p.m., what was his average rate of speed on the trip to Zilchville?

☆25. Two inlet pipes can fill a certain tank in 24 minutes. The larger of the two pipes can fill the tank in 14 minutes less time than the smaller pipe. How long does it take each pipe to fill the tank?

☆26. One workman can do a certain job in 60 hours. With the help of another workman, the same job can be completed in 40 hours. How long would it take the second workman to do the whole job if he worked alone?

☆27. For assembly programs in the school gymnasium, a 672-member student body is seated in rows, with the same number of students in each row. If 2 more persons were seated in each row, it would take 6 fewer rows. How many students were seated in each row in the original arrangement?

H. Expand.

1. $(b + c)^2$
2. $(3d - 7)(3d + 7)$
3. $(3d - 5)(3d - 5)$
4. $(x - 15)(x + 10)$
5. $(3n + 1)(4n + 3)$
6. $(a - 4b)(a - 3b)$
7. $c(3a + 5b)$
8. $e(c - 4)(c + 6)$
9. $f(5g - 1)^2$
10. $(8 - h)(8 + h)$
11. $j(k + 11)(k - 11)$
12. $(3m + 2)(5m - 4)$
13. $\left(\frac{1}{3}n - \frac{1}{4}p\right)\left(\frac{1}{3}n + \frac{1}{4}p\right)$
14. $\left(\frac{1}{2}a + 4\right)\left(\frac{1}{2}a - 3\right)$
15. $\left(\frac{1}{2} - q\right)^2$
16. $(r + 2)(r + 7)$
17. $(s + 3)(s + 1)$
18. $(4t - 3)(7t - 2)$
19. $(x - w)(x - w)$
20. $(2u + v)(u + 2v)$
21. $(4x - 3)(2x + 1)$
22. $2\pi r(r + h)$
23. $\frac{1}{2}h(b_1 + b_2)$
24. $z(3x + 1)(x - 2)$
25. $(5n - 3a)(3n - 4a)$
26. $(6c - 5)^2$
27. $(10 - 3d)^2$
28. $ab(c + .5d)^2$
29. $(7e + 4f)(7e + 4f)$
30. $\left(\frac{k}{2} - 6\right)\left(\frac{k}{3} + 10\right)$

I. Factor.

1. $n^2 + 5n - 36$
2. $a^2 - 2a - 63$
3. $c^2 - 4c - 45$
4. $d^2 + 3d - 40$
5. $a^2 + 11a + 24$
6. $b^2 - 10b - 24$
7. $c^2 - 25c + 24$
8. $d^2 + 14d + 24$
9. $\frac{1}{6}y^2 - \frac{1}{6}y - 1$
10. $\frac{1}{9}x^2 + 4x + 36$
11. $e^2 + 5e - 24$
12. $g^2 - 2g - 24$
13. $h^2 - 11h - 60$
14. $j^2 + 17j + 60$
15. $2k^2 + 32k + 96$
16. $x^2 - 4xy - 21y^2$
17. $ab^2 - 8ab - 20a$
18. $2n^2 - 9n - 35$
19. $8p^2 - 18p + 9$
20. $8q^2 + 26q + 21$

J. Simplify.

1. $7x + 4x$
2. $.8v + (-12v)$
3. $12g - 9g + 5g$
4. $8\triangle - \bigcirc - 12\triangle$
5. $-4m + 9n - 11m - 15q$
6. $2.5s + 7.5r + 9.4s - 11.6r$
7. $6y - 5z - 5y + 6z$
8. $3.2 + 16b - c + 4.7b - 2.3c$
9. $\frac{1}{5}a + \frac{1}{4}a + \frac{3}{4}a$
10. $\frac{1}{5}x - \frac{1}{4}y + \frac{4}{5}x - \frac{2}{7}y$
11. $6y + 5y$
12. $-3r + 4r$
13. $11u - u + 2u$
14. $7s + 12r - 4t - s$
15. $\frac{1}{5}x + \frac{1}{10}x - 10x$
16. $\frac{1}{3}m - \frac{2}{5}n + \frac{3}{4}m - \frac{1}{2}n$
17. $6\bigcirc + 4\square + 5\square - 3\bigcirc$
18. $9u - 2v + 12 - 19u - 12v$
19. $2.6a + 5.4b + 3.7a - 4.5b$
20. $4.8r - 9.6s - 5.4r + s$
21. $3pq - 5p + 7pq$
22. $9ac - 6a + 5a$
23. $6.5xyz - 2.7yxz + 12.6xy$
24. $\frac{5}{6} + 7m - 2n - \frac{2}{5} + 6m - 2n$
25. $\frac{1}{5} + 9x - 3y + 7xy - \frac{1}{4} - 3x$
26. $2.0rr - 6.8 + 5.5rr - 7.3rr$
27. $8y^2 + 7y - 5y^2 - 4y - 5$
28. $3ab - 4a^2 - 5ab + 3a^2 + 15b$
29. $3.1uv - 2.9uv + 10.1v^2 - .2u^2$
30. $5.5 - 7.0y^2 + 6.4y - 4.1y^2 + 8.6y$
31. $7(x + 2y) - 4(x - y)$
32. $4(c - d) + 2(d - c)$
33. $6(2a - 4b) - 10a + 18b$
34. $9r - 4r - 3(3r - 2p)$
35. $7(-w + 3u) - 4(-w + u)$
36. $6(a + b) - 14(-a + b)$

37. $2xy(5x - 36) - 4xy(6y - 9x)$

38. $4rs(3r - 6s) - 3rs(2r + 5s)$

39. $-2(\square + 3\triangle) + 5(2\triangle - 3\square)$

40. $(4\triangle - 7\square)3 - 4(-3\triangle + \square)$

41. $4(3.2p - 5.6q) + 10(7.7p - 12.1q)$

42. $5(6.5c - 3.5d) + 7(2.1c - 4.2d)$

43. $\frac{1}{5}(10m - 5n) + \frac{1}{3}(3n - 6m)$

44. $\frac{1}{6}(3w - 6v) - \frac{1}{3}(3v - 9w)$

Sample 1. $(x + 3)^2 + (x - 5)(x + 7)$

Solution. $(x + 3)^2 + (x - 5)(x + 7)$

$= (x^2 + 6x + 9) + (x^2 + 2x - 35)$

$= 2x^2 + 8x - 26.$

45. $(a + 5)^2 + (a - 2)(a + 9)$

46. $4(c - 1) - (c - 4)^2$

47. $(2a + 1)(a - 3) - (2a - 3)^2$

48. $(n - 7)^2 + (n - 3)(n + 8)$

49. $3(3 - 4d) + (2d - 5)(d + 6)$

50. $(5r - 2)(r + 3) - (5 - 2r)^2$

51. $\frac{-4bc}{-2bc}$

52. $\frac{-3hk}{12h}$

53. $\frac{-8xy^2}{4xy}$

54. $\frac{4gt^2}{2gt}$

55. $\frac{.6x^2yz}{3xy^2z}$

56. $\frac{5rs^2t}{1.5r^2st}$

57. $\frac{bc - bd}{b}$

58. $\frac{\pi R^2 - \pi r^2}{\pi}$

59. $\frac{2k^2 - 4k}{2k}$

60. $\frac{9n^2 + 6n}{-3n}$

61. $\frac{a}{b^2} \cdot \frac{bc}{a^2}$

62. $\frac{x^2}{y} \cdot \frac{3y^2}{2x}$

63. $\frac{3cd}{4c} \cdot \frac{4a}{2c}$

64. $\frac{7x^2}{6yz} \cdot \frac{9}{14x}$

65. $\frac{5n^2r}{7r^2s} \cdot \frac{21rs}{20nr^2}$

66. $\frac{128x^2y^2z}{31a^2bc} \cdot \frac{124ab^2c^2}{16xyz^2}$

67. $56\left(\frac{n}{4} + \frac{n}{8}\right)$

68. $6y\left(\frac{7}{2y} + \frac{3}{y}\right)$

69. $72\left(\frac{2b + 1}{6} + \frac{b - 5}{8} - \frac{2b}{9}\right)$

70. $40\left(\frac{d + 7}{8} - \frac{2d + 1}{5} + \frac{3d - 2}{4}\right)$

(continued on next page)

71. $84\left[\frac{1}{7}(x + 3) - \frac{1}{4}(x - 2) + \frac{1}{12}\right]$

72. $120\left(\frac{3c - 2}{10} + \frac{5t + 6}{12} - \frac{7t - 3}{15}\right)$

73. $\frac{15cd}{21de^2} \div \frac{5c}{3de}$

74. $\frac{-14a^2n}{12np^2} \div \frac{2an}{-3p}$

75. $\frac{x^2y^2}{ar^2} \div \frac{ay^2}{r^2x}$

76. $\frac{a^2b^2c}{mn^2p^2} \div \frac{ab^2c}{m^2np}$

77. $\frac{3a}{8} - \frac{a}{4}$

78. $\frac{5}{6a} - \frac{3}{2a}$

79. $\frac{n + 5}{2} + \frac{n - 2}{4}$

80. $\frac{x + 7}{6} + \frac{x - 2}{3}$

81. $\frac{3x - 1}{8} - \frac{x + 1}{4}$

82. $\frac{m + n}{5} - \frac{7n - m}{20}$

83. $\frac{5}{x - 2} + \frac{7}{2 - x}$

84. $\frac{12}{c - d} - \frac{5}{d - c}$

Sample 2. $\frac{5}{3x^2} - \frac{1}{2x} + 7$

Solution. $\frac{5}{3x^2} - \frac{1}{2x} + 7$

$$= \frac{6x^2\left(\frac{5}{3x^2} - \frac{1}{2x} + 7\right)}{6x^2}$$

$$= \frac{10 - 3x + 42x^2}{6x^2}. \qquad [x \neq 0]$$

Sample 3. $\frac{12}{3y + 1} - 9 + \frac{5}{2y - 1}$

Solution. $\frac{12}{3y + 1} - 9 + \frac{5}{2y - 1}$

$$= \frac{(3y + 1)(2y - 1)\left(\frac{12}{3y + 1} - 9 + \frac{5}{2y - 1}\right)}{(3y + 1)(2y - 1)}$$

$$= \frac{(2y - 1)12 - (3y + 1)(2y - 1)9 + (3y + 1)5}{(3y + 1)(2y - 1)}$$

$$= \frac{24y - 12 - (6y^2 - y - 1)9 + 15y + 5}{(3y + 1)(2y - 1)}$$

$$= \frac{24y - 12 - 54y^2 + 9y + 9 + 15y + 5}{(3y + 1)(2y - 1)}$$

$$= \frac{-54y^2 + 48y + 2}{(3y + 1)(2y - 1)}. \qquad [y \neq -\tfrac{1}{3},\ y \neq \tfrac{1}{2}]$$

85. $\dfrac{a}{a + b} + 6$

86. $\dfrac{n}{a - n} - 9$

87. $\dfrac{x + y}{x} + \dfrac{y}{2x^2} - 3$

88. $\dfrac{r}{3s^2} - 5 + \dfrac{r - s}{2s}$

89. $\dfrac{d}{d + 7} - \dfrac{d}{d - 3}$

90. $\dfrac{3c}{5 - c} - \dfrac{c}{8 + c}$

91. $13 - \dfrac{4}{n - 3} + \dfrac{2}{n + 3}$

92. $\dfrac{3}{k + 1} - \dfrac{2}{k - 1} + 17$

93. $\dfrac{y}{y - 3} - \dfrac{7}{3y}$

94. $\dfrac{2}{t + 3} - \dfrac{1}{2t}$

95. $\dfrac{2}{3k} - \dfrac{1}{2k^2} + 7 - \dfrac{3}{8k}$

96. $1 - \dfrac{2}{7s^2} + \dfrac{1}{2} - \dfrac{3}{5s}$

97. $\dfrac{5}{b} + \dfrac{3}{b - 8} + 2$

98. $\dfrac{3}{d} - 4 - \dfrac{d}{d + 2}$

99. $\dfrac{t + 1}{t + 3} - 5 + \dfrac{3t}{2t + 5}$

100. $\dfrac{u + 8}{u - 2} + 3 - \dfrac{u + 2}{3u - 4}$

Sample 4. $\dfrac{\dfrac{5x}{2} - \dfrac{1}{3x}}{\dfrac{1}{2x} - \dfrac{5x}{6}}$

Solution. $\dfrac{\dfrac{5x}{2} - \dfrac{1}{3x}}{\dfrac{1}{2x} - \dfrac{5x}{6}}$

$$= \frac{6x\left(\dfrac{5x}{2} - \dfrac{1}{3x}\right)}{6x\left(\dfrac{1}{2x} - \dfrac{5x}{6}\right)}$$

$$= \frac{15x^2 - 2}{3 - 5x^2}. \qquad \left[x^2 \neq \frac{3}{5},\ x \neq 0\right]$$

101. $\dfrac{1 + \dfrac{2}{n}}{n - \dfrac{4}{n}}$

102. $\dfrac{\dfrac{5}{a} - 3}{\dfrac{7}{a} + a}$

103. $\dfrac{\dfrac{x}{y} - \dfrac{y}{x}}{1 - \dfrac{y}{x}}$

104. $\dfrac{r - \dfrac{r}{s}}{\dfrac{r}{s} - \dfrac{s}{r}}$

105. $\dfrac{\dfrac{10k}{3} - \dfrac{2}{5k}}{\dfrac{3}{5k} - \dfrac{4k}{7}}$

106. $\dfrac{1 - \dfrac{m}{n}}{\dfrac{n}{m} - 1}$

TEST

I. In the blanks at the left, indicate with a 'T' or an 'F' whether each statement is true or false.

_______ 1. For each a, for each b, if $b \neq 0$, then $\frac{a}{b} \geq 0$.

_______ 2. For each r, for each s, if $s \neq 0$, then $-\frac{r}{s} = -\frac{-r}{-s}$.

_______ 3. For each c, for each d, $6(c + 3d) - 2(c - d) = 4c + 16d$.

_______ 4. $\{9, -9\}$ is the solution set of the sentence '$xx = 81$'.

_______ 5. $\{a: a + 8 = 20\} = \{a: a = 28\}$.

_______ 6. $\{n: n + 6 = n + 4\} = \{d: 0 > d > 2\}$.

_______ 7. $\{5, 2, ^-3\} \subseteq \{-4, 0, 1, ^-3, 7, 2, 5\}$.

_______ 8. $\{r: r^2 = 25\} \subseteq \{t: |t| = 5\}$.

_______ 9. $\{x: 3x^2 = 15x\} \subseteq \{y: 65 - 5y = 8y\}$.

_______ 10. $\forall_x \forall_y$ if $x < 0$ and $y < 0$, then $x + y = |x| + |y|$.

II. Complete the following to make true sentences.

1. The midpoint of $\overleftrightarrow{3, 35}$ is _______.

2. $\forall_x \forall_y$ the midpoint of $\overleftrightarrow{5x + 3y, 9x - 13y}$ is _______.

3. For each number of arithmetic $s > 3$, if the length of one side of a square is $s - 3$ inches, the perimeter is _______.

4. For each number of arithmetic $\square$, for each number of arithmetic $\triangle$, for each number of arithmetic $\bigcirc$, if $\square$ dozen eggs cost $4\triangle$ cents, then the cost of $\bigcirc$ eggs is _______ cents.

5. $\forall_{x \neq 0} \forall_{y \neq 0}$ the sum of the reciprocals of x and y is _______. [Write a single fraction in the blank.]

6. For each whole number of arithmetic $d > 3$, the number of cents in d dimes and 5 less than twice that many quarters is _______ cents.

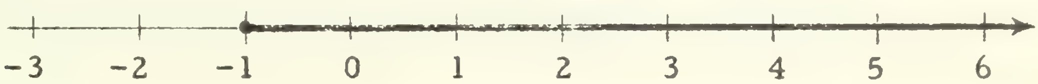
7. For each number of arithmetic A, if Jim's age now is A years, his age in 8 years will be _______ years.

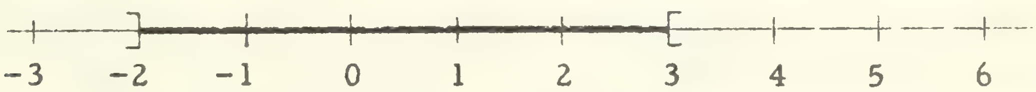
8. For each number of arithmetic t, if t dollars are invested at $3\frac{1}{2}\%$ and \$3000 more than t dollars are invested at 5%, the number of dollars annual interest on the two investments is _______ dollars.

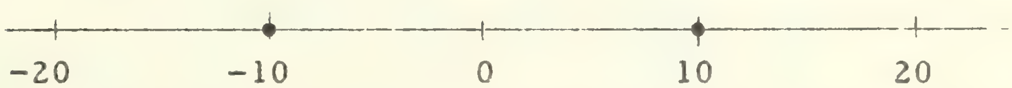
9. For each number of arithmetic m, for each number of arithmetic r, if Mr. Beecher drives at a rate of r + 5 miles per hour, it will take him _______ hours to drive m miles.

10. For each number of arithmetic p, if p pounds of hard candy selling at 29 cents per pound are mixed with 3 more than twice as many pounds of candy selling at 35 cents a pound, the resulting mixture is worth ______ cents a pound.

III. Each exercise contains a picture of the number line with a graph marked on it. For each exercise, give <u>two</u> descriptions of the set of numbers which are the coordinates of the points in the graph.

1. 

2. 

3. 

IV. (a) Use geometric language to name the loci of the following sentences.

<u>Sample</u>. $x \geq 1$

<u>Solution</u>. $\overleftrightarrow{1, 2}$

1. $n < 2$ 2. $-4 \leq a \leq 3$ 3. $|x + 1| = 5$

4. $-d \geq 2$ 5. $k > -3$ or $k > 2$ 6. $-3 > b > 1$

(b) Use brace-notation ['{x: ___}'] to name the sets with these loci.

7. $\overline{-5, 2}$ 8. $\{0\}$ 9. $\overrightarrow{4, -4}$ 10. $\overline{1, -3}$

V. Solve these equations and inequations.

1. $3 + 2n = 7 + 6n$
2. $-6a + 11 = 2a + 43$
3. $\frac{2c}{3} + 1 = \frac{c}{2}$
4. $\frac{2d + 5}{4} - \frac{10d + 13}{8} = 2d + 1$
5. $\frac{1}{10}(7c + 1) - \frac{1}{4}(c - 9) = 1$
6. $\frac{7}{2a} - 6 = 9 - \frac{5}{4a}$
7. $\frac{2}{a + 5} = \frac{1}{a - 7}$
8. $\frac{13}{a - 7} = \frac{13}{7 - a}$
9. $x^2 = 9x$
10. $x^2 + 6x = 27$
11. $9 - 3x > 15x + 2$
12. $6 + 4(2 - x) < \frac{1}{2}x - 5$

VI. (a) Solve each of the following equations for the pronumeral indicated.

1. $A = 3a + 2b$; b
2. $x = -7y + w$; y
3. $V = \ell wh$; w
4. $N = a + (x - 1)b$; b
5. $\frac{1}{a} + \frac{1}{b} = 10$; b
6. $\frac{12}{m - a} = \frac{6}{m + a}$; a

(b) Make the indicated substitutions and solve the resulting equations. [The solution for Exercise 10 should be given as an approximation correct to the nearest 0.01.]

7. $A = \frac{h(a + b)}{2}$, '136' for 'A', '15' for 'a', '17' for 'b'.
8. $R = \frac{gs}{g + s}$, '8' for 'R', '12' for 'g'.
9. $\ell = a + (n - 1)d$, '6' for 'a', '76' for 'ℓ', '21' for 'n'.
10. $D = 89\sqrt{H}$, '250' for 'H'.

VII. Expand.

1. $(n + 7)(n - 9)$
2. $(a + 10)(a + 8)$
3. $(2c - 3)(c - 2)$
4. $(5 - 4n)(5 + 4n)$
5. $(3x_1 - 2)(3x_1 - 2)$
6. $x(3y - 7)(y + 2)$

7. $a(5 - 3c)^2$

8. $r(2s_1 - \frac{1}{3})(3s_1 + \frac{1}{2})$

9. $3ab(a - 5b)$

10. $(y - 3)(\frac{y + 5}{y - 3} + \frac{4}{y - 3} - 5)$

*

11. Use the principles for real numbers to prove:

$$\forall_a \forall_b \ (a + b)^2 = a^2 + 2ab + b^2.$$

VIII. Factor.

1. $c^2 - 9c + 20$

2. $d^2 + 15d - 34$

3. $k^2 - k - 72$

4. $4r^2 - 20rs + 25s^2$

5. $3a^2 + 3a - 36$

6. $na^2 + 3an + 2n$

7. $5x^2 - 180$

8. $xy^2 - 14xy + 49x$

9. $b^2 - b + \frac{1}{4}$

10. $\frac{n^2}{25} + \frac{1}{15}nd + \frac{d^2}{36}$

*

11. Use the principles for real numbers to prove:

$$\forall_a \forall_b \ a^2 - b^2 = (a + b)(a - b).$$

IX. (a) Simplify.

1. $\sqrt{128}$ [$128 = 64 \times 2$]

2. $\sqrt{\frac{1}{4}a^2}$

3. $\sqrt{32} \times \sqrt{2}$

4. $\sqrt{150} \div \sqrt{6}$

5. $\sqrt{r^2 - 12rs + 36s^2}$

6. $\sqrt{(n + 8)^2(n - 1)^2}$

7. $(2\sqrt{7} - \sqrt{6})(2\sqrt{7} + \sqrt{6})$

8. $(\sqrt{3} + 5)^2 + (\sqrt{3} - 2)^2$

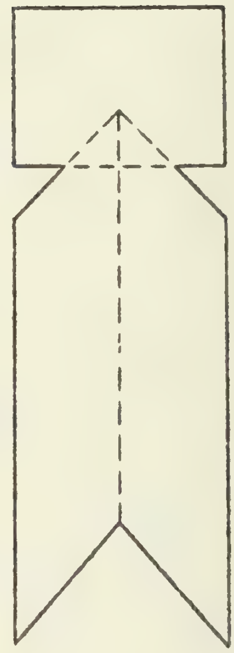
(b) Find the approximation correct to the nearest 0.01 by first transforming the expression into a simpler one which contains a '$\sqrt{2}$' or a '$\sqrt{3}$'.

9. $\sqrt{18} + \sqrt{50}$

10. $\sqrt{108} - \sqrt{75}$

X. Solve these problems.

1. An 11-sided figure is made up of two parallelograms of exactly the same size, and one rectangle, placed as in this picture. The width of the rectangle is $\frac{3}{4}$ its length; the shorter side of each parallelogram has the same measure as the measure of the width of the rectangle; the longer side of each parallelogram is twice the length of the rectangle; the three figures overlap in such a way that one half of one short side of each parallelogram is in the interior of the rectangle, and one half of one long side of the rectangle is in the interior of the parallelograms.



   (a) Find a formula for the perimeter of the 11-sided figure.

   (b) If the perimeter is 37, what is the measure of the length of the shorter side of one of the parallelograms?

2. The perimeter of a rectangle is 34; its area measure is 66. Find the measure of its length.

3. Mr. Abel invests a total of \$7000 in three enterprises. One enterprise returns 5% on the investment, another 4%, and a third 3.5%. He invests \$3000 more in the 4% enterprise than in the 5% one, and his total income from the three investments is \$280. How much does he invest in each of the three enterprises?

SUPPLEMENTARY EXERCISES

A. Sketch the graph of each of the following sentences.

1. $3s + 7 = 16$

   $-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

2. $3x + 6 \leq 0$

   $-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

3. $y \leq y + \frac{1}{5}$

   $-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

4. $|x - 2| + 3 < 4$

   $-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

5. $\frac{1}{3}(x + 5) \geq 2$

   $-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

6. $xx - 9 \leq 0$

   $-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

7. $1.5 \leq x < 2.5$

   $-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

8. $x = -2$ or $x = 0$ or $x = 1$

   $-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

9. $|x| < 1$ or $|x| > 2$

   $-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

10. $|x| \geq 1$ and $|x| \leq 2$

    $-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

11. $(x < 2$ and $x < 6)$ or $x = 3$

    $-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

12. $x < 2$ and $(x < 6$ or $x = 3)$

    $-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

B. Find the roots of these equations.

1. $7a - 4 = 17$	2. $3 + 2x = 19$	3. $5 - 6y = -7$
4. $-10 = -x - 6$	5. $7y - 1 = -50$	6. $28 = 8t + 4$
7. $.5x + 9 = 11$	8. $-6 = 3 + 9y$	9. $3 - 9y = 6$
10. $-6 = 2 - k$	11. $6x - 2 = 2$	12. $4 = 8y + 4$
13. $\frac{5}{7}x = 10$	14. $\frac{5x}{7} = 10$	15. $\frac{5}{7x} = 10$
16. $\frac{2}{5}y = 6$	17. $\frac{2y}{5} = 6$	18. $\frac{2}{5y} = 6$
19. $\frac{1}{3}y - 17 = 4$	20. $3 - \frac{1}{2}x = 0$	21. $\frac{1}{2}y + 1 = -\frac{1}{2}$
22. $-6 + 6x = 12$	23. $8 = 2y - 3$	24. $-73 = 7k + 4$
25. $-1 = -y + 6$	26. $5 + 5y = 5$	27. $5 = 8 - 4t$
28. $9 = 7 - 2k$	29. $-2m + 8 = 5$	30. $9 + 5z = 0$
31. $\frac{8}{z} = 16$	32. $\frac{8}{-z} = 16$	33. $\frac{8}{3z} = 24$
34. $\frac{7}{x} = 1$	35. $\frac{7}{2x} = 1$	36. $\frac{7}{-x} = 1$
37. $\frac{1}{3} = \frac{2}{5} - k$	38. $9 + \frac{k}{10} = 11$	39. $\frac{1}{2}(3 + 4x) = \frac{3}{2}$
40. $\frac{1}{3}(s + 6) = 22$	41. $5 = \frac{2}{5}(3 - x)$	42. $1 + \frac{3}{2}t = 25$
43. $\frac{5 + 3x}{4} = 20$	44. $\frac{8 - 3y}{7} = 2$	45. $1 = \frac{-2x - 7}{5}$
46. $\frac{9}{2y} = \frac{9}{2}$	47. $\frac{18}{7x} = \frac{18}{7}$	48. $\frac{19}{6y} = \frac{19}{3}$
49. $\frac{8}{x - 3} = 4$	50. $\frac{8}{3 - x} = 4$	51. $\frac{8}{\vert 3 - x \vert} = 4$
52. $16 + 4f = -16$	53. $5 = 1 + 3k$	54. $2 - 5s = 4$
55. $2x + 2 = 10$	56. $6 - 3y = 9$	57. $14 = 28 + 7x$
58. $2(x + 1) = 10$	59. $3(2 - y) = 9$	60. $14 = 7(4 + x)$

61. $2(x + 1) = 13$
62. $3(2 - y) = 18$
63. $42 = 7(4 + x)$
64. $3(x + 1) = 3x$
65. $2(x - 5) = 2$
66. $3(2 + x) = 3x + 6$
67. $|4 + x| = 12$
68. $|-2x + 5| = 5$
69. $|8 + 4x| = 32$
70. $3 + |a - 2| = 8$
71. $8 - |3 - n| = 13$
72. $|3 - d| - 8 = 13$

C. Solve.

1. $3x - 5 + 8x - 12 = 5$
2. $6m - 4 - 4m + 8 = 4$
3. $7A - 6 + 13A + 18 = 8$
4. $9b - 21 - 3b - 2 = 1$
5. $3(n - 2) + 5 - 6n = 17$
6. $9x + 7 - 2(x - 7) = -21$
7. $2(6k - 3) + 4(5k - 4) = 42$
8. $a - (6 - 2a) = -15$
9. $y - (9 - y) - (8 - y) = -8$
10. $2b - 2(3b - 7) = 2$
11. $5(7r - 1) - 6(7r - 1) = 15$
12. $7 + 8(x - 4) = 15$
13. $13(s - 4) + 5(3s - 3) = -11$
14. $12 - 3(s + 5) - 2(9s - 7) = 95$
15. $7(x - 2) - 8x - 3 = 100$
16. $3(x - 5) - 3x = 6$
17. $t - (1 - t) + (t - 1) - 1 = 0$
18. $6(2 - 9y) + 4(2y - 9) + y = -114$
19. $18m - (m + 5) - 3(m - 2) + 4(m - 5) = -55$
20. $9(x - 1) - 9(x + 1) + 2(x - 2) - 2(2 + 2x) = 0$
21. $x(3 - 2x) - 5x(x - 3) + 7x(x + 2) - 18x = 42$
22. $10(10 + yy) - 9(yy + 2) - 2(1 + yy) = -1$
23. $2aa + 2(6a - 3) + 6(2a - 1) - a(2a + 3) = -5$
24. $11(2 + s) + 2(11 - s) + 2 - 11s = 4$
25. $5 + 4(5 - 2r) - 6r - 3(5r + 2) + 4 = 103$
26. $(5n - 2) - 7 - (n - 2) + 7n - 5(2n - 5n) = 6$
27. $7rr + 5r(r - 1) - 12r(5 + r) - 100 = 30$
28. $2w + 2w(w - 2) + 2(2 - ww) + 4w = 4$
29. $7x - 34(2x - 5) - 9(x - 7) - 64x = -1442$
30. $m - 12m(m - 3) + 16m(\frac{3}{4}m - 11) + 175m = 252$

D. Solve.

1. $3x = x + 8$	2. $7k = k - 12$
3. $5y = 18 - 4y$	4. $4t = -6t - 25$
5. $-5x = 6 + x$	6. $3y = -12y - 30$
7. $-11z = 50 - 6z$	8. $5A = 9A - 8$
9. $0 = 7 - 3b$	10. $m = 12 + 2m$
11. $10k = -7k$	12. $20t = 19 + t$
13. $2y = 20 - 8y$	14. $4x + 12 = -2x + 49$
15. $51x - 13x - 76 = 0$	16. $7m - 5 = 3m - 25$
17. $5y - 3y + 17 = 35 - 4y$	18. $8t + 30 - 3t + 260 = 5 + 2t + 1185$
19. $4x + 15x - x - 5 = x + 6 + 18x - 6$	
20. $9y - 25 = 4y - 5$	21. $t - 6 = 18 - 3t$
22. $11R - 7 - 4R = 5R - 19$	23. $7V + 2 = 24 - 4V$
24. $8x - 5 = 7 + 9x$	25. $3.6y = 26 + y$
26. $7 - 14x + 8 = x$	27. $12a - 3 = 10a + 15$
28. $x + 52 = 648 - x$	29. $3r + 2.5r = 132121$
30. $4t + 2t + t = 2100$	31. $300 + d = 16000 - 3d$
32. $g - 820 = .8g$	33. $5K + 7 = 3K + 6$
34. $3x + 1 = 5x - 8$	35. $5y + 2 = 3y + 7$
36. $7c + 2 - 5c = 8 - 9c + 3c$	37. $2a - 8 + 5a = 2a + 7$
38. $3x - 7 = x + 3 + 2x$	39. $13 - 5c + 7 = 3 - 2c$
40. $5c - 9 = 9c + 42$	41. $5m - 5 = 12m - 9$
42. $7k - 4 + k = 3k - 10$	43. $6t - 1 = 2t + 8$
44. $.6y - 11 = 1.6y + 1$	45. $4.43a - 11 - 2.15a = 34.6$
46. $4.1S + .60 = .7S - .08$	47. $.80y + .84 = -.76$

E. Solve.

1. $60x = 40(x + 1)$
2. $50x = 40(x + 1)$
3. $90x = 60(x + \frac{1}{2})$
4. $90x = 60(x + 1)$
5. $5x + 9 = 2(x + 9)$
6. $20 - x = 2(15 - x)$
7. $16 + x = 2(6 + x)$
8. $2s - 6 = 8(s - 6)$
9. $60t = 40(t + 1.5)$
10. $x = 1 - (x - 3)$
11. $100f = 80(9 - f)$
12. $5y = 73 - (2y + 3)$
13. $n + 1 = 5(6 - n) + 1$
14. $5d = 6(5 + d) - 2$
15. $25(628 - y) + 50y = 21100$
16. $5(142 - t) + 10t = 1110$
17. $5c + 2(4 - c) = 32$
18. $80s + 60(100 - s) = 6400$
19. $90(10 - b) = 110b$
20. $5r = 4(27 - r)$
21. $5n + 10(700 - n) = 5000$
22. $a + 2a + 8a + 5(2a - 5) = 101$
23. $50x + 80(50 - x) = 50 \cdot 60$
24. $5n + 10(2n - 4) = 385$
25. $6(50 - c) = 624 - 15c$
26. $100(x + 4) + 10x + x = 111x + 400$
27. $2(d - 3) = 8 - 3(d - 2)$
28. $2(2x + 1) = 3(x - 5)$
29. $3(x - 1) = 2(x + 2)$
30. $x + 2(x + 1) = 2(x + 2) - 10$
31. $9(160 - p) = 11p$
32. $2t + 3(t + 2) - 1 = 8$
33. $3(2x - 9) = 5(10 - x)$
34. $3(5y - 2) - 5y = 8 - 2(y - 5)$
35. $9a = 7 + 2(1 + 4a)$
36. $4(3c + 7) - 7c = 2(c - 7) - 3$
37. $8 + 3x = 2(5x - 3) + 7x$
38. $4(y - 1) = 5y - 1 + 4(3y - 4)$
39. $10(3a - 1) - 2(a - 3) = 15(2a + 3)$
40. $4(2x - 3) - 24 = 5(x - 1) - 6(2x + 1)$
41. $8(5 - a) - 5(8 - a) = 5(8 - a) + 8(a - 5)$

F. Solve.

1. $\frac{x}{2} - \frac{x}{3} = \frac{x}{6}$

2. $\frac{x}{2} + \frac{x}{3} = \frac{x}{6}$

3. $1 = R - \frac{19}{4}$

4. $x - \frac{1}{5} = \frac{1}{6}$

5. $-\frac{x}{2} = 6$

6. $-\frac{1}{3} = \frac{x}{4}$

7. $\frac{s}{10} = \frac{13}{100}$

8. $\frac{a}{0.3} = 1.2$

9. $-2.1 = \frac{b}{-2.1}$

10. $\frac{x}{4} - 3 = 2$

11. $10C = \frac{1}{4}C - 39$

12. $\frac{x + 3}{4} = 2.5$

13. $\frac{\frac{2}{3}x}{\frac{1}{3}} = \frac{5}{3}$

14. $\frac{2 - x}{\frac{3}{5}} = \frac{5}{3}$

15. $\frac{\frac{1}{2}x}{\frac{1}{2}} = \frac{1}{2}$

16. $\frac{s}{4} = -\frac{1}{2}$

17. $\frac{2x}{9} = \frac{1}{5}$

18. $\frac{x - 1}{3} = \frac{1}{5}$

19. $\frac{7}{4} = \frac{5}{4}x + \frac{1}{2}$

20. $\frac{3}{2} - x = \frac{4}{3}$

21. $\frac{y + 3}{6} = \frac{3}{2}$

22. $\frac{3K - 2}{15} = \frac{2}{3}$

23. $\frac{1}{5}(3y - 12) = \frac{3}{5}$

24. $3a = \frac{1}{2}a + 1$

25. $\frac{1}{2}x - \frac{1}{3}x = \frac{1}{12}x$

26. $\frac{4}{5}(t + 2) = t$

27. $\frac{1}{3}(3x + \frac{x}{2}) = x$

28. $4(\frac{x}{3} + \frac{1}{2}) = \frac{x}{2} + 7$

29. $\frac{\frac{1}{5}R - R}{2} = 2$

30. $\frac{n + 16}{6} = \frac{n + 4}{3}$

31. $2(s + \frac{s}{2}) = 300\%$ of s

32. $\frac{n - 10}{15} = 0$

33. $\frac{s + 2}{12} = \frac{s}{3} - \frac{s}{4}$

34. $\frac{x + 0.5x}{2} = x + 1$

35. $\frac{P + 10\% \text{ of } P}{11} = P$

36. $\frac{6 - x}{2} = -7$

37. $\frac{P + 10\% \text{ of } P}{11} = 5$

38. $\frac{x}{10} = \frac{x}{2} - \frac{x + 1}{2}$

39. $\frac{x}{2} - \frac{x}{5} = 5$

40. $6(x - 2x + 3) = \frac{0}{2}$

41. $40 - x = \frac{3}{5}(50 + x)$

42. $\frac{a - 3}{9} = \frac{3 + a}{8}$

43. $\frac{3}{4}z = \frac{z + 6}{2}$

44. $\frac{a + 6}{7} = \frac{a - 9}{2}$

45. $a = \frac{3}{4}(210 - a)$

46. $\frac{5}{6}b = \frac{3}{4}b + 1$

47. $\frac{m}{40} = \frac{m}{50} + 1$

48. $\frac{r}{7} + \frac{r}{4} = 11$

49. $\frac{w}{3.5} + \frac{w}{35} = 2.2$

50. $\frac{h}{2} + \frac{h}{5} = 1$

51. $.08(1 + x) = .10$

52. $[.7q - .7(1)] + 1 = .85q$

53. $\frac{n + 4}{4} - \frac{3n - 9}{7} = \frac{1}{2}$

54. $\frac{7n + 4}{10} = \frac{n + 11}{4} - 1$

55. $.03x + .02(15 - x) = .0275(15)$

56. $x + 70 = 1.20[(x + 5) - 70]$

57. $2(y - 6) = \frac{1}{2}(7 + 4y)$

G. Solve.

1. $\frac{3.5}{r} = 1$

2. $\frac{3(x + 1)}{2} = 3$

3. $\frac{5 + a}{8 + a} = \frac{4}{5}$

4. $\frac{180}{n + 5} = \frac{160}{n}$

5. $\frac{b + 4}{b - 8} = 5$

6. $\frac{2}{3} = \frac{m}{2m + 4}$

7. $\frac{500}{5m} + \frac{1}{2} = \frac{125}{m}$

8. $\frac{1}{7} + \frac{1}{5\frac{1}{2}} = \frac{1}{x}$

9. $\frac{1}{6} + \frac{1}{4} = \frac{1}{t}$

10. $\frac{300}{4r} = \frac{300}{r} - 4\frac{1}{2}$

11. $\frac{360}{4c} = \frac{360}{c} - 6$

12. $\frac{150}{d} - 5 = \frac{150}{3d}$

13. $\frac{3c}{2} + \frac{8 - 4c}{7} = 3$

14. $\frac{4}{m - 3} = \frac{3}{m}$

15. $\frac{5}{b - 3} = \frac{7}{b + 3}$

16. $\frac{3}{2x + 3} = \frac{5}{6x - 1}$

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17. $\frac{9}{d + 2} = \frac{6}{d + 5}$

18. $\frac{12}{3k - 7} = \frac{3}{7 - k}$

19. $\frac{182}{r + 12} = \frac{182}{r + 12}$

20. $\frac{23}{t - 7} = \frac{23}{7 - t}$

21. $\frac{1\frac{1}{3}}{r} + \frac{1\frac{1}{3}}{3} = 1$

22. $\frac{u}{5} + \frac{2u}{5} = 18$

23. $\frac{z}{4} - \frac{z}{3} = \frac{11}{12}$

24. $\frac{d + 1}{2} = \frac{d - 5}{3}$

25. $3e - \frac{4e}{5} = 22$

26. $\frac{s - 2}{4} - \frac{s - 4}{6} = \frac{2}{3}$

27. $\frac{6q - 7}{5} = \frac{3q - 1}{2}$

28. $\frac{f + 5}{f - 3} + \frac{4}{f - 3} = 5$

29. $\frac{n + 3}{2n} = 5$

30. $\frac{-3}{r + 10} = \frac{1}{r}$

31. $\frac{n}{n + 7} = \frac{2}{3}$

32. $\frac{h}{200} + \frac{h}{160} = 9$

33. $\frac{2 + k}{7} + \frac{2 + k}{13} + \frac{k}{7} = \frac{1}{13}$

34. $\frac{2p + 1}{2p} - \frac{3p - 2}{3p} = \frac{7}{12}$

35. $\frac{g + 2.1}{g + 3} = \frac{1.90}{2}$

36. $\frac{3 + j}{8 + j} = \frac{2}{3}$

37. $\frac{d + 5}{4d + 2} = \frac{1}{2}$

38. $\frac{r + 18}{4} - \frac{3r}{7} + \frac{r + 2}{8} = 4$

39. $\frac{120}{t} = \frac{120}{\frac{3}{4}t} - 1$

40. $\frac{y - 3}{4} = \frac{2y - 5}{5} + 1$

41. $\frac{2}{\frac{2}{3}r} + \frac{45}{r} = 8$

42. $\frac{1}{3}(e + 5) - 4 = \frac{e - 8}{4} - \frac{1}{2}$

43. $\frac{28.4}{9 - n} = \frac{4n}{9 - n} - \frac{18}{5}$

44. $\frac{3}{4}\left(\frac{2}{1 - a}\right) + \frac{1}{2} = \frac{24a - 31}{4(1 - a)}$

H. Solve each of the following equations for the pronumeral indicated.

1. $P = 3s$; s
2. $P = 2n + b_1 + b_2$; n
3. $C = 2\pi r$; r
4. $P = 2n + b_1 + b_2$; b_1
5. $P = 4a + 5b$; a
6. $s = att + bt$; b
7. $y = 3x - b$; x
8. $2x = y + b$; y
9. $mx - a = 6b$; a
10. $bx - y = d$; y
11. $N = 3y - z + 2x$; y
12. $N = 5r - s + 2t$; s
13. $d = 3(c + d) - a$; a
14. $P = 4(a - b) - 3c$; b
15. $S = 2\pi rh$; r
16. $S = 2\pi rh$; h
17. $S = \frac{1}{2}ap$; a
18. $x = \frac{y}{z}$; y
19. $I = \frac{c}{d}$; c
20. $\frac{m}{a} = c$; a
21. $rg = a + S(r - 1)$; S
22. $rg = a + S(r - 1)$; r
23. $K = \frac{1}{6}\pi h(a + b)$; h
24. $K = \frac{1}{6}\pi h(a + b)$; a

I. Solve each of these equations for 'y'.

1. $x - y = 2$
2. $2a + 3y = 7$
3. $n - y + 18 = 0$
4. $4z + 3y = 6$
5. $m - 5y = 35$
6. $3y - 5r = 24$
7. $7a + 3 - 2y = 12 - 5y - 4a$
8. $9c + 10y - 6 = 12y - 3c + 8$
9. $7(e - 6) + 8(y - 2) = 12$
10. $5(r + y - 3) + 9(y - r + 2) = 0$
11. $\frac{3}{2c} - \frac{4}{3y} = \frac{4}{3}$
12. $1 = \frac{d}{5} - \frac{y}{3}$
13. $\frac{1}{m} + \frac{1}{n} = \frac{1}{y}$
14. $\frac{1}{m} + \frac{1}{y} = \frac{1}{n}$
15. $\frac{1}{y} + \frac{1}{c} = 8$
16. $\frac{5}{2a} - \frac{7}{3y} = \frac{1}{6}$
17. $\frac{c}{y} = \frac{10}{1 + y}$
18. $\frac{12}{n - y} = \frac{13}{n + y}$

J. Complete with the simplest expressions you can to make true sentences.

1. For each k, the sum of the number 3 less than k and k is ____________.

2. For each x, the sum of x and the number $^{-}5$ more than the quotient of 0 by x is __________.

3. For each e, the number which is 3.5 times larger than e is ____________.

4. For each w, the difference of a number $^{-}3$ times as large as w from w is ____________.

5. For each q, the number which exceeds q by 77 is __________.

6. For each a, the number which is 1% of a is ____________.

7. For each s, the number which is 200% [of s] larger than s is ____________.

8. For each z, 50% of the number which exceeds z by $^{-}3$ is ____________.

9. For each f, the number which is 35% less than f is __________.

10. For each c, for each d, the product of 100% of c and 40% of d is __________.

11. For each g, the product of $-6g$ and $\frac{1}{30}$ is ____________.

12. For each j, j less than $^{-}2$ is ____________.

13. For each r, the difference of the quotient of $^{-}5r$ by 10 from the product of r and 3 is ____________.

14. For each $m \neq 0$, the quotient of $^{-}45$ by the quotient of $^{-}25m$ by 5 is ____________.

15. For each number of arithmetic $x > 7$, if George is x years old now, he was ____ years old 7 years ago.

16. For each number y of arithmetic, if Herb is $y + 2$ years old now and Henry is three times as old as Herb, then Henry will be ____ years old 3 years from now.

17. For each number of arithmetic $z > 20$, if John's age is 10 years less than one half Dotty's age, and Dotty is z years old, then John is ____ years old.

18. For each number of arithmetic $p < 26$, if Ruth is 28 years old now and she was 2 years older than Nancy p years ago, then Nancy is now ____ years old.

19. For each number y of arithmetic, if a 45 r.p.m. record costs y cents, and a 78 r.p.m. record costs 10 cents less than a 45 r.p.m. record, then the cost of four 45 r.p.m. records and three 78 r.p.m. records will be ____ dollars.

20. For each number n of arithmetic, there are ____ quarts in n gallons.

21. For each number p of arithmetic, p pints and three times that many quarts together contain ____ cups. [Hint: there are 2 cups in one pint.]

22. For each number s of arithmetic, there are ____ yards in s inches.

23. For each number f of arithmetic, f feet, 5 times as many yards, and 6 times as many inches [as feet] together make ____ inches.

24. For each number d of arithmetic, there are ____ cents in d dimes.

25. For each whole number n of arithmetic, for each whole number q of arithmetic, there are ____ cents in a total of n nickels and q quarters.

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26. For each whole number r of arithmetic, there are ____ dollars in r nickels.

27. For each whole number g of arithmetic, if Joe buys erasers at the rate of 3 erasers for 10 cents and sells them to his friends at the rate of 5¢ each, then his profit on the sale of 3g erasers is ____ cents.

28. For each number c of arithmetic, if the cost price of an article is c dollars and the margin is 22% of the cost price, then the selling price is _____ dollars.

29. For each number g of arithmetic, if one of the shorter sides of a rectangle is g units long, and a side of an equilateral triangle has the same length as this shorter side of the rectangle, then the perimeter of the equilateral triangle is ____.

30. For each number of arithmetic $a \geq 5$, if a longer side of a rectangle is a units, and a shorter side of this rectangle is 2 units less than one half this longer side, then the perimeter of the rectangle is ____.

31. For each t, the sum of t and 7 multiplied by the product of 5 by t is ____.

32. For each y, the difference of the product of 9 by y from $\frac{1}{8}$ of y is ____.

33. For each number of arithmetic $N > 80$, if Mr. Smith earns N dollars per year and Mr. Harris earns $60 less than three fourths of what Mr. Smith earns, then Mr. Harris earns ____ dollars a year.

34. For each number r of arithmetic, you can walk ____ miles in r hours at the average rate of 2 miles per hour.

35. For each number m of arithmetic, you can travel ____ miles in $5\frac{1}{2}$ hours at the average rate of m miles per hour.

36. For each number w of arithmetic, for each number s of arithmetic, you can travel ____ miles in w hours at the average rate of s miles per hour.

37. For each number m of arithmetic, it takes ____ hours for a freight train to travel m miles if its average rate is $45\frac{1}{2}$ miles per hour.

38. For each number p of arithmetic, to travel p miles in $6\frac{1}{2}$ hours, you must walk at an average rate of ____.

39. For each number y of arithmetic, if you invest y dollars at $4\frac{1}{2}\%$, the annual income on y dollars is ____.

40. For each number of arithmetic $r \leq 2500$, the annual income on $(2500 - r)$ dollars invested at 5.5% is ____ dollars.

41. For each number of arithmetic $k \leq 5000$, the total annual income on $5000, k dollars of which is invested at $3\frac{1}{2}\%$, and the rest at 2%, is ____ dollars.

42. For each number of arithmetic $y > \frac{4}{3}$, if the base of an isosceles triangle is y inches long and the perimeter is $3y - 4$, then the length of one of the two sides of equal length is ____.

43. For each number b of arithmetic, if the circumference of a circle is 49b, a diameter measures ____.

44. For each number v of arithmetic, if a square has perimeter v, then a pentagon, each of whose sides is 2 units more than a side of the square, will have perimeter ____.

45. For each number of arithmetic $n > 0$, if Mary can clean the house in n hours, then she can clean ____ of the house in 1 hour.

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46. For each number p of arithmetic, if Ned can paint a certain barn in 12 hours, then he can paint ____ of this barn in p hours.

47. For each number p of arithmetic, if Ned can paint a certain barn in 12 hours and Guy can paint this barn in 10 hours, then together they can paint ____ of this barn in p hours.

48. For each number of arithmetic $w \geq 5$, if an inlet pipe can fill a tank in 5 hours and an outlet pipe can empty this tank in w hours, then when both pipes are turned on [starting with an empty tank], ____ of the tank is filled at the end of 1 hours.

49. For each number of arithmetic $r > 0$, if Bruce runs $\frac{2}{3}$ as fast as Cecil, and if Cecil runs r feet per second, then Bruce takes ____ seconds to run 150 feet.

50. For each whole number c of arithmetic, a pile of nickels, dimes, and quarters which contains c dimes, three times as many nickels, and 6 more quarters than nickels, is worth ____ cents.

51. For each number h of arithmetic, if the measure of the width of a rectangle is $\frac{3}{4}$ the perimeter h then the length measures ____.

52. For each whole number of arithmetic $w > 96$, if 48 more than one half of the w students in a study hall are dismissed for lunch at 11:15, and one third of the remaining students are dismissed at 11:25, then there are ____ students left in the study hall.

53. For each number f of arithmetic, if a person can climb a smokestack in f minutes and descend 3 times as fast, the total time required for the trip up and down [no resting] is ____ minutes.

54. For each number p of arithmetic, if p pounds of peaches at 49 cents per pound are mixed with 2 more than 3 times as many pounds of smaller peaches at 39 cents per pound, then the resulting pile of peaches is worth ____ cents per pound.

55. For each number k of arithmetic, if k pounds of cookies worth 36 cents per pound are mixed with 10 pounds of cookies worth 30 cents per pound, the resulting mixture contains ____ pounds worth 40 cents per pound.

56. For each number of arithmetic $q < 34$, the total cost of q pounds of coffee at 87 cents per pound and $(34 - q)$ pounds of coffee at 95 cents per pound is ____ cents.

57. For each a, if a is an integer, ____ is the second larger integer.

58. For each n, if n is an even integer, ____ is the largest even integer smaller than n.

59. For each s, if s is an odd integer, the sum of the next two consecutive odd integers is ____.

60. For each number g of arithmetic, g gallons of a salt and water solution is 20% salt, so the solution contains ____ gallons of salt.

61. For each number g of arithmetic, g gallons of a 40% salt solution contains ____ gallons of water.

62. For each number d of arithmetic, if 5 gallons of a 20% salt solution are added to d gallons of a 25% salt solution the new mixture contains ____ gallons of salt.

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63. For each number j of arithmetic, if j liquid ounces of a 5% argyrol solution are added to 3 liquid ounces of a 10% argyrol solution, the new mixture is a ____ per cent argyrol solution.

64. For each number b of arithmetic, if b ounces of Brazil nuts are added to 2.5 pounds of a nut mixture which contains 20% Brazil nuts, the new mixture contains ____ pounds of Brazil nuts.

65. For each number r of arithmetic, if r quarts of a 50% antifreeze solution are mixed with twice as many quarts of a 75% antifreeze solution, the result is a mixture containing ____ quarts of antifreeze.

66. For each number n of arithmetic, if 5 quarts of an n% alcohol solution are combined with 2 quarts of pure alcohol, the new mixture is ____ per cent alcohol.

67. For each number of arithmetic $d > 0$, if it takes Sandy d hours to ride her bicycle 12 miles and if Jody can ride her bicycle $\frac{1}{3}$ mile per hour faster than Sandy, then it takes Jody ____ hours to ride $\frac{1}{3}$ as far as Sandy.

68. For each whole number p of arithmetic, if p people share equally in the cost of a $55 camping trip, a trip for 2 more than three times this number of people should cost ____ dollars.

69. For each number p of arithmetic, if 2000 pounds of milk containing 5% butterfat are added to p pounds of milk containing 2% butterfat, the new mixture will contain ____ pounds of butterfat.

K. Solve these problems.

1. How many dimes and quarters are there in a collection of 60 of these coins if the collection is worth \$11.40?

2. How many pounds of cleaning compound worth 10¢ a pound should be mixed with cleaning compound worth 25¢ a pound to make a mixture of 60 pounds worth 19¢ a pound?

3. Jerry selected a number, divided it by 4, subtracted -3 from the result, multiplied the difference by 2. His final result was 16. What number did he select?

4. Six years ago Walter was 12 years older than Eddie. In 7 years Walter will be twice Eddie's age then. How old is each now?

5. Mr. King has 29 minutes to catch a train 18 miles from home. He starts out at a rate that will enable him to get to the depot 5 minutes early. However, he has an unexpected 11-minute delay at a railroad crossing 6 miles from home. For the remainder of the trip, he increases his speed, hoping to catch the train; however, he arrived at the depot just as the train pulled out. At what rate did he travel during the last part of the trip?

6. One gallon of an ammonia solution contains 22% ammonia, the rest water. How much water must be added to make the solution contain 18% ammonia?

7. Two trains are 480 miles apart at 6 p.m. and are traveling toward each other on parallel tracks. If one is traveling 20 miles an hour faster than the other, at what time will they meet?

(continued on next page)

8. A truck makes a trip at a uniform speed. Had the rate been 5 miles per hour faster, the trip would have taken 24 minutes less time, or the driver could have gone 22 miles farther. At what speed was the truck traveling?

9. How much candy worth 40¢ a pound should be mixed with candy worth 60¢ a pound to make 200 pounds of mixture worth 52¢ a pound?

10. Two planes are 30 miles apart and going in opposite directions. One travels 25 miles per hour faster than the other. At the end of 4 hours they are 1018 miles apart. If during this time each has maintained a steady rate, what is the rate of each?

11. A painter could paint a certain house by himself in 16 hours. It would take his assistant about 24 hours to paint the same house. The painter begins at 8 a.m. At 10 a.m. his assistant joins him and they finish painting the house. How long did the assistant work?

12. A man invests a certain amount of money at 4%. He invests $1000 more than twice that amount at 5%. If the total amount of interest he receives at the end of a year is $540, how much did he invest at each rate?

13. It takes a man who rows 6 miles an hour in still water 10 hours longer to row a certain distance upstream than it does for him to row back the same distance. If the complete trip takes 24 hours, what is the rate of the current?

14. Water is flowing out of a tank through a single pipe. A second pipe is then opened so that the rate of outflow is increased by 20%. With both pipes open the amount of water which flows out in 24 minutes is 18 gallons. At what rate was the water flowing out of the first pipe?

15. John has some dimes and nickels; the number of dimes is 6 more than twice the number of nickels. These coins are worth \$2.90. How many of each denomination does he have?

16. A tank has 2 inlet pipes and one outlet pipe. Each of the 2 inlet pipes alone could fill the tank in 15 minutes. The outlet pipe could empty the tank in 12 minutes. If all three pipes are open, how long would it take to fill the tank?

17. If a number is divided by $-\frac{1}{2}$, the result is 6. What is the number?

18. A man needs 90 yards of fencing for his lot. He needs twice as much for the northern edge as he does for the western. He needs 5 more yards for the southern than he does for the northern edge. The eastern edge will take as much as the western and the southern together. How much does he need for each side?

19. John leaves home for school which is 2 miles away. At the end of 5 minutes he figures that the distance from home to school is $\frac{1}{2}$ mile more than twice as far as he has already come. How far has he come?

20. On Monday, Dan spent $\frac{1}{4}$ of his weekly allowance for a haircut and new shoelaces. On Tuesday, he bought a sundae for 30¢ and then paid 80¢ more than $\frac{1}{3}$ of what remained for gas and one quart of oil. A movie Thursday evening lacked 10¢ of costing him $\frac{1}{2}$ of what he had left. A hamburger and Coke afterward cost 35¢. Friday he had to pay $\frac{2}{3}$ of what was left to have a flat fixed. This left him only 15¢. What was his weekly allowance?

21. Ralph's car is 2 years less than 3 times as old as Bob's and $1\frac{1}{2}$ years more than one half as old as Gene's. Last year Ralph's car was $\frac{3}{5}$ as old as Gene's was. How old is each car now?

(continued on next page)

22. An alloy of 60 pounds of zinc and copper is 65% copper. How much copper must be added to make an alloy which is 79% copper by weight?

23. A grain dealer mixed wheat costing $1.64 a bushel with wheat costing $1.53 a bushel, and sold the mixture for $1.77 a bushel. He sells 660 bushels at this price and makes a profit of $99. How many bushels of each did he use?

24. A solution which is 82% sodium dichromate salt, by weight, and the rest water is mixed with 16 pounds of a mixture containing 42% sodium dichromate salt and the rest water. In order to decrease the percentage of water in the final mixture to 26%, 4 pounds of water are evaporated. How many pounds of the first solution were used?

25. On an algebra test, 32 more pupils passed than failed. On the second test, one student who had passed the first test failed the second but 3 of those failing the first passed the second. Altogether, on the last test, 95% passed. How many students took the algebra tests?

26. A life insurance company offers its salesmen a commission of 55% of the first year's premium on each policy they sell, 10% of the second year's premium, and 5% of each succeeding year's premium for the next 8 years. If a holder of a certain policy pays the same yearly premium for 10 years, the salesman receives a total of $420 in commission on this policy. What was the yearly premium on the policy?

27. The ratio of Jim's hourly wage to John's hourly wage is 4:5. If they could each work 8 hours, the total amount they would receive would be $43.20. How much does Jim earn per hour?

28. A company that has failed in business is able to pay its creditors at the rate of 25 cents on the dollar. If the company had been able to collect a certain debt of $1200, it could have paid 28 cents on the dollar. How much did the company owe at the time of failure?

29. A dump cart can haul enough gravel to fill a pit in 8 days. A motor truck can do it in 6 days. How long would it take 2 dump carts and one truck working together to fill the pit?

30. A container is filled with a solution which is 82% alcohol and 18% water. All but 8 quarts are spilled. The container is then filled with pure alcohol making the new solution 92% alcohol. The solution is then mixed with 12 quarts of a solution of 72% alcohol. What is the per cent alcohol in this new solution?

☆31. A well digger can shovel as much dirt in 3 hours as his partner who stays above ground can shovel into a truck in 5 hours. Assume that they both work an eight-hour day. At the end of how many hours must the faster man stop digging to help the slower one shovel the dirt into the truck so that all of the dirt is in the truck at quitting time? [Assume that the faster man shovels dirt into the truck at the same rate as he shovels it out of the hole.]

☆32. Two boys run around the block in which their school gymnasium is situated; when they run in the same direction, the faster boy overtakes the slower boy every 48 seconds. When they run in opposite directions, they meet every 16 seconds. How long does it take the slower boy to run around the block?

L. Expand.

1. $(x + 5)(x + 7)$
2. $(n - 8)(n + 2)$
3. $(c + 12)(c - 5)$
4. $(y + 6)^2$
5. $(y + .6)^2$
6. $(y - 6)^2$
7. $(d - 11)(d + 1)$
8. $(r - 4)(r + 5)$
9. $(3a + 2)(4a - 3)$
10. $(k + 12)^2$
11. $(x - 13)^2$
12. $(z + 4)^2$
13. $(p - 2)^2$
14. $(r + 11)^2$
15. $(m + 8)^2$
16. $(t - \frac{1}{8})^2$
17. $(q - .3)^2$
18. $(b + \frac{1}{4})^2$
19. $(4x + 3)^2$
20. $(5y + 1)^2$
21. $(3z - 1)^2$
22. $(6n + 4)^2$
23. $(3m + 2)^2$
24. $(3y - 7)^2$
25. $(b - 2)(b + 1)$
26. $(x + 7)(x - 3)$
27. $(y - 7)(y + 3)$
28. $(a - 10)(a + 10)$
29. $(g - 14)(g + 4)$
30. $(y + 13)(y + 2)$
31. $(x + \frac{3}{4})^2$
32. $(13r_1 + 2r_2)^2$
33. $(p + q)^2$
34. $(p - q)^2$
35. $(5 - s)^2$
36. $(q - p)^2$
37. $(2x + 3)(x + 6)$
38. $(3n - 1)(6n + 7)$
39. $(7e + 2)(2e - 6)$
40. $(9u - 5)(9u - 5)$
41. $(10f + 3)(4f - 7)$
42. $(60x - 2)(2x + 1)$
43. $(5 + 2p)^2$
44. $(9 - 4t)^2$
45. $(11c + 3)^2$
46. $(3d - 8)^2$
47. $(10W - v)^2$
48. $(2a_1 - 9a_2)^2$
49. $r(16q + 6)(3q - 2)$
50. $(5b - 4)(6b - 7)$
51. $(6 + 5y)(8 - 3y)$
52. $(-x + 3)(x + 11)$
53. $(15 + r)^2$
54. $(3p + 2q)^2$
55. $n(2r + 3)(r - 2)$
56. $c(2a + 5b)^2$
57. $y(7x + 3)(1 + 5x)$
58. $(-4p - 2)(6p + 8)$
59. $(-10x + 11)(-x - 3)$
60. $s(11r + 1)(11r - 1)$

61. $(3m - \frac{1}{2})(3m + \frac{1}{2})$

62. $(7k - 2)(k + 2)$

63. $(rs + 7)(rs - 7)$

64. $(a + b)(c + d)$

65. $(\square + a)(\triangle + a)$

66. $(\triangle + b)^2$

67. $(\triangle + \square y)^2$

68. $(\hexagon + \square c)(\bigcirc + \triangle c)$

M. Factor.

1. 24
2. 36
3. 393
4. 16
5. $42x$
6. $18a^2b$
7. $50d$
8. $57xy$

*

9. $3x + 9$

10. $6y - 24$

11. $14p + 42$

12. $9x + 81y$

13. $13q - 65r$

14. $10s - 2t + 22u$

15. $ab - ac$

16. $4yz - 20z$

17. $15pq - 18pr - 6p$

18. $9b^2 - 21b$

19. $11xy - 44x^2$

20. $-7mn^2 - 56mn^2o$

*

21. $k^2 + 7k + 12$

22. $y^2 + 4y + 4$

23. $x^2 + 9x + 8$

24. $p^2 + 6x + 8$

25. $b^2 + 23b + 22$

26. $a^2 + 47a + 90$

27. $q^2 + 12q + 35$

28. $r^2 + 20r + 36$

29. $x^2 + 18x + 32$

30. $c^2 + 29c + 100$

31. $g^2 + 11g + 18$

32. $e^2 + 4e - 5$

33. $n^2 + 22n + 131$

34. $s^2 + 24s + 144$

35. $t^2 + 40t + 144$

*

(continued on next page)

36. $v^2 + 7v - 18$

37. $k^2 - 7k - 18$

38. $w^2 + w - 12$

39. $x^2 - 6x + 8$

40. $y^2 - 6y + 9$

41. $p^2 - 49$

42. $q^2 + 14q + 33$

43. $t^2 - 9t + 18$

44. $a^2 - 15a + 36$

45. $c^2 + 9c - 36$

46. $f^2 - 4$

47. $h^2 - h - 42$

48. $i^2 + 20i + 51$

49. $b^2 - 15b + 50$

50. $d^2 - 3d + 28$

51. $p^2 - 169$

52. $s^2 + 2s - 15$

53. $27 + 12u + u^2$

54. $55 - 16w + w^2$

55. $40 - 3e - e^2$

56. $x^2 + 27x + 72$

57. $14 - 5y - y^2$

58. $\triangle^2 - 16$

59. $x^2 - 2xy + y^2$

60. $m^2 - m - 2$

61. $75 + 22n - n^2$

62. $p^2 + 33p + 62$

*

63. $35t^2 + 46t + 15$

64. $16x^2 + 58x + 7$

65. $12x^2 + 41x + 24$

66. $25 + 70y + 49y^2$

67. $3r^2 + 10r + 3$

68. $12u^2 + 56u + 15$

69. $48m^2 + 86m + 35$

70. $100t^2 + 60t + 9$

71. $100t^2 - 9$

72. $35p^2 + 35p - 24$

73. $81m^2 - 1$

74. $121r^2 - 44r + 4$

75. $25k^2 + 10km + m^2$

76. $9x^2 - 6xy + y^2$

77. $56p^2 - 113p + 56$

78. $1 - 64n^2$

79. $3k^2 + 10kj + 3j^2$

80. $6m_1^2 - 13m_1m_2 - 15m_2^2$

N. Solve these equations.

1. $n^2 - 5n - 24 = 0$
2. $12 + a^2 + 13a = 0$
3. $r^2 = 64r$
4. $b^2 + 49 + 14b = 0$
5. $3s^2 = 14 - 19r$
6. $2c^2 = 7c + 9$
7. $x^2 + x = 20$
8. $14 + 3d = 2d^2$
9. $6x^2 - 32x + 10 = 0$
10. $x(x + \frac{1}{6}) = \frac{35}{6}$

O. For each exercise, find the approximation correct to 3 decimal places, and the approximation correct to the nearest 0.01.

1. $\sqrt{47}$
2. $-\sqrt{34}$
3. $\sqrt{6}$
4. $\sqrt{5}$
5. $\sqrt{287}$
6. $\sqrt{0.003}$
7. $\sqrt{4786}$
8. $\sqrt{264.1}$
9. $\sqrt{7}$
10. $-\sqrt{23}$
11. $\sqrt{0.51}$
12. $\sqrt{3782}$

P. Simplify.

1. $\sqrt{18}$
2. $\sqrt{27}$
3. $\sqrt{128}$
4. $\sqrt{242}$
5. $\sqrt{45}$
6. $\sqrt{180}$
7. $\sqrt{8}$
8. $\sqrt{12}$
9. $\sqrt{288}$
10. $\sqrt{500}$
11. $\sqrt{363}$
12. $\sqrt{108}$
13. $\sqrt{.49}$
14. $\sqrt{.98}$
15. $\sqrt{192}$
16. $\sqrt{1.92}$
17. $\sqrt{18} + \sqrt{8}$
18. $\sqrt{5} + \sqrt{20}$
19. $\sqrt{27} + 2\sqrt{12}$
20. $\sqrt{28} \times \sqrt{7}$
21. $\sqrt{6} \times \sqrt{24}$
22. $\sqrt{\frac{1}{3}} \times \sqrt{300}$
23. $\sqrt{72} \div \sqrt{24}$
24. $\sqrt{180} \div \sqrt{5}$
25. $\sqrt{300} + \sqrt{192} - \sqrt{75}$
26. $\sqrt{108} - \sqrt{12} - \sqrt{3}$
27. $\frac{1}{3}\sqrt{27} + \frac{1}{2}\sqrt{48} - \frac{1}{6}\sqrt{108}$
28. $\sqrt{96} - 2\sqrt{6} + 3\sqrt{24}$
29. $5\sqrt{2} \times 3\sqrt{50} \times 4\sqrt{16}$
30. $4\sqrt{75} \times 5\sqrt{48} \times 4\sqrt{36}$
31. $(\sqrt{70})^2 \times (2\sqrt{5})^2$
32. $(\sqrt{6} - \sqrt{5})(\sqrt{6} + \sqrt{5})$

Q. Simplify.

1. $\sqrt{25n^2}$
2. $\sqrt{144b^2}$
3. $-\sqrt{9r^2}$
4. $-\sqrt{121t^2}$
5. $\sqrt{169x^2y^2}$
6. $-\sqrt{225c^2d^2}$
7. $-\sqrt{196(h+2)^2}$
8. $\sqrt{a^2 + 12a + 36}$
9. $\sqrt{n^2 - 8n + 16}$
10. $\sqrt{p^2 - 16p + 64}$
11. $\sqrt{49 - 14r + r^2}$
12. $\sqrt{d^2 - 2dc + c^2}$
13. $\sqrt{p^2 + 2pq + q^2}$
14. $\sqrt{16y^2 - 40y + 25}$
15. $\sqrt{9n^2 - 48na + 64a^2}$
16. $\sqrt{9r^2 - 24rs + 16s^2}$
17. $\sqrt{(a-7)^2(a+2)^2}$
18. $\sqrt{5b^2(b^2 + 20b + 100)}$
19. $\sqrt{a^2 + b^2}$
20. $\sqrt{a^2 - b^2}$

R. Solve these equations and inequations.

1. $\frac{15}{a} + 2 = a$
2. $-8 + \frac{16}{y} = 15y$
3. $1 - \frac{15}{n} + \frac{26}{n^2} = 0$
4. $\frac{29}{4c} + 1 = \frac{6}{c^2}$
5. $\frac{18}{x-3} - 6 = x + \frac{x+4}{x-3}$
6. $1 + \frac{5}{d+4} = \frac{d+6}{d+4} - d$
7. $(s+6)^2 = 2(6s + 30.5)$
8. $r(r+7) = 6 + 6(11 + r)$
9. $\frac{2x-5}{2x+5} = \frac{x-1}{2x+2}$
10. $\frac{8x+9}{3x+4} = \frac{7x+3}{5x+9}$
11. $14(x^2 + 2) \geq 5(1 - x)$
12. $|x|^2 - 5|x| + 6 < 0$

HIGH SCHOOL MATHEMATICS

Unit 4.

ORDERED PAIRS AND GRAPHS

UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS

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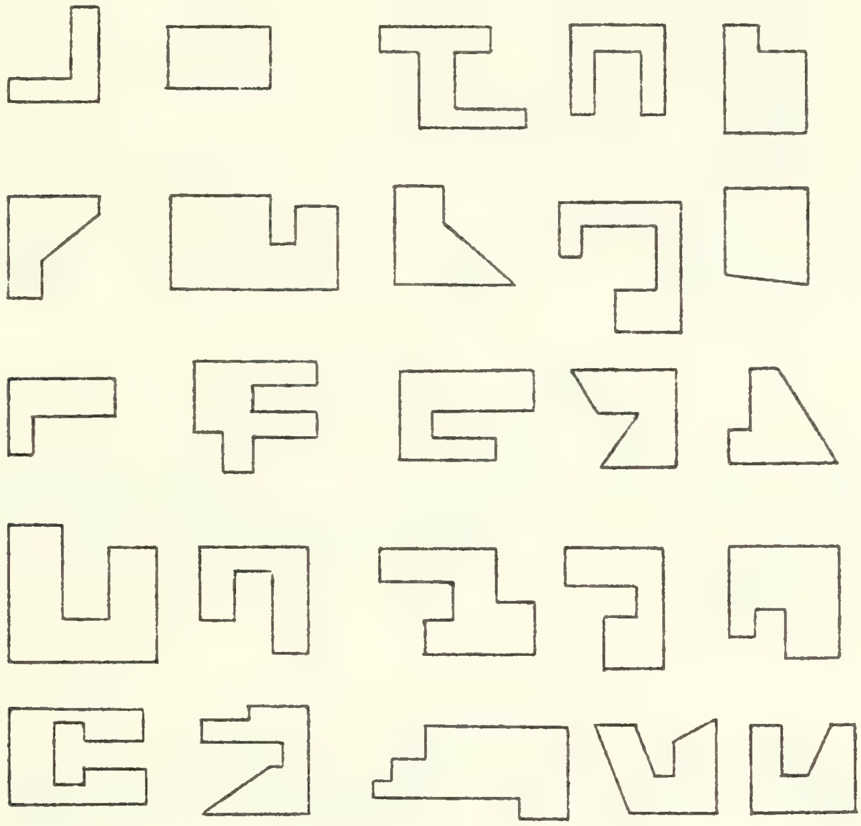
TABLE OF CONTENTS

Locating buildings.--Suppose that a big manufacturing plant has 25 buildings on its grounds. A map of the grounds looks like this:



Suppose you had to make up 25 names for these buildings so that a newcomer to the plant could find his way around as quickly and as easily as possible. Of course, you could just use 25 names like 'tool and die building' or 'administration building' or 'spare parts warehouse'. Such names would tell a newcomer something about what went on in the buildings, but they would not help him learn where the buildings were located.

Names which are easy to remember and which could be used in locating the buildings are the numerals for whole numbers from 1 through 25. These numerals are already known (to most people) and, if they were used to name the buildings in some order, say, like this:

21	22	23	24	25
16	17	18	19	20
11	12	13	14	15
6	7	8	9	10
1	2	3	4	5

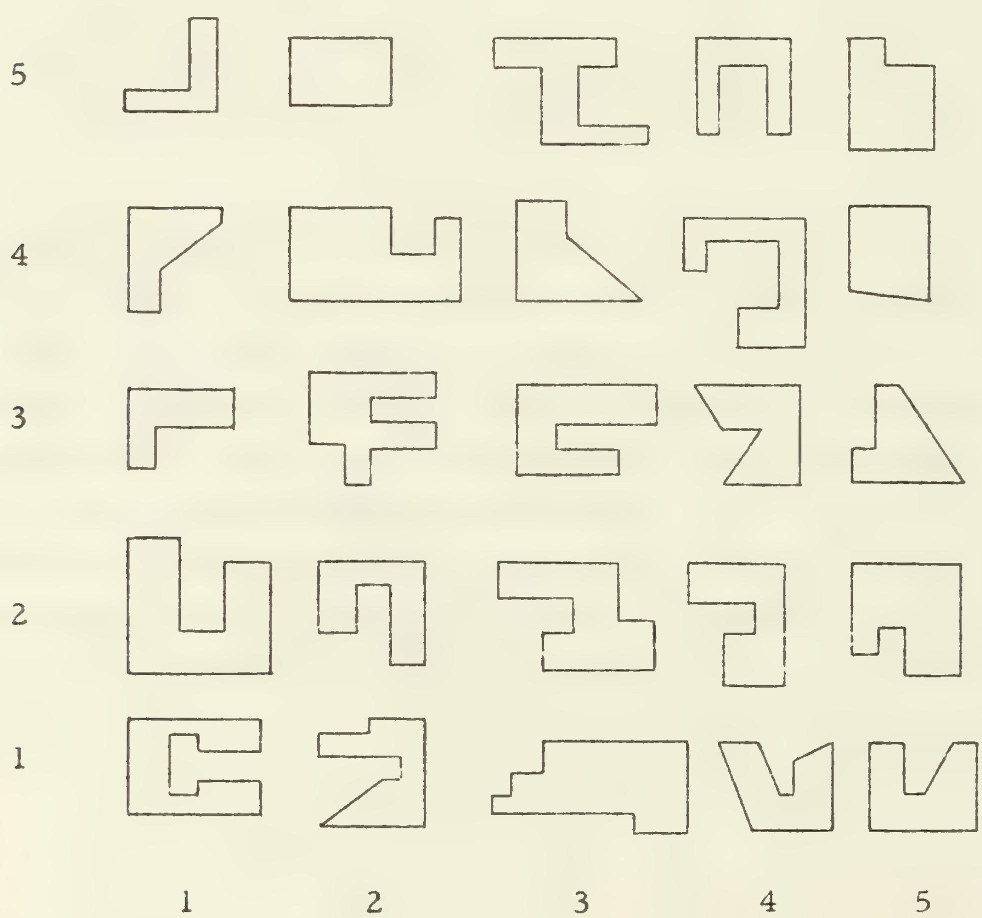
it would be easy to learn where the buildings were located. If you told a newcomer to go to building 14, he would think:

> Building 14; there are 5 buildings in each row, so the first row is 1 to 5, the second row is 6 to 10, and the third row is 11 to 15. So, building 14 is the fourth building in the third row.

Notice that to tell himself the location of building 14 he used a single number, 14, to get a pair of numbers:

building 4 in row 3.

Since you think of the building in terms of "which building in which row", you might just as well have named them that way in the first place. Say, like this:



Instead of 'building 14', you will now say: building 4 in row 3. Instead of assigning a single number to each building, you are describing the location of each building in terms of two numbers, its "column number" and its "row number". But just the two numbers themselves are not enough. To use them to locate a building, you must know which of the two numbers is its column number [and which is its row number]. So, it would not be correct to say that you are now locating each building merely by assigning a set of numbers to it [instead of, as at first, a single number]. To direct a newcomer to a particular building, you must say:

Go to building 4 in row 3.

But, for those in the plant who are used to this numbering system, you would probably get in the habit of saying:

Go to building 4, 3

or even:

Go to 4, 3.

You would expect that such people understood your convention that the number you mention first is the column number of a building and the number you mention second is its row number. What this amounts to is that you are now assigning to each building an ordered pair of numbers. In describing the location of a building [in the case of the instructions above] you do not use the set whose members are the numbers 3 and 4, but the ordered pair whose first component is 4 and whose second component is 3. It is customary to name this ordered pair '(4, 3)'. Although $\{4, 3\} = \{3, 4\}$,

$$(4, 3) \neq (3, 4)$$

because the first component of (4, 3) is not the first component of (3, 4). [Also, $(4, 3) \neq (4, 2)$ because (4, 3) and (4, 2) have different numbers as second components.] In general,

$$\forall_a \forall_b \forall_c \forall_d \ [(a, b) = (c, d) \text{ if and only if } (a = c \text{ and } b = d)].$$

EXERCISES

A. Arrange 16 dots on a sheet of paper in a four-by-four square array. Label the columns '1', '2', '3', and '4' from left to right; label the rows '1', '2', '3', and '4' from bottom to top. Agree that the first component of an ordered pair of numbers will tell you the column, and the second component will tell you the row. Find the dots which correspond with these ordered pairs. [Darken the corresponding dot and write the name of the ordered pair next to it.]

1. (3, 2)	2. (2, 3)	3. (4, 1)
4. (3, 3)	5. (1, 2)	6. (2, 1)
7. (8 - 6, 3)	8. (4, 10 - 6)	9. $(2 \div 2, 15 \div (5 \times 1))$

B. The set of numbers consisting of 3 and 5 is the same as the set consisting of 5 and 3. That is,

$$\{3, 5\} = \{5, 3\}.$$

But,

$$(3, 5) \neq (5, 3).$$

Explain.

C. Consider the sets A and B where

$$A = \{3, 5, 8, 9\}$$

and

$$B = \{4, 5, 6, 7, 8\}.$$

We can build ordered pairs from these sets by selecting first components from A and second components from B. On the next page is a list of all of the ordered pairs which can be built from these sets in the described way.

(3, 4)	(5, 4)	(8, 4)	(9, 4)
(3, 5)	(5, 5)	(8, 5)	(9, 5)
(3, 6)	(5, 6)	(8, 6)	(9, 6)
(3, 7)	(5, 7)	(8, 7)	(9, 7)
(3, 8)	(5, 8)	(8, 8)	(9, 8)

1. List all the ordered pairs whose first components are elements of B and whose second components belong to A.
2. List all the ordered pairs each of whose components is a member of A.
3. Suppose M is a set consisting of 5 numbers and N is a set consisting of 7 numbers. What is the total number of ordered pairs whose first components belong to M and whose second components belong to N?

*

The set of all ordered pairs with first components from A and second components from B is called the cartesian product of A by B, and is named by the symbol:

$$A \times B \qquad \text{[read as 'A cross B']}.$$

For example, if $A = \{1, 3, 4\}$ and $B = \{1, 2\}$, then

$$A \times B = \{(1, 1), (1, 2), (3, 1), (3, 2), (4, 1), (4, 2)\}$$

and

$$B \times A = \{(1, 1), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4)\}.$$

Here are other examples of cartesian products.

$$\{5, 7\} \times \{8, 1\} = \{(5, 8), (5, 1), (7, 8), (7, 1)\}$$

$$\{9\} \times \{3, 8, 9\} = \{(9, 3), (9, 8), (9, 9)\}$$

['cartesian' is derived from the Latinized form of the name of Rene Descartes (dā - kȧrt′), a French mathematician and philosopher who lived in the first half of the seventeenth century.]

*

4. Describe each cartesian product by listing its elements (as on page E).

(a) $\{4, 1, 7\} \times \{8, 5\}$ (b) $\{3, 1\} \times \{9, 1, 3\}$

(c) $\{8, 5\} \times \{4, 1, 7\}$ (d) $\{5, 7\} \times \{5, 7\}$

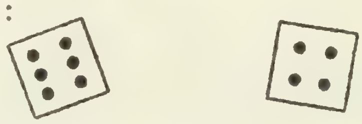
(e) $\emptyset \times \{2, 8\}$ (f) $\{2, 8\} \times \emptyset$

5. If M is a set of m numbers and N is a set of n numbers, $M \times N$ consists of __________ ordered pairs of numbers and $N \times M$ consists of __________ ordered pairs of numbers.

6. Describe a set S and a set T for which $S \times T = T \times S$.

7. A baseball squad has 6 pitchers and 3 catchers. How many possible batteries are there? [A battery is a pair consisting of a pitcher and a catcher.]

8. If you throw two dice, you get a pair of numbers. For example, if you throw this:



you get the pair $\{4, 6\}$. This is not an ordered pair, and, usually, you don't care because you add the numbers anyway. [Explain.]

You could use the dice to obtain an ordered pair, by agreeing, for example, that the die which lands first gives the first component of the ordered pair.

An easier way to get ordered pairs of numbers from the dice is to have one die red and one white, and agree that the red die gives the first component of the ordered pair. In the picture below, assume that the shaded die is the red one. Tell, in each case, the ordered pair given by the dice.

(a)  (b) 

(c)  (d) 

(e)  (f) 

(g) How many ordered pairs of numbers could you get from two such dice?

D. Obtain a pair of dice with one die red and one die white. [Any two colors will do, but if you don't use red and white, you will have to interpret the instructions accordingly.] Make 36 dots in a 6-by-6 square array, placing the dots about one inch apart. Assign an ordered pair of numbers to each of these dots, following the convention that the first component in the ordered pair tells you the column the dot is in, and the second component tells you the row the dot is in. The columns are to be numbered from 1 to 6 starting at the left, and the rows are to be numbered from 1 to 6 starting at the bottom. So, the first and second components of the 36 ordered pairs are chosen from $\{1, 2, 3, 4, 5, 6\}$. Now, if we agree that the red die gives us the first component of an ordered pair then each throw of this pair of dice gives us one of these 36 ordered pairs of numbers.

Throw the dice. Make a small tally mark next to the dot which corresponds with the ordered pair given by the dice. Repeat this process several hundred times. [This could be a class project with each student making and recording, say, 25 throws and combining all the results.]

E. Refer to a 6-by-6 chart like the one in Part D in answering these questions.

1. What are the chances that you will get a dot in the third row when you make a throw? First row? Fourth row? First column? Third column?

2. What are the chances that you will get the dot corresponding to (2, 3) when you make a throw? (4, 2)? (1, 5)? (2, 4)? (3, 3)? Any particular dot?

3. What are the chances of getting in one throw either (3, 2) or (1, 5)? Either (1, 6) or (6, 1)? Either (1, 2) or (1, 3) or (1, 4)? Either (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), or (2, 6)?

4. Pick out any subset of this set of 36 dots. Describe a method for telling the chances that you will get a dot in this subset in one throw.

(continued on next page)

5. Next to each of the dots in your diagram write a numeral for the sum of the components in the ordered pair corresponding with this dot. How many dots are there for which the sum is 12? What are the chances of getting 12 when you throw a pair of dice? How many dots are there for which the sum is 7? What are the chances of getting 7 when you throw two dice?

   Make a table showing the chances of getting each of all possible sums for the two dice. Indicate the chances by giving the probability of getting each sum. You compute the probability like this. If the chances are, say, 6 in 36, then the probability is the number $\frac{6}{36}$. Add up all the probabilities listed in your table for the 11 sums.

   What is the probability of getting any one of the sums 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or 12 when you throw two dice? What is the probability of getting the sum 89 when you throw two dice?

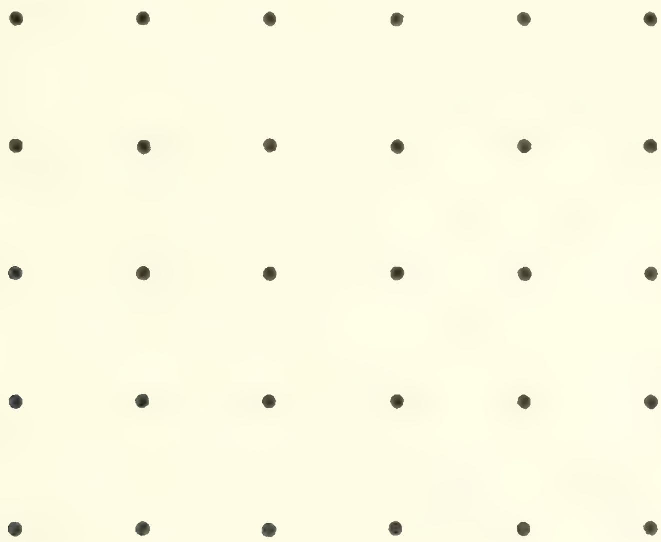
6. Find the subset of dots where the sum is either 6, 7, or 8. What is the probability of getting in this subset when you throw two dice? What is the probability of getting any one of the sums 6, 7, or 8 when you throw two dice? What is the probability of getting either the sum 3 or the sum 10 when you throw two dice? What is the probability of getting both the sums 3 and 10 in one throw of the dice?

7. (a) What is the probability of getting a sum which is an even number? An odd number? An even number or an odd number?

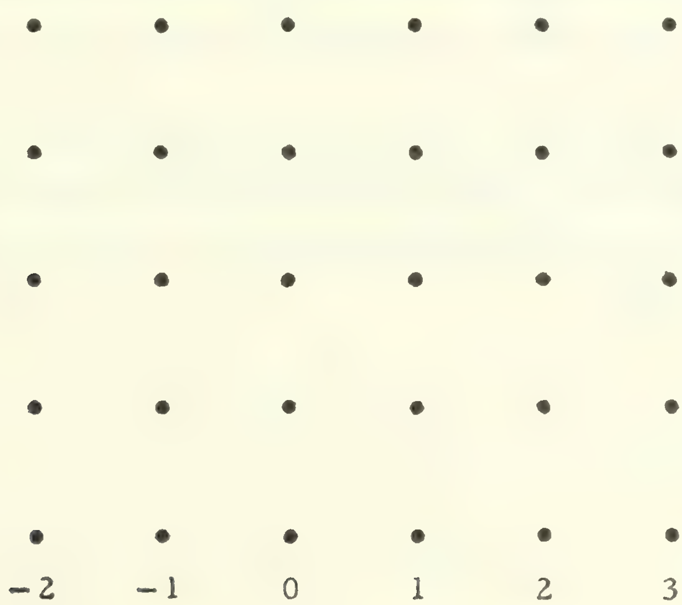
   (b) What is the probability of getting an ordered pair with both components even numbers? With both components odd numbers? Is the sum of these probabilities the same as the probability of getting a sum which is an even number?

8. If you were to throw three dice at once, what is the probability that you would get the sum 3? 18? 5?

4.01 Lattices.--The rectangular arrays of dots with which you worked in the preceding section are pictures of lattices. Each dot in such a picture corresponds with an ordered pair of numbers. Here is a picture of a lattice.



An easy way to assign ordered pairs of numbers to these dots is to number the columns and the rows. Pick one of the columns and label it with a '0'. Then label the columns to the right of this one with a '1', a '2', a '3', and so on, and label the columns to the left of the 0-column with a '−1', a '−2', etc.



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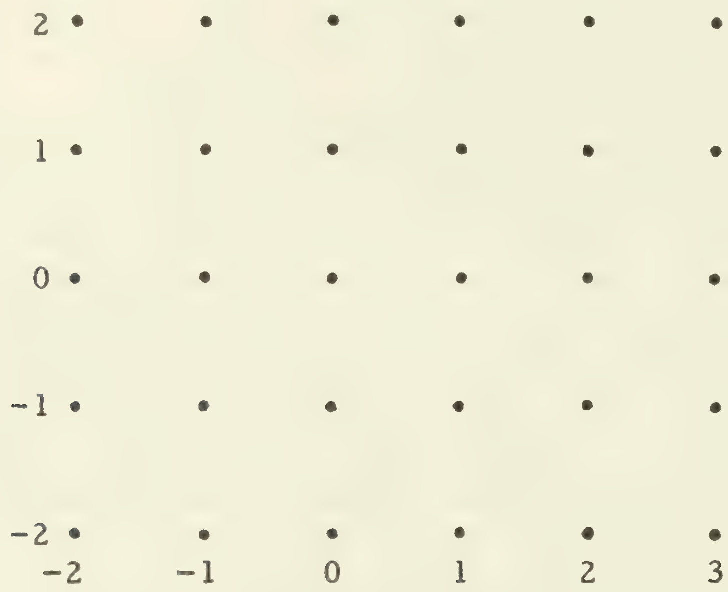
Next, pick one of the rows as the 0-row, and label the rows above it with a '1', a '2', and so on, and label the rows below it with a '-1', a '-2', and so on.

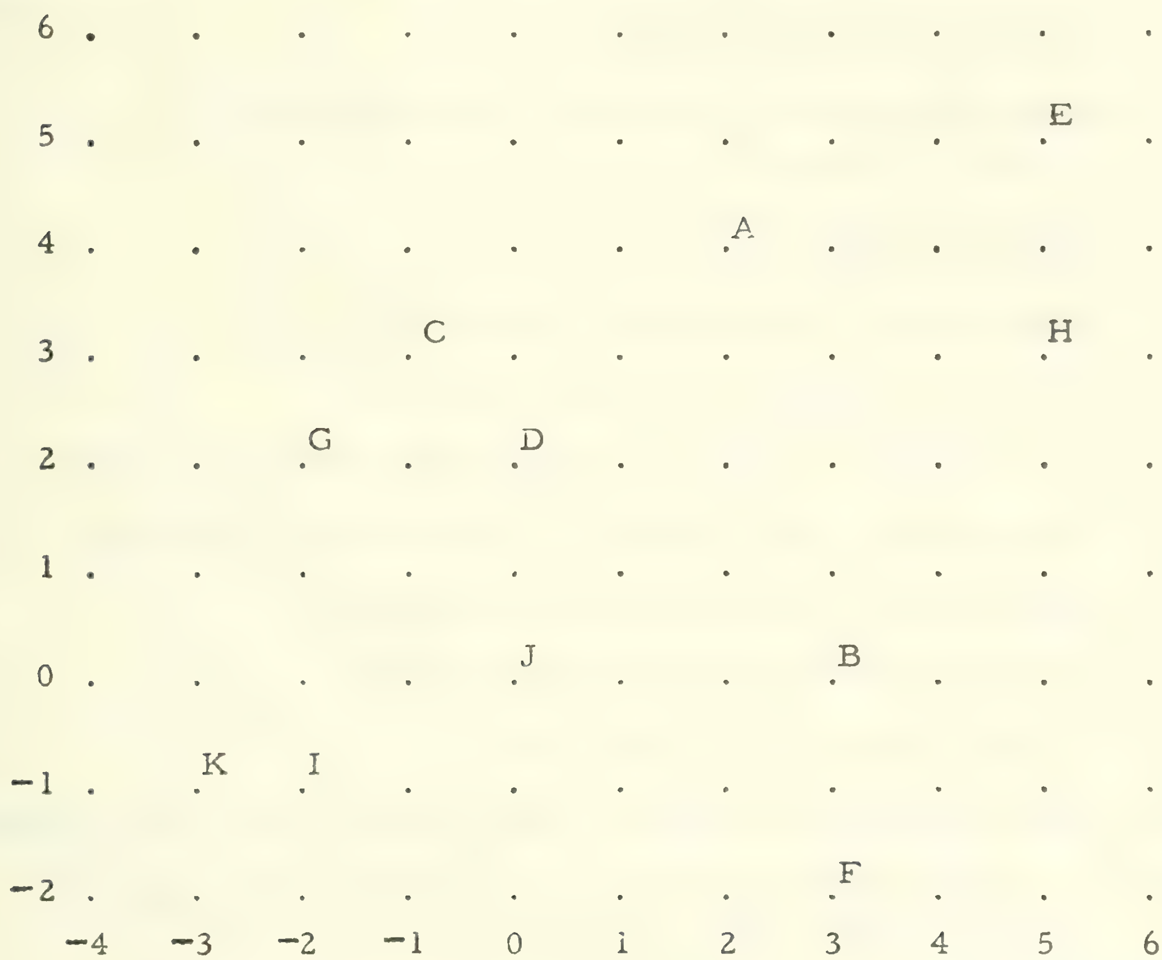


Now, we can assign an ordered pair of numbers to each of these 30 dots by agreeing that the first component of the ordered pair tells the column the dot is in and that the second component tells the row the dot is in. So, for example, the ordered pair (3, 2) corresponds with the dot in the upper right hand corner. Point to the dot which corresponds with (1, 2). With (2, 1). With (0, 0). With (0, 2). With (2, 3).

Each of the 30 dots in the picture is the <u>graph</u> of the corresponding ordered pair of numbers, and the first and second components of each of the 30 ordered pairs are the <u>first coordinate</u> and <u>second coordinate</u> of the corresponding dot.

EXERCISES

A. Study this picture of a lattice, and answer the questions which follow.



1. Give the ordered pairs of numbers which correspond with the eleven labeled dots. [Sample. A: (2, 4)]

2. Give the first coordinate of each labeled dot.

3. Give the second coordinate of each labeled dot.

4. How many dots in the picture have first coordinate equal to second coordinate?

5. Label with an 'L' the graph of the ordered pair (4, 3).

6. Label with an 'M' the dot whose first coordinate is 3 and whose second coordinate is 4.

7. Draw a dashed line which connects all the dots which have first coordinate 0.

8. Draw a dashed line which connects all the dots which have second coordinate 0.

9. Label with a 'W' the graph of (0, 1).

B. Consider the cartesian product

$\{-3, -2, -1, 0, 1, 2, 3\} \times \{-5, -4, -3, -2, -1, 0, 1, 2, 3\}$.

Make a picture of this lattice.

Tell how many dots are graphs of ordered pairs with

(1) first component 2 .

(2) second number −3 .

(3) first number greater than or equal to 2 .

(4) second component less than or equal to −1 .

(5) first number greater than 1 and second number less than 2 .

(6) first number greater than 1 or second number less than 2 .

(7) first component equal to second component .

(8) first number 1 more than second number .

(9) second number twice first number .

(10) second component equal to 3 more than 2 times first component .

(11) second number 5 more than 2 times first number and with second number 3 more than first number .

[More exercises are in Part A, Supplementary Exercises.]

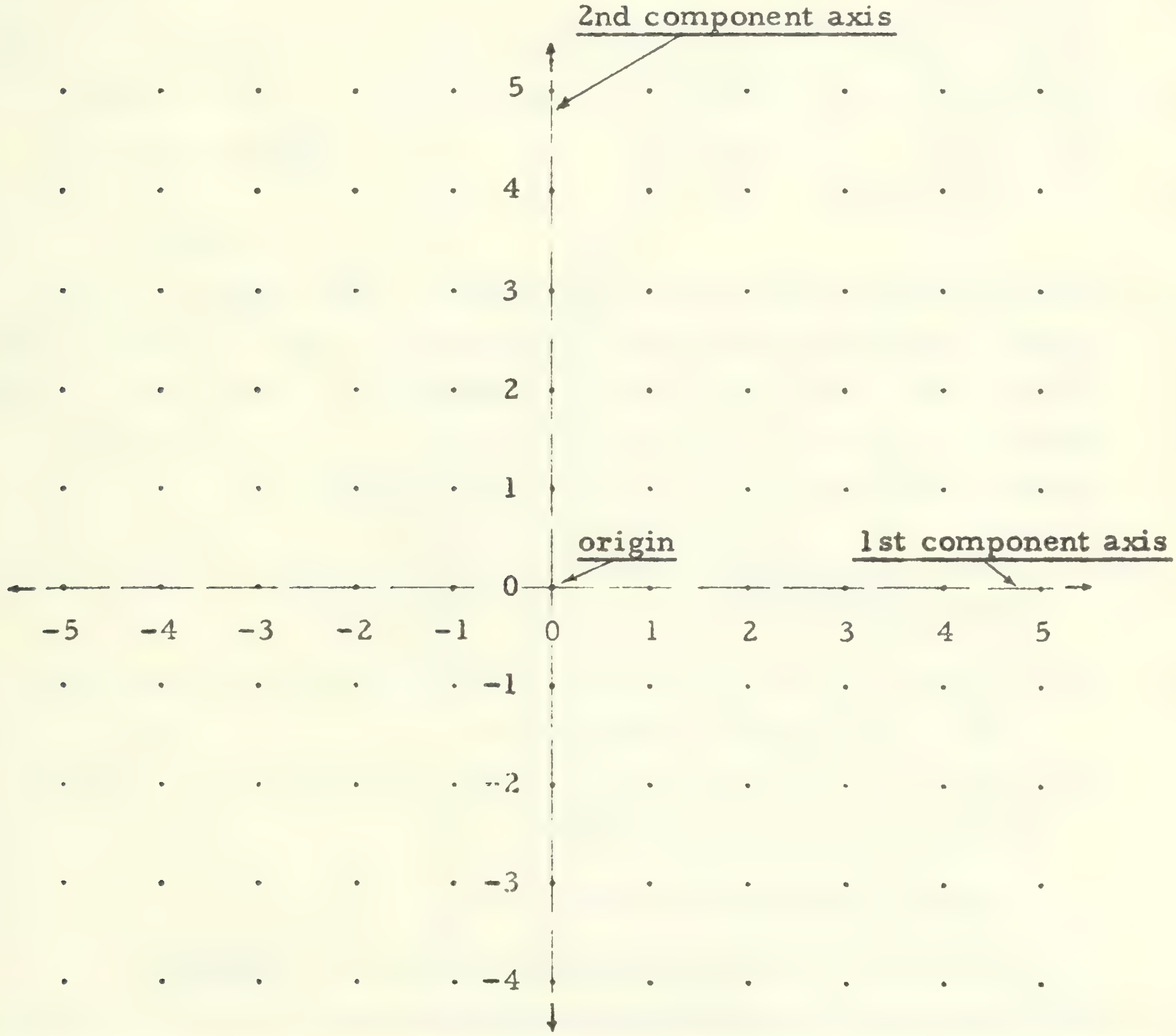
THE NUMBER PLANE LATTICE

Imagine the set of all the ordered pairs whose first and second components are integral real numbers [for short: real integers], that is, the numbers

.... −3, −2, −1, 0, +1, +2, +3, ...

This set is the cartesian product of the set of real integers by itself [that is, the cartesian square of the set of real integers]. We call this cartesian product the number plane lattice. ['plane' because we picture it as a flat surface like a sheet of paper, and flat surfaces are called planes.]

Just as you could draw pictures of only part of the number line, it is possible to draw pictures of only part of the number plane lattice. Still, we shall call such pictures pictures of the number plane lattice. When drawing such a picture, you may first select a 0-column and a 0-row. [It's helpful if you draw a dashed line connecting the dots in the 0-column and a dashed line connecting the dots in the 0-row.] Then, number the rows and columns by writing numerals close to the dots in the 0-column and 0-row. You can put little arrows at the ends of this column and this row to show that you have a picture of only part of the lattice.



The dots in a picture of the number line correspond with points of the number line, that is with real numbers; the dots in a picture of the number plane lattice correspond with points of the number plane lattice, that is, with ordered pairs of real integers. The process of locating the dot which corresponds with a given ordered pair is called graphing the ordered pair or plotting the point.

The set of points in the lattice each of which has second component 0 is called the first component axis. The dots in the picture which correspond with these points are lined up horizontally; they are the dots which have 0 as second coordinate. What is meant by the second component axis? The first and second component axes have one point in common, and this point is called the origin. What ordered pair is this point?

EXERCISES

A. Draw a picture of the number plane lattice. Plot the points listed below and label them with the given letters.

A: (3, 5)	B: (2, 5)	C: (−3, 1)	D: (2, 0)
E: (0, 0)	F: (0, −2)	G: (6, 1)	H: (1, 6)
I: (−3, −4)	J: (4, −3)	K: (10, −10)	L: (−9, −9)

B. Draw a picture of the number plane lattice [your diagram should contain enough dots so that you can plot the points (−8, 8), (−8, −8), (8, 8), and (8, −8).] Draw the sets of points described below. Mark the dots in some particular fashion so that you can tell the sets apart. [We abbreviate 'real integers' to 'integers'.]

1. The set of all ordered pairs of integers with first component equal to 1 less than the second component.

2. The set of all ordered pairs of integers with first component 3 more than second component.

3. The set of all ordered pairs of integers such that the sum of the components of each ordered pair is 9.

4. The set of all ordered pairs of integers such that 8 is the sum of the second component and twice the first component.

5. The set of all ordered pairs of integers with second component 7.

6. The set of all ordered pairs of integers which correspond with dots which have first coordinate −3.

7. The set of all ordered pairs of integers such that the first component is less than −5 and the second component is greater than 6.

8. Repeat Exercise 7 using 'or' instead of 'and'.

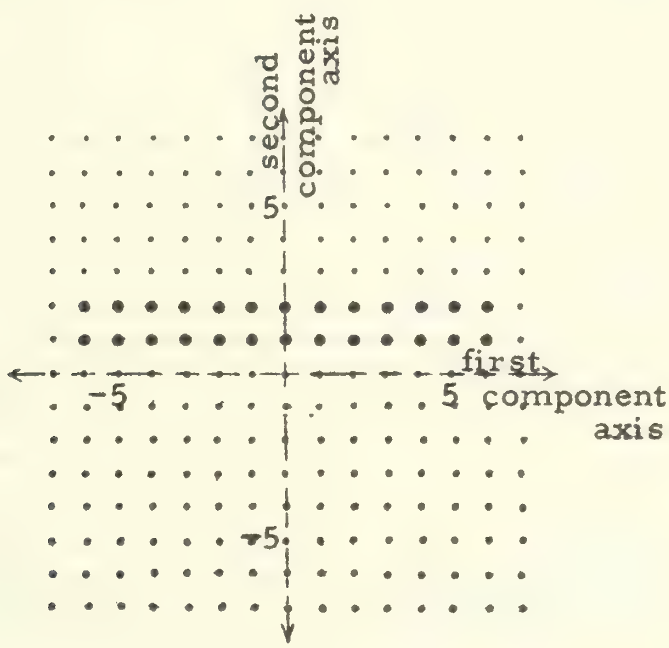
9. The set of all ordered pairs of integers such that the second component is less than 2 but greater than −2 and the first component is greater than 3 but less than 6. [How many points are there in this set?]

10. The set of all ordered pairs of integers such that the first component is greater than −5 but less than −2 and the second component is 6.

[More exercises are in Part B, Supplementary Exercises.]

C. Here are pictures of sets of ordered pairs of integers. Write descriptions of the sets pictured, using the type of wording of the exercises of Part B.

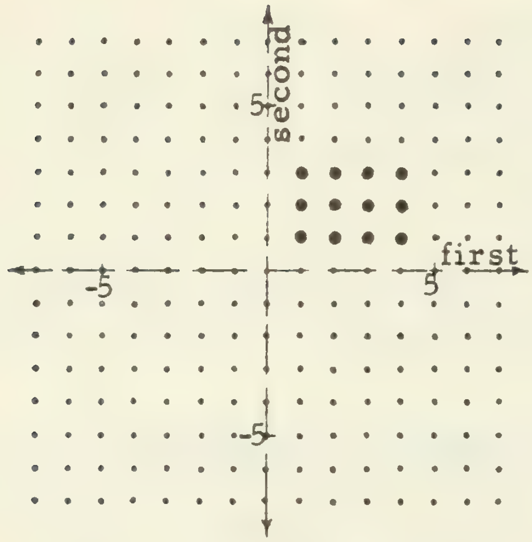
Sample.



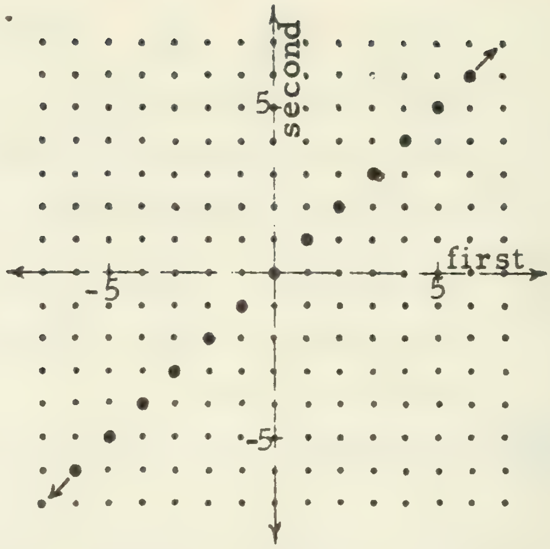
Solution. The first coordinate of each of these dots is an integer between −7 and 7; the second coordinate is either 1 or 2. So, a description of this set is: the set of all ordered pairs of integers such that the first component is greater than −7 but less than 7 and the second component is greater than 0 but less than 3.

(continued on next page)

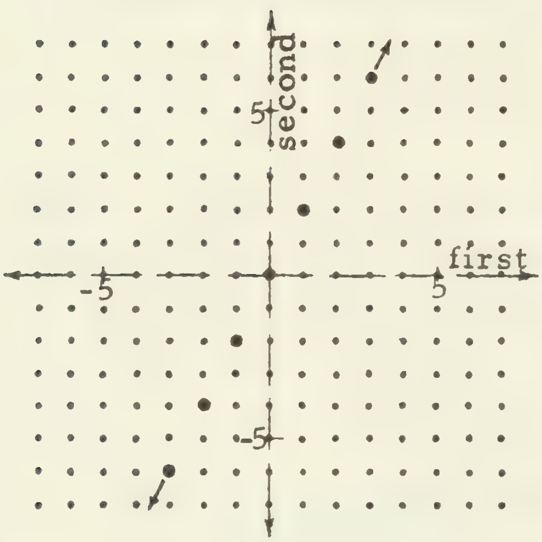
1.



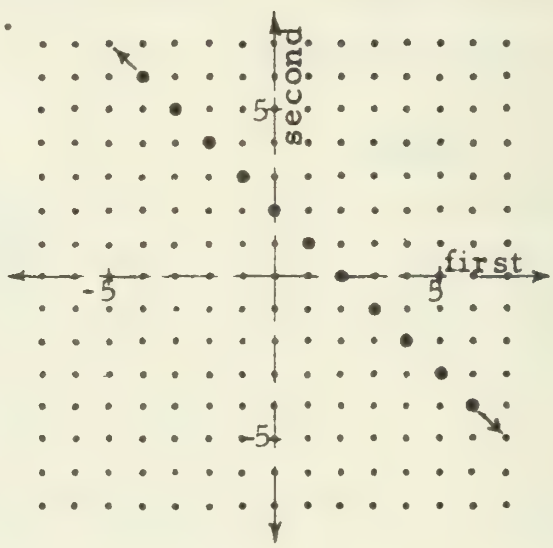
2.



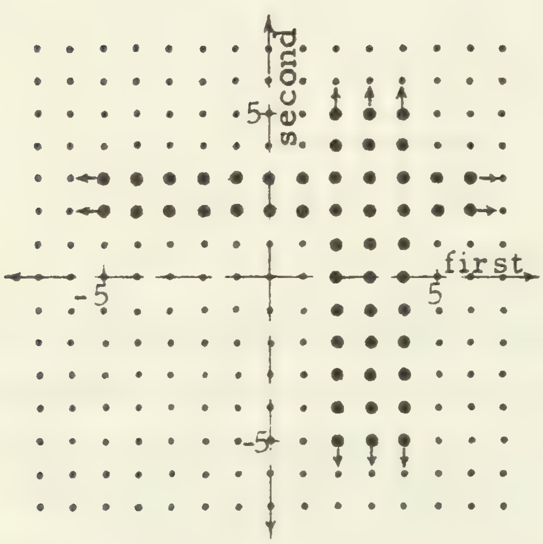
3.



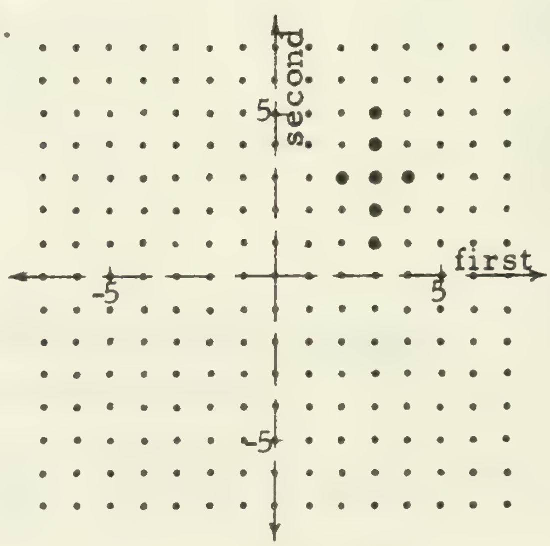
4.



5.



6.



[More exercises are in Part C, Supplementary Exercises.]

* * *

As you have probably noticed, these word-descriptions of sets can be quite complicated. We can simplify the job of describing sets by using the notation introduced in Unit 3. For example, consider the description given in the first exercise of Part B:

> The set of all ordered pairs of integers
> with first component equal to 1 less
> than the second component.

We use a pair of braces, '{' and '}', to show that we are talking about a set, an index of pronumerals such as '(x, y)' or '(p, q)' to show that we are talking about ordered pairs of real numbers, a restriction to indicate what kind of real numbers we are interested in, and an open sentence which is satisfied by those ordered pairs (and only those) which belong to the set in question. Thus, the set described in words above is also described by:

$$\{(x, y),\ x \text{ and } y \text{ integers}:\ x = y - 1\}.$$

[Read this as 'the set of ordered pairs of integers such that the first component is 1 less than the second component' or, more simply, as 'the set of ordered pairs (x, y) of integers such that $x = y - 1$'.]

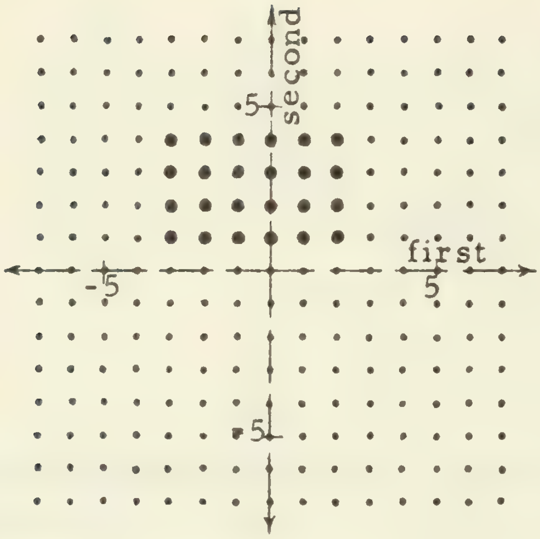
* * *

<u>D</u>. 1. Rewrite the word-descriptions of Exercises 2-10 of Part B using the brace-notation mentioned above.

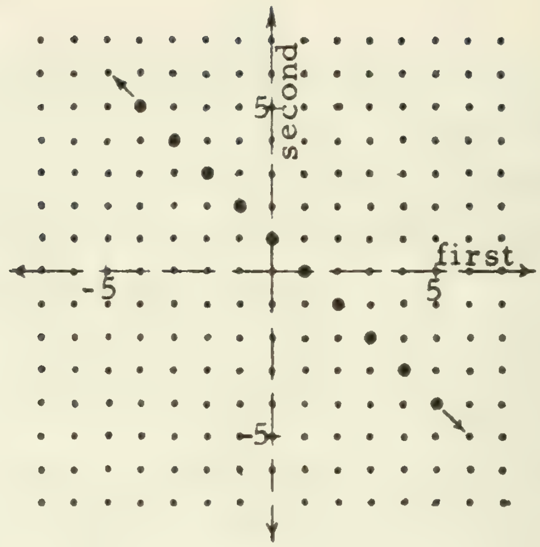
2. Rewrite the descriptions of the sets pictured in the exercises of Part C.

3. Describe the sets pictured on page 4-10.

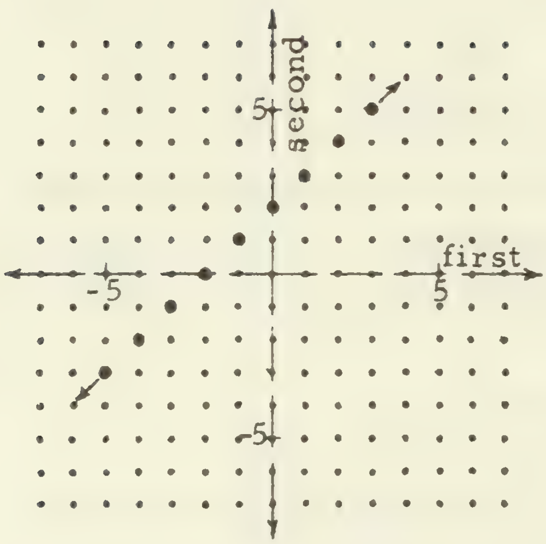
(a)



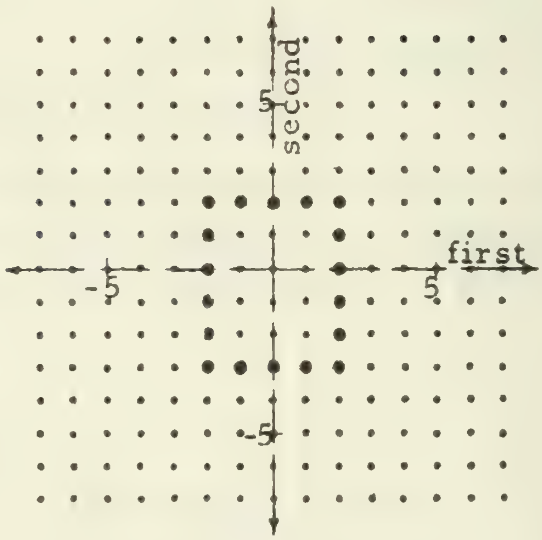
(b)



(c)



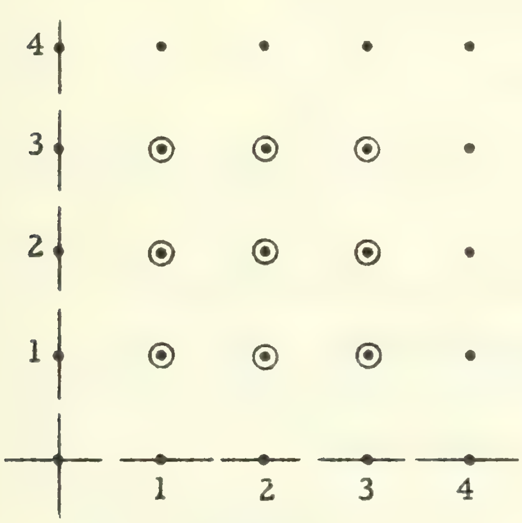
(d)



[More exercises are in Part D, Supplementary Exercises.]

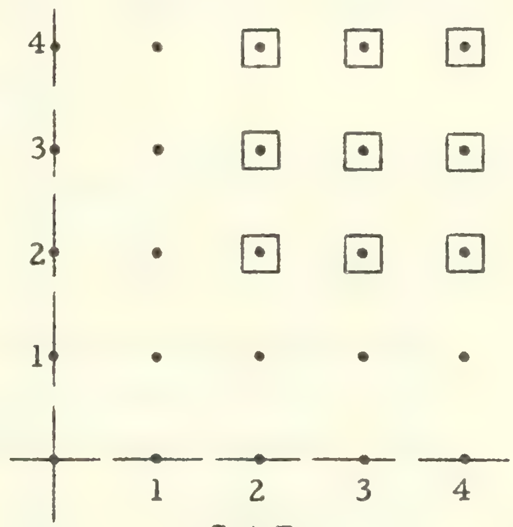
INTERSECTIONS AND UNIONS

Below are four pictures of the same part of the number plane lattice. Each picture shows a set. The first picture shows set A, the picture in the upper right shows set B, the picture in the lower left shows the set consisting of all the points which belong to both A and B, and the fourth picture shows the set of all points which belong either to A or to B.



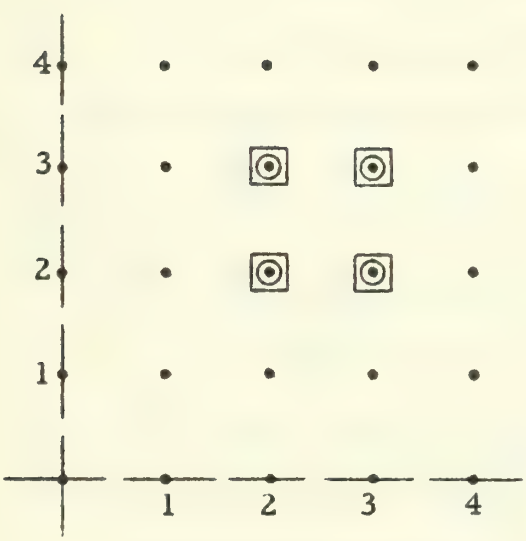
Set A

(9 points)



Set B

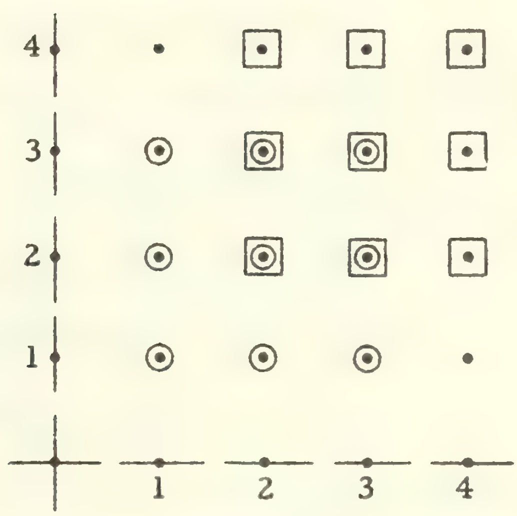
(9 points)



Intersection of

Set A and Set B

(4 points)



Union of

Set A and Set B

(14 points)

Notice the words intersection and union. The intersection of two sets is the set which consists of the elements which belong to both of the given sets. The union of two sets is the set which consists of the elements which belong to either of the given sets.

In the illustration on page 4-11, A and B each have 9 points. The intersection of A and B consists of the 4 points each of which belongs to both A and B. The union of A and B consists of the 14 points each of which belongs to at least one of the sets A and B.

EXERCISES

A. In each of the following exercises you are given descriptions of a set A and a set B. For each exercise,

(a) plot the points in each set on the same diagram,

(b) tell the number of points in each set,

(c) tell the number of points in the intersection, and

(d) tell the number of points in the union.

1. $A = \{(x, y),\ x \text{ and } y \text{ integers}: \ -2 < x\ 3 \text{ and } 3 < y < 6\}$

   $B = \{(x, y),\ x \text{ and } y \text{ integers}: \ 0 < x < 6 \text{ and } -1 < y < 5\}$

   the number of points in A = ________

   the number of points in B = ________

   the number of points in the intersection of A and B = ________

   the number of points in the union of A and B = ________

2. $A = \{(x, y),\ x \text{ and } y \text{ integers}: \ -7 < x < -2 \text{ and } -4 < y < 3\}$

   $B = \{(x, y),\ x \text{ and } y \text{ integers}: \ -8 < x < 0 \text{ and } -4 < y < 0\}$

   n(A) = ____________ ['n(A)', read as 'en of A', means the number of elements in set A.]

   n(B) = ________

   n(the intersection of A and B) = ________

   n(the union of A and B) = ________

3. $A = \{(x, y),\ x \text{ and } y \text{ integers}:\ |x| < 3 \text{ and } |y| < 2\}$

$B = \{(x, y),\ x \text{ and } y \text{ integers}:\ 3 < x < 6 \text{ and } |y| < 2\}$

n(A) = ________, n(B) = ________

n(the intersection of A and B) = _______, n(the union of A and B) = _____

4. $A = \{(2, 0), (2, 1), (3, 0), (3, 1)\}$

$B = \{(x, y),\ x \text{ and } y \text{ integers}:\ 1 < x < 4 \text{ and } -1 < y < 2\}$

n(A) = ________, n(B) = ________

n(the intersection of A and B) = ______, n(the union of A and B) = _____

5. $A = \{(x, y),\ x \text{ and } y \text{ integers}:\ -4 < x < -3 \text{ and } -2 < y < 0\}$

$B = \{(x, y),\ x \text{ and } y \text{ integers}:\ 3 < x < 6 \text{ and } 1 < y < 3\}$

n(A) = ________, n(B) = ________

n(the intersection of A and B) = ______, n(the union of A and B) = _____

6. $A = \{(x, y),\ x \text{ and } y \text{ integers}:\ -1 < x < 0 \text{ and } 5 < y < 6\}$

$B = \{(x, y),\ x \text{ and } y \text{ integers}:\ 5 < x < 6 \text{ and } -1 < y < 0\}$

n(A) = ________, n(B) = ________

n(the intersection of A and B) = _____, n(the union of A and B) = _____

7. $A = \{(x, y),\ x \text{ and } y \text{ integers}:\ x = y,\ |x| < 2, \text{ and } |y| < 2\}$

$B = \{(x, y),\ x \text{ and } y \text{ integers}:\ |x| < 3 \text{ and } |y| < 3\}$

n(A) = ________, n(B) = ________

$n(A \cap B)$ = ________, $n(A \cup B)$ = ________

['$A \cap B$', read as 'A intersection B', means the intersection of sets A and B;

'$A \cup B$', read as 'A union B', means the union of sets A and B.]

(continued on next page)

8. $A = \{(x, y), \text{ x and y integers}: x + y \leq 5, x \geq 0, \text{ and } y \geq 0\}$

$B = \{(x, y), \text{ x and y integers}: x + y > 5, x < 6, \text{ and } y < 6\}$

$n(A) =$ ____, $n(B) =$ ____, $n(A \cap B) =$ ____, $n(A \cup B) =$ ____

9. $A = \{(x, y), \text{ x and y integers}: |x| + |y| \leq 5\}$

$B = \{(x, y), \text{ x and y integers}: |x| \leq 5\} \cap$

$\{(x, y), \text{ x and y integers}: |y| \leq 5\}$

$n(A) =$ ____, $n(B) =$ ____, $n(A \cap B) =$ ____, $n(A \cup B) =$ ____

10. $A = \{(x, y), \text{ x and y integers}: -3 < x < 0 \text{ or } 3 < y < 6\}$

$B = \{(x, y), \text{ x and y integers}: 2 < x < 6 \text{ or } -4 > y > -7\}$

[Note that neither A nor B is <u>finite</u>, that is, you can't count the elements in these sets. Sets which are not finite are called <u>infinite sets</u>. When you are asked the number of elements in a set which is not finite, just answer that the question doesn't make sense because the set is infinite. [In a later course you may learn about other numbers which can be used to tell how many elements there are in an infinite set.]]

$n(A) =$ ____, $n(B) =$ ____, $n(A \cap B) =$ ____, $n(A \cup B) =$ ____

11. $A = \{(x, y), \text{ x and y integers}: x + y = 0\}$

$B = \{(x, y), \text{ x and y integers}: x - y = 0\}$

$n(A) =$ ____, $n(B) =$ ____, $n(A \cap B) =$ ____, $n(A \cup B) =$ ____

12. $A = \{(x, y), \text{ x and y integers}: 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$

$B = \{(x, y), \text{ x and y integers}: y = x, y = x^2, y = x^3, \text{ and } y = x^4\}$

$n(A) =$ ____, $n(B) =$ ____, $n(A \cap B) =$ ____, $n(A \cup B) =$ ____

13. $A = \{(x, y),\ x \text{ and } y \text{ integers}:\ |x| + |y| = 5\}$

$B = \{(x, y),\ x \text{ and } y \text{ integers}:\ 5 > x > 0 \text{ and } 7 > y > 2\}$

$n(A) =$ ____, $n(B) =$ ____, $n(A \cap B) =$ ____, $n(A \cup B) =$ ____

14. $A = \{(x, y),\ x \text{ and } y \text{ integers}:\ y - |x| = 6\}$

$B = \{(x, y),\ x \text{ and } y \text{ integers}:\ 2 \leq x < 4 \text{ or } 4 < y < 9\}$

$n(A) =$ ____, $n(B) =$ ____, $n(A \cap B) =$ ____, $n(A \cup B) =$ ____

15. $A = \{(m, n),\ m \text{ and } n \text{ integers}:\ m = 3n\}$

$B = \{(m, n),\ m \text{ and } n \text{ integers}:\ n = 3m\}$

$n(A) =$ ____, $n(B) =$ ____, $n(A \cap B) =$ ____, $n(A \cup B) =$ ____

16. $A = \{(r, s),\ r \text{ and } s \text{ integers}:\ s = r^2\}$

$B = \{(p, q),\ p \text{ and } q \text{ integers}:\ q = p + 2\}$

$n(A) =$ ____, $n(B) =$ ____, $n(A \cap B) =$ ____, $n(A \cup B) =$ ____

17. $A = \{(x, y),\ x \text{ and } y \text{ integers}:\ x^2 + y^2 = 25\}$

$B = \{(x, y),\ x \text{ and } y \text{ integers}:\ |x| + |y| = 7\}$

$n(A) =$ ____, $n(B) =$ ____, $n(A \cap B) =$ ____, $n(A \cup B) =$ ____

18. $A = \{(x, y),\ x \text{ and } y \text{ integers}:\ 2y + 2x = 1\}$

$B = \{(x, y),\ x \text{ and } y \text{ integers}:\ 3y - 3x = 1\}$

$n(A) =$ ____, $n(B) =$ ____, $n(A \cap B) =$ ____, $n(A \cup B) =$ ____

[More exercises are in Part E, Supplementary Exercises.]

B. Number Plane Lattice Games

A number plane lattice game consists of a series of moves from point to point of the lattice with each move made according to a given rule.

Sample 1. Rule: A move from (x, y) takes you to $(x + 1, y + 1)$.

Start at $(-4, -3)$ and make 5 moves. Where do you finish?

Solution. First move: From $(-4, -3)$ to $(-4 + 1, -3 + 1)$, or $(-3, -2)$.

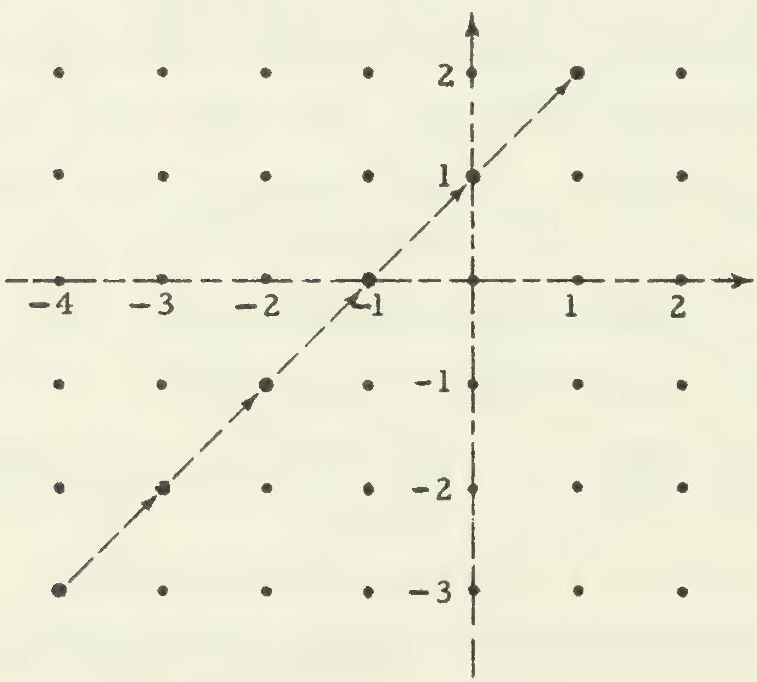
Second move: From $(-3, -2)$ to $(-3 + 1, -2 + 1)$, or $(-2, -1)$.

Third move: From $(-2, -1)$ to $(-2 + 1, -1 + 1)$, or $(-1, 0)$.

Fourth move: From $(-1, 0)$ to $(0, 1)$.

Fifth move: From $(0, 1)$ to $(1, 2)$.

So, starting at $(-4, -3)$, after five moves you reach $(1, 2)$. Here is a picture showing the moves.



<u>Sample 2</u>. Rule: A move from (x, y) takes you to (y, x).

Consider the set whose elements are the points

$$(2, 1), (4, 0), (6, -1).$$

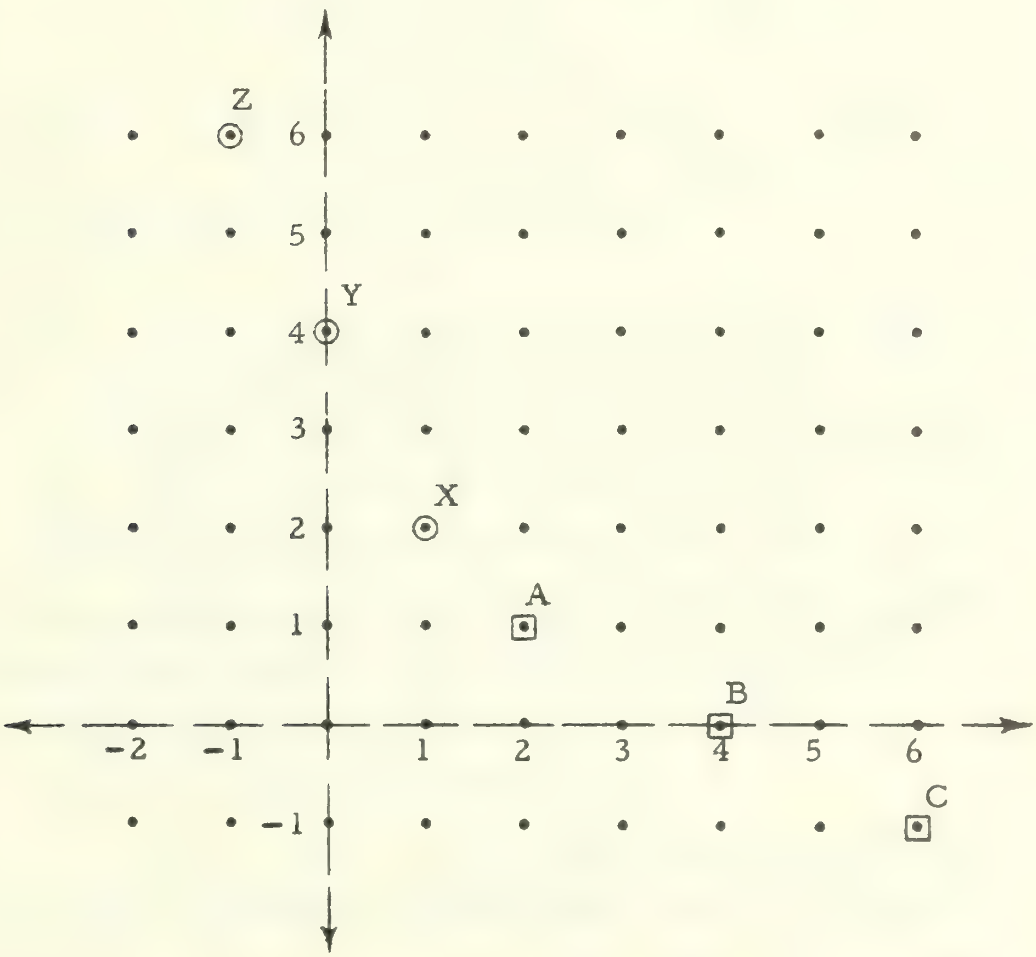
Make one move from each of these points and describe the new set of points.

<u>Solution</u>. From A, $(2, 1)$, move to X, $(1, 2)$.

From B, $(4, 0)$, move to Y, $(0, 4)$.

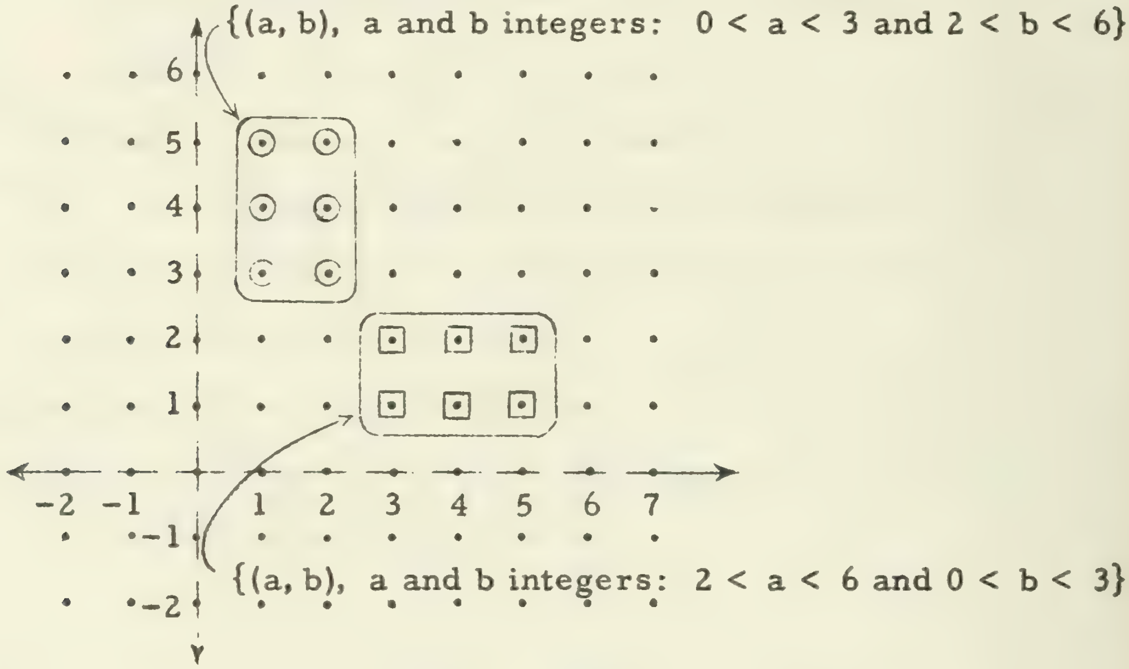
From C, $(6, -1)$, move to Z, $(-1, 6)$.

The new set is $\{(1, 2), (0, 4), (-1, 6)\}$.



Sample 3. Rule: A move from (x, y) takes you to (y, x).
Consider the points in
$\{(a, b),\ a \text{ and } b \text{ integers}:\ 2 < a < 6 \text{ and } 0 < b < 3\}$.
Describe the set obtained by making one move from each point of the given set.

Solution. The points in the given set are shown by boxed dots, and the points in the new set are shown by circled dots.



1. Rule: A move from (x, y) takes you to $(x, y - 2)$. Start at $(0, 4)$ and make 3 moves. What is the final point?

2. Rule: A move from (x, y) takes you to $(x + 2, y - 3)$. Start at $(3, 3)$ and make 3 moves. What is the final point?

[Note: In the following exercises we shall give an abbreviated form for each rule. For example, the statement of the rule in Exercise 2 could have been abbreviated as:

$$(x, y) \to (x + 2, y - 3).]$$

3. Rule: $(x, y) \to (2x, 2y)$.
Start at $(1, 2)$ and make 5 moves. What is the final point?

4. Rule: $(x, y) \to (3x, 2y)$.
Start at $(0, 0)$ and make 10 moves. What is the final point?

5. Rule: $(j, k) \to (3j - 5, 2k + 3)$.
Start at $(4, -3)$ and make 4 moves. What is the final point?

6. Rule: $(x, y) \to (x - 3, y + 2)$.
After making 3 moves, the final point is $(-4, 1)$. Give the starting point.

7. Rule: $(u, v) \to (2u + 5, 3 - 4v)$.
After making 3 moves, the final point is $(43, -89)$. Give the starting point.

8. Rule: $(x, y) \to (x + y, x - y)$.
Start at $(3, -3)$ and make 4 moves. What is the final point?

9. Rule: $(s, t) \to (|s|, |t|)$.
Make one move from each point in

$$\{(a, b), \text{a and b integers: } a = b \text{ and } -5 \leq a \leq 0\},$$

and describe the set of points which you obtain.

10. Rule: $(x, y) \to (y, x)$.
Make one move from each point in $\{(2, 0), (2, -1), (2, -2), (2, -3)\}$, and describe the new set of points.

11. Rule: $(x, y) \to (y, x)$.
   (a) Start at each point in $\{(4, 3), (5, -1), (6, -2)\}$, and make two moves. Describe the resulting set.
   (b) Repeat (a) but, starting at each point in the given set, make three moves.
   (c) Repeat (a) but with four moves.
   (d) Repeat (a) but with any even number of moves.
   (e) Repeat (a) but with any odd number of moves.

12. Rule: $(y, x) \to (y + 1, x + 1)$.
Make one move from each point in

$$\{(m, n), \text{m and n integers: } m + n = n + m\},$$

and describe the resulting set.

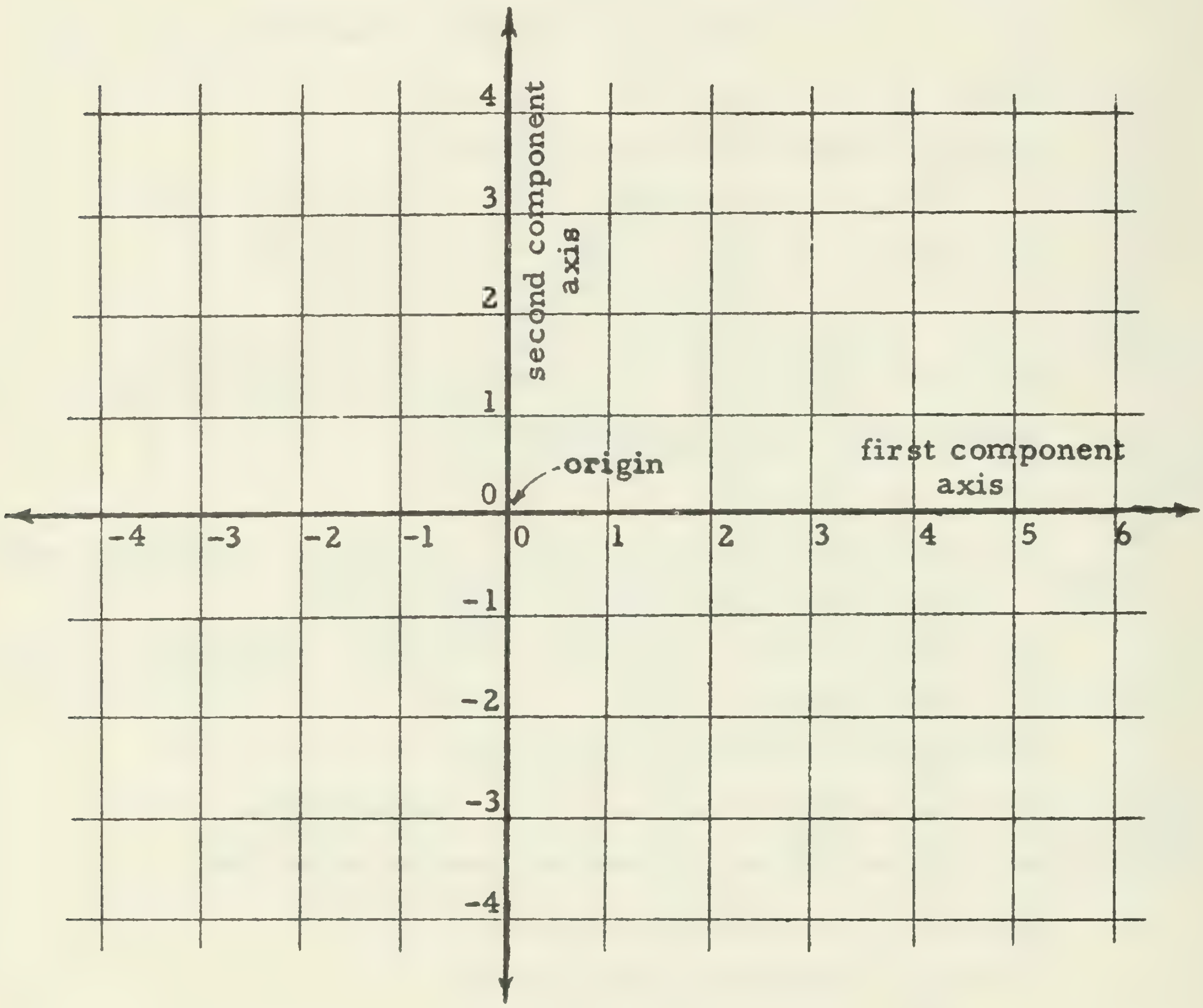
13. Rule: $(x, y) \to (x + 2, y - 3)$.
Start at each point in

$$\{(c, d), \text{c and d integers: } c^2 + d^2 = 25\}$$

make two moves, and describe the resulting set.

4.02 The number plane.--Up to now you have been working with the number plane lattice--that is, the cartesian square of the set of integral real numbers. Now we shall consider the cartesian square of the set of all real numbers. This cartesian square is called the number plane. [Do you see that the number plane lattice is a subset of the number plane?]

As in the case of the number line, you can draw a picture of only a part of the number plane. The usual procedure is illustrated here.



Notice that the pictures of the component axes are made more prominent than the other columns and rows in the picture. These other lines are called grid lines. A vertical grid line pictures a set of ordered pairs of real numbers which have the same first component, and a horizontal grid line pictures a set of ordered pairs which have the

same second component. Just as in the case of the number line, we must frequently make estimates when plotting points. For example, the ordered pair (2.5, 3.1) corresponds with a dot which is midway between the 2-column and the 3-column, and one-tenth of the way from the 3-row to the 4-row. So, we would have to make estimates to mark this dot on the picture.

An ordered pair of real numbers is a point of the number plane [you recall that an ordered pair of integers is a point of the number plane lattice], and corresponds with a dot which can be marked on the picture. The dot is the graph of the ordered pair, and the components of the ordered pair are the coordinates of the dot. The first coordinate of a dot is sometimes called its abscissa, and the second coordinate is sometimes called its ordinate.

Study the diagram on page 4-20 and follow these instructions.

(1) Point to the grid line which contains dots with abscissa (first coordinate) 3.

(2) Point to the grid line which contains dots with ordinate 2.

(3) Point to the grid line which contains dots with ordinate 4.

(4) Point to the grid line which contains dots with abscissa -1.

(5) Point to the grid line which contains dots with ordinate 0.

(6) Draw the grid line which contains dots with abscissa $\frac{1}{2}$.

(7) Draw the grid line which contains dots with ordinate $-2\frac{1}{3}$.

(8) Point to the grid lines which contain the graph of (5, 3). Of (2, -1). Of (-3, -4). Of (4, 4). Of ($\frac{1}{2}$, 2). Of (0, 0).

(9) Draw the grid lines which contain the graph of ($3\frac{1}{2}$, $2\frac{1}{2}$). Of (-2.5, 1.5). Of (-3.5, -3.5).

In making a picture of part of the number plane you are completely free in your choice of which grid lines to draw. [It is customary, however, to include pictures of the component axes.] Since the only purpose of the grid lines is to help you plot points, the components of the points to be plotted in a particular problem determine the selection of grid

lines. For example, the diagram on page 4-20 would be completely useless if you wanted to plot the graph of, say, (10, 17). When you use cross section paper [or graph paper, as it is commonly called], you will find equally spaced grid lines already printed on the paper. Choose two of these grid lines to represent the component axes and use a pencil to make them more prominent. Then decide upon the numbers to be assigned to each grid line. You do this by selecting the dots which are to correspond with (1, 0) and (0, 1). In other words, you select a scale for each axis. Here are several examples. Note that the scales differ from diagram to diagram even though the smallest distance between parallel grid lines is the same for all of the diagrams.

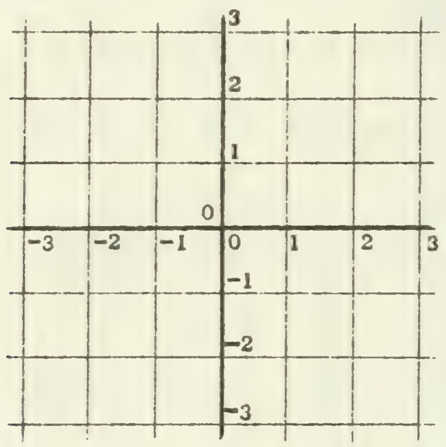


Figure 1.

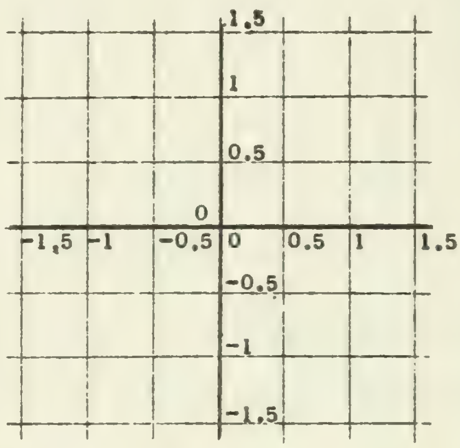


Figure 2.

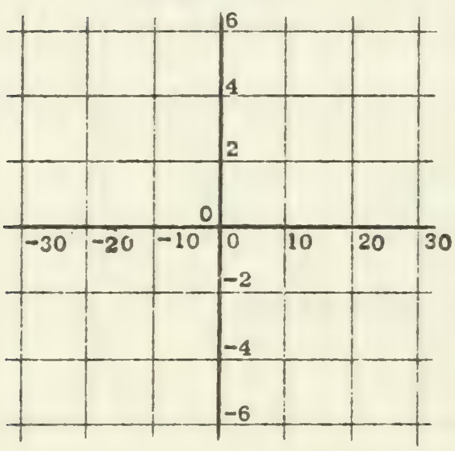


Figure 3.

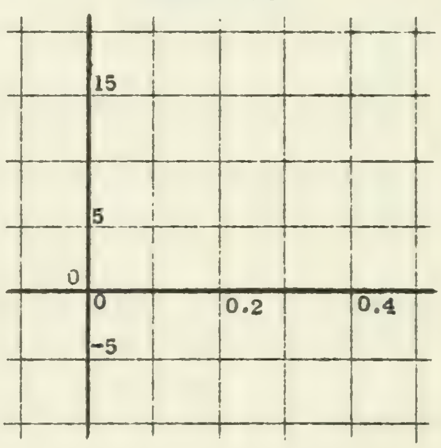


Figure 4.

In Figure 1 the same scale is used on both axes. The same scale is used on both axes in Figure 2. The scales for the axes differ in Figure 3 and 4. Note [Figure 4] that it is not necessary to label each grid line. When the grid lines are equally spaced, you can always tell the numbers to be assigned to "in-between" grid lines.

EXERCISES

A. For each exercise, draw a picture of part of the number plane and graph the ordered pairs given. Select scales so that all the pairs in the exercise can be conveniently plotted on the same picture.

1. (3, 5), (2, −1), (4, −3), (0, −2), (−3, 4)
2. (2.5, 3.5), (−1.5, 1.5), (0, −2.5), (−3.5, 1.5), (3, −2)
3. (20, 3), (−40, 5), (30, −4), (−50, 1), (0, −2)
4. (1, 1), (−1, −1), (2, 8), (−3, −27), (4, 64)
5. (0.3, 7), (−0.2, −8), (−1, 3), (−0.8, 0), (0.5, −5)

B. At the end of this unit you will find several sheets of paper showing four regions of the number plane. Use two sides of one sheet for the five exercises below and on the next page. [You will need the rest of the sheets for later exercises.]

1. Graph each of these ordered pairs, and label the graph with the appropriate letter.

H: (1 000 003, 1 000 003)	J: (−156 616, −2)
K: (5, 135 410)	L: (999 999, −450 003)
M: (1 000 008, 999 998)	N: (−156 620, 0)
O: (0, 135 406)	P: (156 602, 4)
Q: (135 411, −2)	R: (0, 0)

2. Draw a picture of part of the number plane [include the two component axes] and make a rough sketch of the location of the four regions.

(continued on next page)

3. (a) Is there a straight line in the number plane which intersects in a nonempty set each of the regions A and B?

   (b) Is there a straight line in the number plane which intersects in a nonempty set each of the regions C and D?

4. Suppose you were in Region A and you met a stranger who asked the direction in which he should walk to get to the origin. Indicate (by drawing an arrow on the picture for Region A) which way you would point to show him the location of the origin.

5. Repeat Exercise 4 for each of the other three regions.

C. Number Plane Games.

Use a picture of the number plane to keep a "running" record of the moves.

Sample. Rule: $(x, y) \rightarrow (x + 2, -\frac{1}{2}y)$.
Start with (1, 8) and make 4 moves. What is the final point?

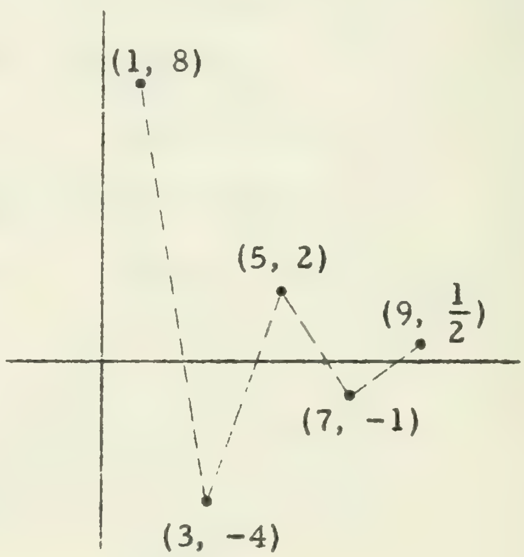
Solution. $(1, 8) \rightarrow (3, -4)$

$(3, -4) \rightarrow (5, 2)$

$(5, 2) \rightarrow (7, -1)$

$(7, -1) \rightarrow (9, \frac{1}{2})$

The final point is $(9, \frac{1}{2})$.



1. Rule: $(x, y) \rightarrow (x + 3, \frac{1}{2}y)$.
Start at (−5, −8) and make 4 moves.
What is the final point?

2. Rule: $(x, y) \to (2x - 1, y + 1)$.
Start at $(1, -3)$ and make 8 moves. What is the final point?

3. Rule: $(x, y) \to (3x - 8, 9y - 16)$.
Start at $(4, 2)$ and make 20 moves. What is the final point?

4. Rule: $(x, y) \to (x + y, x - y)$.
Start at $(-3, 4)$ and make 6 moves. What is the final point?

5. Rule: $(x, y) \to (2x + 1, y - 4)$.
After 3 moves you are at $(-9, -6)$. What was the starting point?

6. Rule: $(u, v) \to (3u - 2, 2v + 1)$.
After 3 moves you are at $(109, -9)$. What was the starting point?

7. Rule: $(x, y) \to (\frac{1}{2}x, \frac{1}{2}y)$.
How many moves will it take to go from $(64, 64)$ to $(1, 1)$?
How many moves to go from $(1, 1)$ to $(0, 0)$?

8. Rule: $(x, y) \to (x^2, y^2)$.
Start at $(1, 2)$ and make 3 moves. What is the final point?

9. Rule: $(x, y) \to (\sqrt{x}, \sqrt{y})$.
After 3 moves you are at $(2, 3)$. What was the starting point? Can you start this game at $(16, -9)$? Where can you start the game so that after 2 moves you are at $(-2, -3)$?

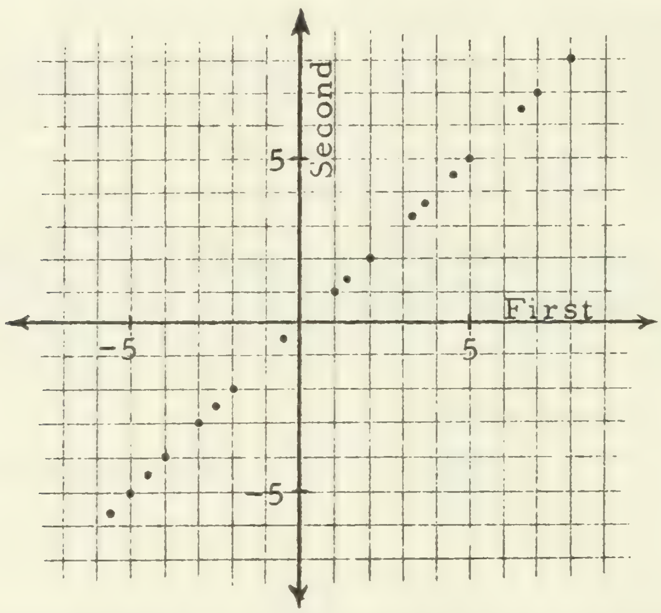
10. Rule: $(x, y) \to (\frac{1}{x}, \frac{1}{y})$.
Start at $(3, 5)$ and make 3 moves. What is the final point? Start at $(3, 5)$ and make 1000 moves. What is the final point? What is the final point if you start at $(3, 5)$ and make 1001 moves? Can you use this rule if you start at $(2, 0)$? Where can you start if you want to end at $(0, 0)$?

EXPLORATION EXERCISES

A. For each of the sets of ordered pairs described below, plot as many of the ordered pairs as you can on a picture of the number plane.

Sample. The set of all ordered pairs of real numbers such that the first component is equal to the second component.

Solution. Some of the ordered pairs in this set are (8, 8), (−2, −2), (1.3, 1.3), (π, π), (−5, −5), and (0, 0). Since this is an infinite set, we can't plot all of them. But, we can plot quite a few of them until we see some sort of pattern.



Do you see a pattern? Can you fill in more dots? Do you see a quick way of getting all the dots?

1. The set of all ordered pairs of real numbers such that the first component is equal to the product of −2 by the second component.

2. The set of all ordered pairs of real numbers such that the first component is 2 more than the second component.

3. $\{(x, y): \ x = \frac{1}{2}y + 3\}$

4. $\{(a, b): \ a + b = 9\}$

5. The set of all ordered pairs of real numbers such that the first component is 4

6. $\{(x, y): \ y = 2\}$

7. $\{(x, y): y > 2\}$

8. $\{(x, y): y \le 3\}$

9. $\{(x, y): x \ge 1\}$

10. $\{(p, q): q > p\}$

11. $\{(q, p): q > 2 \text{ and } p > 3\}$

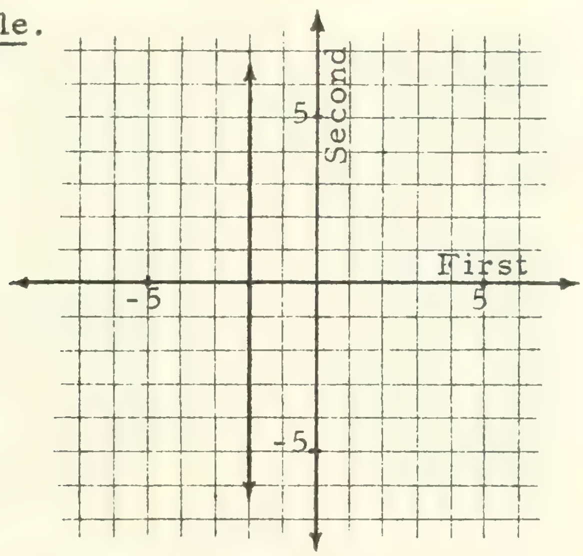
12. $\{(x, y): x \le 1 \text{ and } y \ge 2\}$

13. $\{(x, y): x \le 1 \text{ or } y \ge 2\}$

[More exercises are in Part F, Supplementary Exercises.]

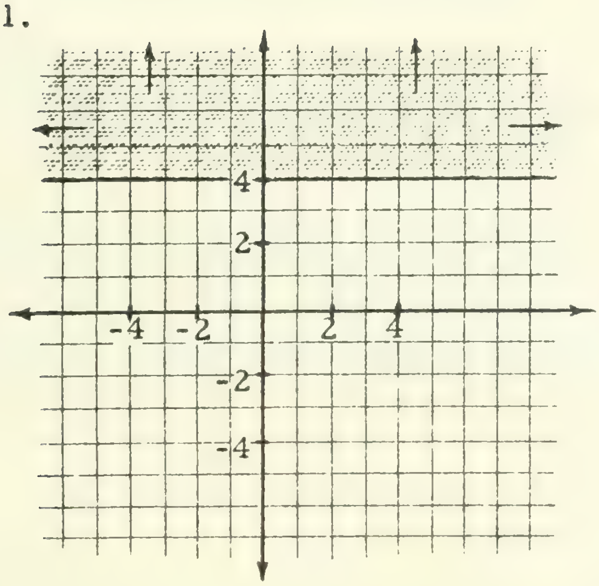
B. For each set pictured below, write its description.

Sample.



Solution. $\{(x, y): x = -2\}$

1.



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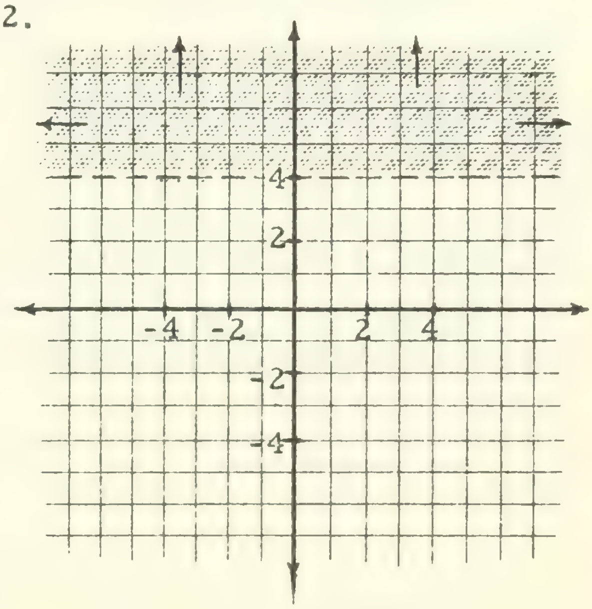

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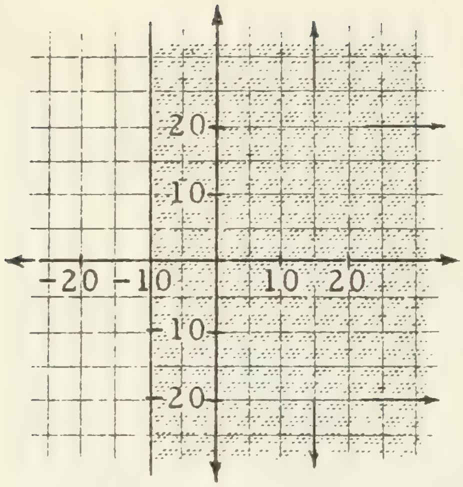
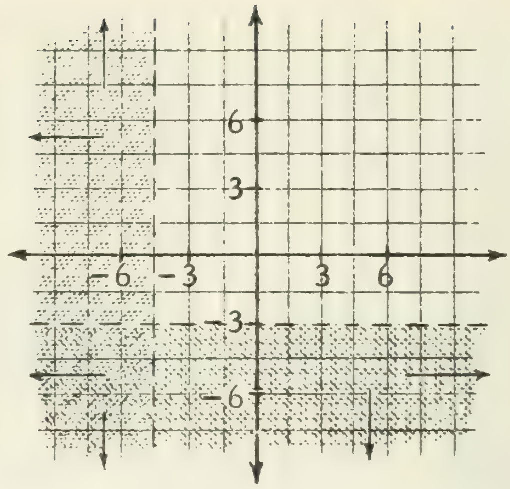
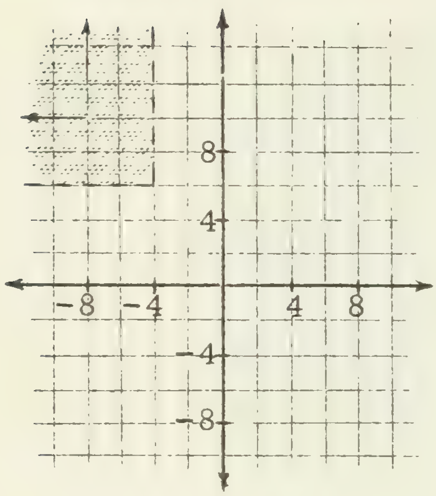
3.



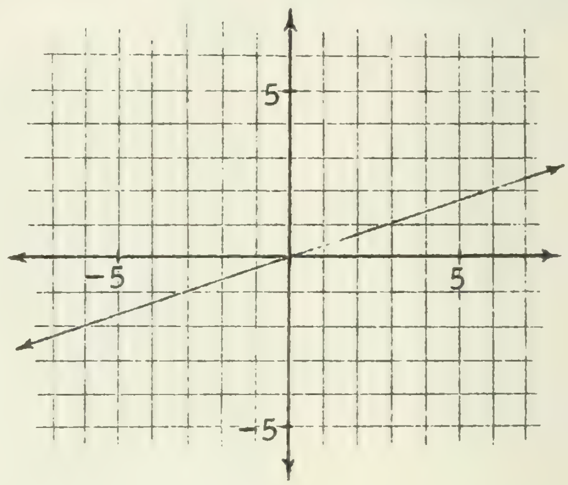
4.



5.



6.



[More exercises are in <u>Part</u> <u>G</u>, <u>Supplementary</u> <u>Exercises</u>.]

LOCUS AND GRAPH

You will recall from Unit 3 that an open sentence such as:

$$x + 5 > 9$$

is satisfied by many numbers. The set of such numbers is the <u>solution set</u> or <u>locus</u> of ‘x + 5 > 9’. The graph [on a number line picture] of ‘x + 5 > 9’ is the picture made up of the graphs of the numbers in the solution set.

An open sentence such as:

$$m + 2k = 9$$

can also be satisfied. For example, we can get a true sentence from it by substituting ‘7’ for ‘m’ and ‘1’ for ‘k’. If we wish to report this fact,

it is not enough to say that the two numbers 1 and 7 satisfy the sentence [Why?]. Here, as in the Introduction, the notion of ordered pair is helpful. If we agree to consider 'm' as the "first" pronumeral and 'k' as the "second" pronumeral then what we mean will be clear if we say that the ordered pair (7, 1) satisfies the sentence. Under this agreement we say that the locus of the sentence:

$$m + 2k = 9$$

is

$$\{(m, k): \ m + 2k = 9\}.$$

Without such an agreement it would make no sense to ask what the solution set of '$m + 2k = 9$' is. However, we can indicate the agreement and, at the same time, ask the proper question by saying:

What is the solution set <u>in</u> <u>(m, k)</u> of '$m + 2k = 9$'?

One answer to this question is the name:

$$\{(m, k): \ m + 2k = 9\}.$$

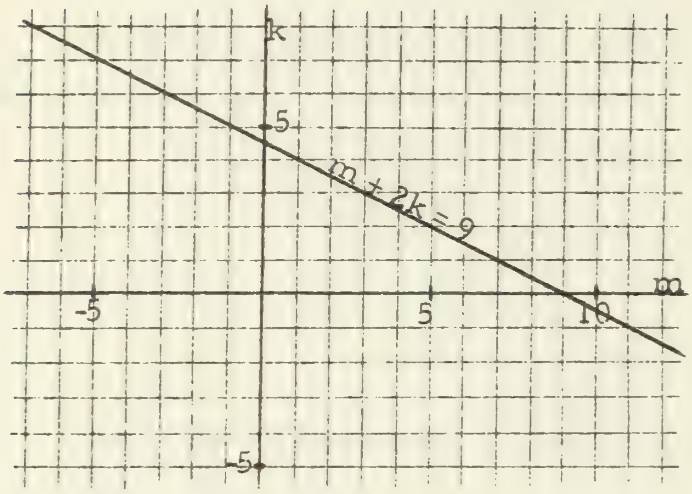
Another name for this locus is:

$$\{(x, y): \ x + 2y = 9\},$$

and still another name is:

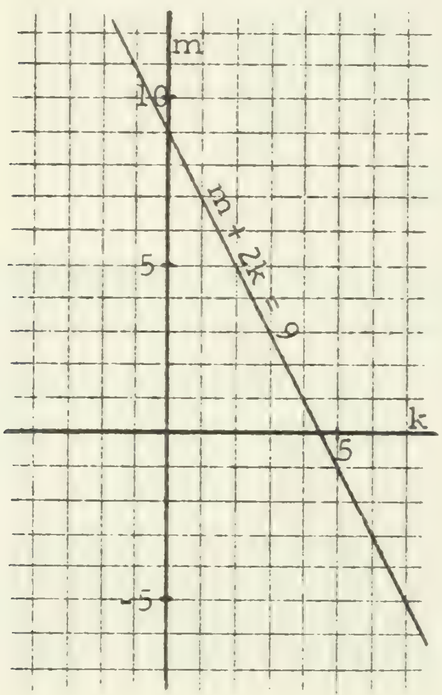
$$\{(k, m): \ k + 2m = 9\}.$$

The graph of a sentence is made up of graphs of the ordered pairs which belong to the solution set of the sentence. So, to draw the <u>graph</u> <u>in</u> <u>(m, k)</u> of '$m + 2k = 9$', you draw a picture of $\{(x, y): \ x + 2y = 9\}$. In drawing such a picture we follow our convention of using a horizontal line to picture the first component axis and a vertical line for the second. It is also customary to use the "first" pronumeral as a label at the right (or "positive") end of the picture of the first component axis, and the "second" pronumeral as a label at the upper (or "positive") end of the picture of the second component axis.



This is a picture (of part) of
$\{(x, y): x + 2y = 9\}$.

The diagram above shows the graph in (m, k) of 'm + 2k = 9'. Compare it with the following diagram which shows the **graph in** (k, m) of 'm + 2k = 9'.



This is a picture (of part) of
$\{(y, x): x + 2y = 9\}$.
[It is also a picture (of part) of
$\{(x, y): y + 2x = 9\}$.]

EXERCISES

A. Make labeled pictures of these loci.

1. the locus in (a, b) of 'a - 2b = 10'
2. the locus in (s, r) of 'r = 2s + 7'
3. the locus in (s, r) of 's = 2r + 7'
4. the locus in (m, n) of 'm = 3'
5. the locus in (x, y) of 'x > 5 and y > 5'
6. the locus in (u, v) of 'u $\geq$ 3 and v $\leq$ 2'
7. the locus in (p, k) of 'p $\geq$ 1 or k $\leq$ 2'
8. the locus in (c, d) of 'c $\geq$ d + 5'

* * *

In dealing with sentences which contain the pronumerals 'x' or 'y', it is customary to regard 'x' as the "first" pronumeral and 'y' as the "second" pronumeral unless the contrary is stated. So, for example, when one says:

the locus of '3x + 2y = 5'

he means

the locus in (x, y) of '3x + 2y = 5'.

This convention is widely observed, and, indeed, the first component axis is often called the x-axis, the second component axis, then, being called the y-axis. In addition, we say that the abscissa of a dot is its x-coordinate, and the ordinate of a dot is its y-coordinate. The number plane is also called the (x, y)-plane.

* * *

B. Graph each of the following sentences. [That is, draw a picture of its locus.]

1. $y \leq 5x$
2. $y \geq x - 5$
3. $y = \frac{1}{3}x + 3$
4. $y = 1.5$

(continued on next page)

5. $x \geq -3$

6. $x + y \leq 2$

7. $3x = y$

8. $y = x + 1$

9. $x + y = 0$

10. $2x + 3 = y$

11. $y = x - 3$

12. $2x - 8 = 0$

13. $y + 5 = 0$

14. $10x + 4y = 0$

15. $y = 5x$

16. $3y = 4x + 1$

17. $3y + 2x = 6$

18. $3y - 2x = 6$

19. $x^2 = 9$

20. $x^2 \leq 9$

21. $|y| < 3$

22. $|xy| > 0$

23. $x + y = 7 + x$

24. $xy = 0$

25. $x = 3$ and $y = 2$

26. $x = 3$ or $y = 2$

27. $x = 3$ and $y > 2$

28. $x = 3$ or $y > 2$

29. $y = 7$ or $y = 5$ or $y = 3$

30. $y = 7$ and $y = 5$ and $y = 3$

31. $|x| + |y| = 10$

32. $|x| + |y| \leq 10$

33. $x + y = y + x$

34. $x - y = y - x$

35. $x^2 + y^2 = 0$

36. $x^2 - y^2 = 0$

37. $xy = yx$

38. $x \div y = y \div x$

39. $\frac{x}{x} + \frac{y}{y} = 2$

C. Use the pages at the end of the unit, and plot the points in each of the four regions which belong to the locus of each of the following sentences.

1. $x = 1\,000\,000$

2. $y = -450\,002$

3. $x > y$

4. $y < 1$

5. $x = y$

6. $y \geq x$

7. $y = -13$

8. $x < y$

9. $y = 2x$

10. $x + y = y + x$

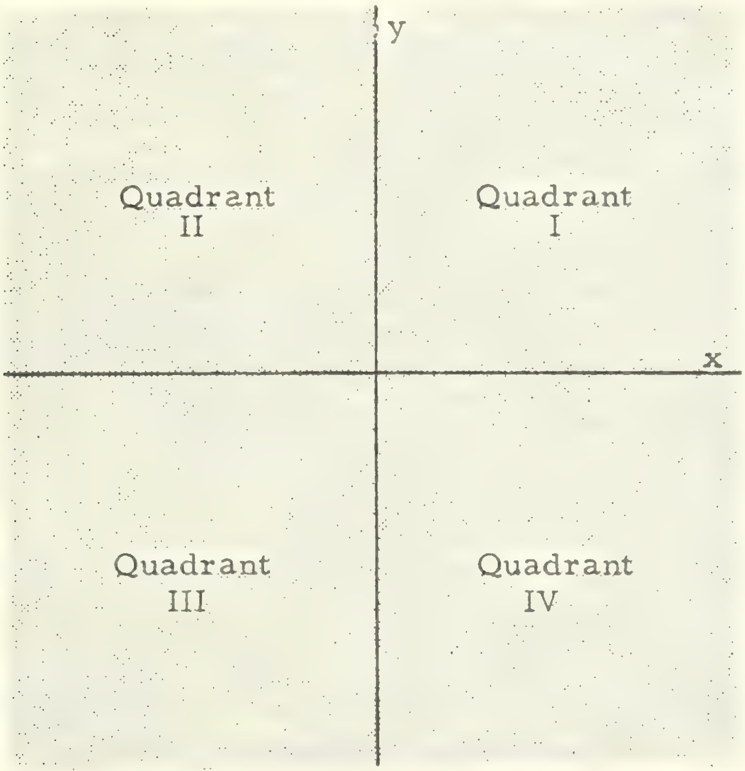
D. The x- and y-axes separate the points in the (x, y)-plane into 4 regions called quadrants. Each quadrant is a set of points.

Quadrant I is $\{(x, y): x > 0 \text{ and } y > 0\}$.

Quadrant II is $\{(x, y): x < 0 \text{ and } y > 0\}$.

Quadrant III is $\{(x, y): x < 0 \text{ and } y < 0\}$.

Quadrant IV is $\{(x, y): x > 0 \text{ and } y < 0\}$.



What is the union of the four quadrants? What is the union of the four quadrants and the x- and y-axes? What is the intersection of any two quadrants? What is the intersection of any quadrant and an axis? What is the intersection of the x-axis and the y-axis?

(continued on next page)

Look at the following table. For each of the sentences in the left-hand column, tell which quadrants contain points of its solution set. Try to answer these questions without making drawings. [The first exercise has been completed for you. Check it.]

	Quadrant			
	I	II	III	IV
1. (a) $y = -2x + 3$	yes	yes	no	yes
(b) $y = -x + 3$				
(c) $y = 3$				
(d) $y = x + 3$				
(e) $y = 2x + 3$				
2. (a) $y = -2x - 3$				
(b) $y = -x - 3$				
(c) $y = -3$				
(d) $y = x - 3$				
(e) $y = 2x - 3$				
3. (a) $y = 0$				
(b) $x = 0$				
4. (a) $x = 4$				
(b) $x = y + 4$				
(c) $x = -y + 4$				
5. (a) $x = -4$				
(b) $x = y - 4$				
(c) $x = -y - 4$				
6. $x^2 + y^2 = 25$				
7. $\lvert x \rvert = 1$ and $\lvert y \rvert = 1$				

E. Each of the following exercises contains a pair of equations. Graph each of the two equations and give the ordered pairs which are in the intersection of their solution sets.

1. (a) $x + y = 6$
   (b) $3x + y = 2$

2. (a) $2x + 5y = -5$
   (b) $y + 3x = 12$

3. (a) $5x = 3 - 2y$
   (b) $y = 5 + x$

4. (a) $x = y$
   (b) $2x = 3 + 3y$

5. (a) $x = 7$
   (b) $y + 8 = 2x$

6. (a) $y = 3$
   (b) $2x = 7 + x$

7. (a) $|x| = 3$
   (b) $y - 2x = 5$

8. (a) $yy = 25$
   (b) $xx = 36$

9. (a) $2x = 3y$
   (b) $2x = 3y + 5$

10. (a) $5x - 2y = 8$
    (b) $6(y + 4) = 15x$

11. (a) $|x| = 5$
    (b) $|y| = 4$

12. (a) $|x - 3| = y$
    (b) $|y + 4| = -x$

13. (a) $x + y = 7.5$
    (b) $2x - 4y = 0$

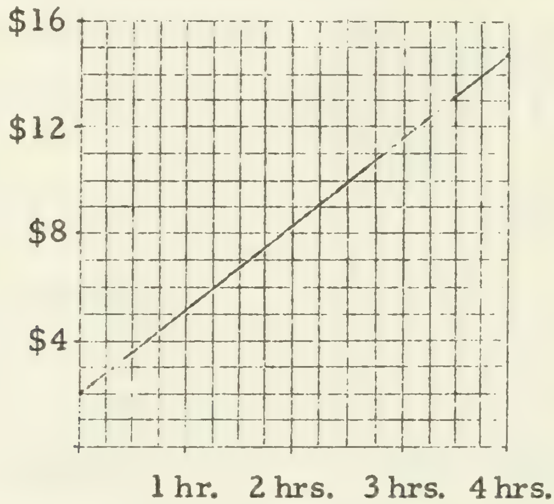
14. (a) $|x + 5| = y$
    (b) $|y - 6| = x$

15. (a) $y = 7 - 2(5 - y)$
    (b) $x = \frac{2}{7}y - 2$

16. (a) $(x - 4)(x + 4) = 0$
    (b) $y^2 - 5y + 6 = 0$

[More exercises are in Part H, Supplementary Exercises.]

4.03 Graphs of formulas.--A repairman uses the following procedure in charging for service calls. He charges $2.00 for going to a home, and he charges $3.20 more for each hour that he works. He could use the following chart to determine the amount to charge.



Use the chart to answer these questions.

(1) What is the charge if the repairman works 2 hours?

(2) What is the charge if the repairman works $3\frac{1}{2}$ hours?

(3) What is the charge if the repairman works 45 minutes?

(4) How long did the repairman work if the charge was $6.25?

(5) How long did the repairman work if the charge was $2.00?

(6) How long did the repairman work if the charge was $1.00?

The repairman could also use a formula instead of a chart to compute his charge. If he uses a 'c' to hold a place for a numeral which tells his charge and a 't' to hold a place for a numeral for the number of hours he works, then he can use the equation:

$$c = 2.00 + 3.20t$$

as a formula for computing his charges. [The equation serves as a formula when one has decided on the "use" of the pronumerals 'c' and 't'.] For example, if he worked 2 hours, he would replace 't' by '2' and obtain:

$$\begin{aligned} c &= 2.00 + 3.20(2) \\ &= 2.00 + 6.40 \\ &= 8.40 \end{aligned}$$

which tells him that his charge for working 2 hours is $8.40.

Use the formula:

$$c = 2.00 + 3.20t$$

to answer these questions.

(7) What is the charge if the repairman works 4 hours?

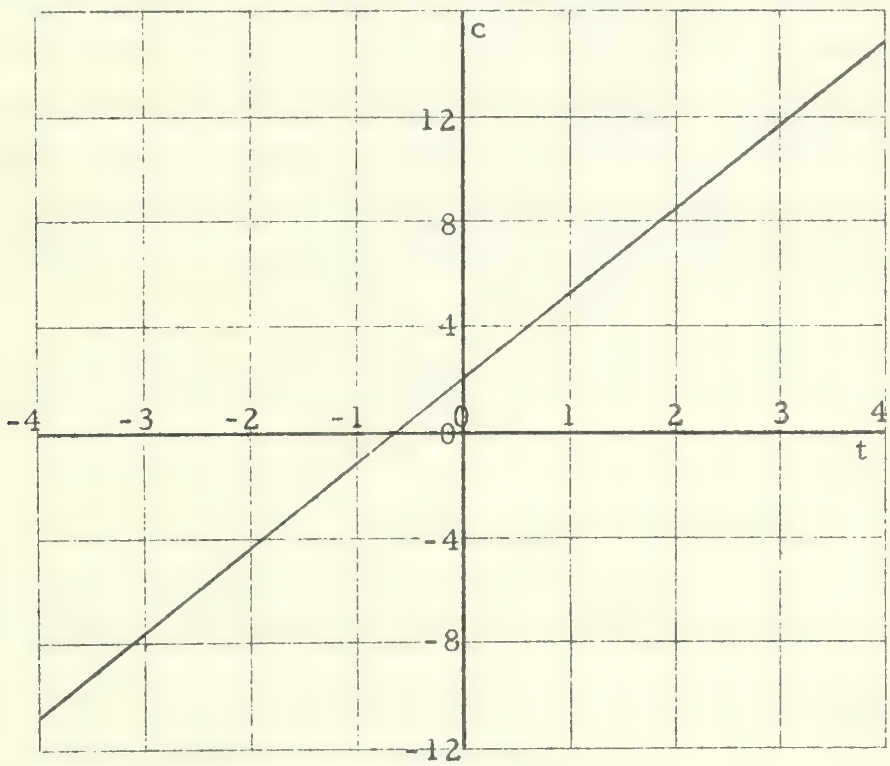
(8) What is the charge if the repairman works $2\frac{1}{2}$ hours?

(9) What is the charge if the repairman works 75 minutes?

(10) How long did the repairman work if the charge was $2.80?

(11) How long did the repairman work if the charge was $.40?

When you used the chart before to find charges, you were using the locus in (t, c) of '$c = 2.00 + 3.20t$'. Now, if you were told to graph '$c = 2.00 + 3.20t$' [using a c-axis for the vertical axis and a t-axis for the horizontal axis], you would get something like this.



This graph contains dots corresponding with some ordered pairs which satisfy the equation but which do not make sense in the problem to which the equation applies. For example, points in the second quadrant would correspond to negative numbers of hours worked. Points in the third quadrant would correspond to negative numbers of dollars charged and negative numbers of hours worked. Points in Quadrants II, III, and IV do not apply in this situation. Therefore, we should not even draw these quadrants [or the negative halves of the axes] when graphing '$c = 2.00 + 3.20t$' as a formula for finding repair charges.

Whenever you make a graph of a formula, you should keep in mind the kinds of numbers which enter into the problems to which the formula applies. In the repairman formula we are interested in charges and in time worked, both of which are measured by numbers of arithmetic. So, the domain of the pronumerals 'c' and 't' in '$c = 2.00 + 3.20t$' is the set of numbers of arithmetic. [The domain of a pronumeral is the set of its values.] Now, in transforming an equation which contains pronumerals whose domain is the set of numbers of arithmetic, we usually find it convenient to pretend that their domain is the set of nonnegative real numbers. Then, in solving, we pay attention only to nonnegative results. Because the nonnegative real numbers behave like the numbers of arithmetic in computation, this reinterpretation of an equation will not lead us into error. Similarly, when graphing a formula whose pronumerals have the set of numbers of arithmetic as domain, we can think of the locus of the formula as a subset of the cartesian square of the set of nonnegative real numbers [$\overrightarrow{0, 1} \times \overrightarrow{0, 1}$]. So, for example, we speak of points in the first quadrant belonging to the locus of the formula even though points in the first quadrant have real number components.

EXERCISES

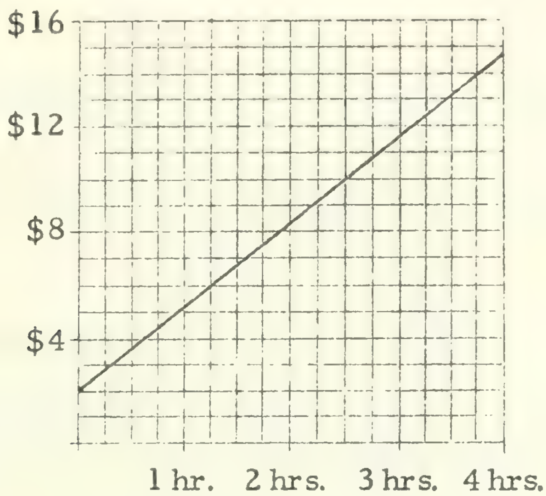
1. A formula for finding an approximation to the circumference of a circle is:

$$C = 6.28r.$$

Because the formula deals with measures of geometric figures, the domain of 'C' and of 'r' is the set of numbers of arithmetic. So, a graph of the formula is conveniently drawn on a picture of $\overrightarrow{0, 1} \times \overrightarrow{0, 1}$. Draw the graph, being sure to label the axes.

2. A towel service charges $1.00 minimum per week for making two calls and for the use of a container. In addition to the minimum, it charges 2 cents for each clean towel supplied. Write a formula for finding the weekly charge (c) in terms of the number (n) of towels used. The domain of 'c' is the set of numbers of arithmetic, and the domain of 'n' is the set of whole numbers of arithmetic. Make a graph for this formula.

3. A repairman uses the following chart to determine charges for his services.



(a) How much does he charge if he just makes the call and doesn't count any time at all?

(b) How much does he charge if he works one hour?

(c) How much does he charge if he works two hours?

(d) Not counting the charge for making the call, what is his charge per hour for labor?

(e) Give a formula which corresponds to the graph. Tell the domains of the pronumerals in the formula.

(continued on next page)

4. Temperature may be measured on a Fahrenheit scale or on a centigrade scale. Thus, the temperature of boiling water may be given as $212^\circ F$ or $100^\circ C$. A formula for finding the Fahrenheit reading when you know the centigrade reading is:

$$F = \frac{9}{5}C + 32.$$

Make a graph for this formula, and answer the following questions. [What is the domain of 'F' and of 'C'?]

(a) What Fahrenheit reading corresponds to a centigrade reading of 0°?

(b) If the Fahrenheit reading is 80°, what is the centigrade reading?

(c) If the centigrade reading is -15°, what is the Fahrenheit reading?

(d) If there is a 10-degree increase in the temperature measured on the centigrade scale, what is the corresponding increase measured on the Fahrenheit scale?

(e) If there is a 20-degree decrease on the Fahrenheit scale, what is the corresponding decrease on the centigrade scale?

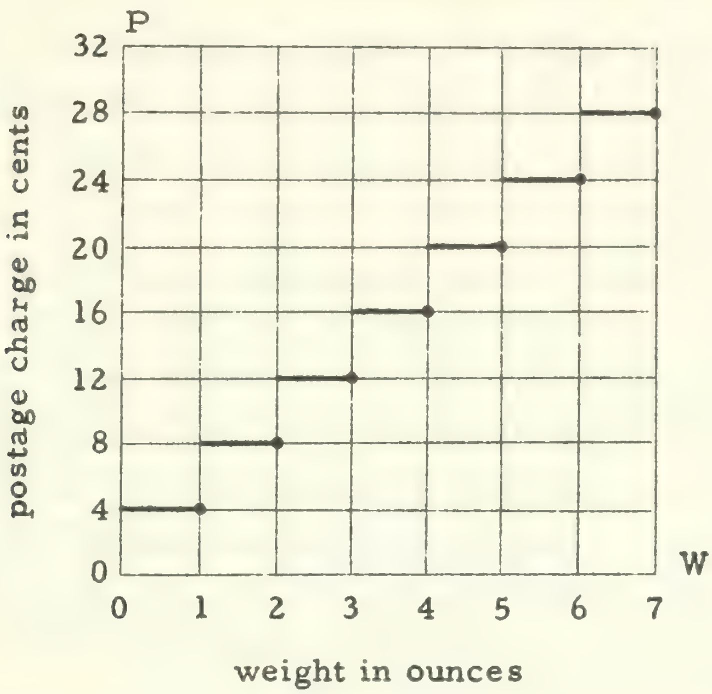
(f) Suppose the out-of-doors temperature rises during the period 12:00 noon to 1:00 p.m. on a certain day. Which of the two scales will show a greater change in readings?

5. The postage charge on first class mail is "4 cents for each ounce or part of an ounce." Use the chart on the next page to find the postage charge for

(a) $2\frac{1}{2}$ ounces (b) $\frac{1}{8}$ ounce

(c) $5\frac{1}{4}$ ounces (d) $\frac{1}{10}$ ounce

[Note: The graph shown in this chart is often called a step-graph.]



What is the domain of each of the pronumerals 'P' and 'W'?

6. The U.S. Post Office has a "book rate" of 9 cents for the first pound and 5 cents for each additional pound or part of a pound. Make a chart for determining the postage charge for all book shipments which do not exceed 15 pounds in weight.

   (a) How much is the postage charge for a 12-pound package of books?

   (b) What is the minimum weight and what is the maximum weight of packages which require exactly 29 cents postage?

   (c) What is the minimum weight and what is the maximum weight of packages which require exactly 22 cents postage?

   (d) What is the minimum weight and what is the maximum weight of packages which could be sent with 27 cents postage?

(continued on next page)

7. The sum of the numbers of years in the ages of two children is 8. Give a formula for finding the age of one child when you know the age of the other. Tell the domain of each of your pronumerals. Make a graph for this formula.

8. A farmer keeps cows and chickens. The total number of legs of these animals is 570. Consider all the ordered pairs in which the first number is a number of cows the farmer might own and the second number is the corresponding number of chickens he would own. For example, one ordered pair is (10, 265). If you listed all such possible ordered pairs, what would be the smallest first component? The smallest second component? The largest first component? The largest second component? Give a formula for finding the number of cows when you know the number of chickens.

4.04 Factors.--In Unit 3 we talked about factoring expressions. Transforming the expression '$x^2 + 6x + 8$' into '$(x + 4)(x + 2)$' is an example of factoring '$x^2 + 6x + 8$'. Each of the expressions '$x + 4$' and '$x + 2$' is a factor of '$x^2 + 6x + 8$'. So, factors of expressions are expressions. In this sense, '3' and '7' are factors of '21' because '21' and '$3 \cdot 7$' are equivalent expressions.

Another use of the noun 'factor' is illustrated by saying that each of the numbers 3 and 7 is a factor of the number 21. Is 5 a factor of 20? Is 1 a factor of 13? Is 3 a factor of 11? Is 6 a factor of 8? Is 10 a factor of 0? Is 0 a factor of 2? In answering these questions you probably considered the facts that

$$20 = 5 \cdot 4, \quad 13 = 1 \cdot 13, \quad 11 = 3 \cdot \frac{11}{3}, \quad 8 = 6 \cdot \frac{4}{3}, \quad 0 = 10 \cdot 0,$$

and that, for each x, $0 \cdot x \neq 2$. It may be easy to get everyone to agree that 5 is a factor of 20, that 1 is a factor of 13, that 10 is a factor of 0, and that 0 is not a factor of 2. But, there may be some doubt about whether 3 is a factor of 11 and whether 6 is a factor of 8. There is a number [4] whose product with 5 is 20, there is a number [13] whose product with 1 is 13 and there is a number [$\frac{11}{3}$] whose product with 3 is 11. But, this last case differs from the preceding two in that 4 and 13 are integers but $\frac{11}{3}$ is not. Most people, when they speak of a factor

of an integer [such as 20, 13, and 11], mean an integer whose product by some integer is the given integer. On the other hand, it may sometimes be convenient to speak of 3 as a factor of 11. So, we need to give a careful definition of 'factors of a number' which will allow us to treat both the situation in which we want 3 to be a factor of 11 and the situation in which we do not want 3 to be a factor of 11. [Believe it or not, this can be done!]

SUBSETS OF THE SET OF REAL NUMBERS

In our definition we shall need to refer to subsets of the set of real numbers. In particular, we shall consider these subsets:

(1) The set of positive integers

These are the real numbers 1, 2, 3, ... A more careful description of this set is that it is the set of real numbers consisting of 1 together with the numbers obtainable by successive additions of 1.

(2) The set of integers

This is the union of 3 sets--{0}, the set of positive integers, and the set whose members are the opposites of the positive integers.

(3) The set of rationals

A rational real number is a real number which is the quotient of real integers. For example, $\frac{3}{4}$ is a rational real number because it is the quotient of the integer 3 by the integer 4. Also, -5 is a rational real number because it is the quotient of 5 by -1. [Do you see that every real integer is a rational real number?] The set of rationals is the set consisting of all the rational real numbers. Is 3.7 a rational? Why? Is $-1\frac{5}{8}$ a rational?

[Note: Not every real number is rational. For example, $\sqrt{10}$ is not rational. Neither is π. Each real number which is not a rational is said to be an irrational real number. A rational real number is the ratio of a real integer to a real integer; an irrational real number is not the ratio of a real integer to a real integer.]

$$\text{real numbers}\begin{cases}\text{rationals}\begin{cases}\text{integers}\begin{cases}\text{positive integers}\\ 0\\ \text{negative integers}\end{cases}\\ \\ \text{nonintegral rationals}\end{cases}\\ \\ \text{irrationals}\end{cases}$$

EXERCISES

A. The numbers named below belong to at least one of these sets:

(I) the set of real numbers
(II) the set of rationals
(III) the set of integers
(IV) the set of positive integers

and you are to tell all the sets to which each number listed belongs.

Sample. $^-3$

Solution. $^-3$ is an integer but not a positive integer. So, $^-3$ belongs to set III but not to set IV. Each integer is a quotient of integers, that is, is a rational number. So, $^-3$ belongs to set II. Each rational number is a real number, so $^-3$ belongs to set I.

Answer. I, II, III.

1. -2	2. $75 + 34$	3. $\sqrt{100}$
4. $2\frac{7}{8}$	5. -3.125	6. $\frac{2}{9}$
7. $-(5/3)$	8. $47 \cdot -13$	9. $(-7)^2$
10. 0	11. $19 - 28$	12. $16 - 35.2$
13 $\frac{17}{19} \cdot \frac{18}{53}$	14. $0.1666...$	15. $0.8333...$

* * *

In Part A you may have had some doubts about whether 0.1666... and 0.8333... are rational numbers. But, do you recall that when you try to find the "decimal equivalent" for 1/6 by dividing 1 by 6, your answer doesn't come out "even"?

$$\begin{array}{r}
0.1666 \\
6\,\overline{)\,1.0000} \\
\underline{6} \\
40 \\
\underline{36} \\
40 \\
\underline{36} \\
40
\end{array}$$

Instead, you keep getting '6's in the quotient numeral. Because of this, we use the expression '0.1666...' [called <u>a</u> <u>repeating</u> <u>decimal</u>] as a name for the rational number 1/6. Similarly, the repeating decimal '0.8333...' is a name for the rational number 5/6.

A common abbreviation for the repeating decimal '0.1666...' is '$0.1\bar{6}$' and for '0.8333...' is '$0.8\bar{3}$'.

Here is another example of a repeating decimal:

$$0.583583583...,$$

or, for short:

$$0.\overline{583}\ .$$

Is $0.\overline{583}$ a rational number? One way to find out is to hunt around for a division problem in which the quotient numeral comes out to be '0.583583583...'. [You might hunt for a long time!] Here is a way to find two integers whose quotient is $0.\overline{583}$ without hunting.

Notice that

$$\begin{aligned} 0.583583583... \times 1000 &= 583.583583583... \\ &= 583 + 0.583583583...\ . \end{aligned}$$

So, 0.583583583... is a solution of the equation:

$$1000k = 583 + k.$$

Since the only solution of this equation is 583/999, it must be the case that

$$0.\overline{583} = \frac{583}{999}\,.$$

[Check by dividing 583 by 999.]

Is $\overline{5.1}$ a rational number?

$$\begin{array}{r}5.15151\ldots \times 100 = 515.15151\ldots \\ \underline{5.15151\ldots \times \quad 1 = \quad 5.15151\ldots} \\ 5.15151\ldots \times \ 99 = 510.\end{array}$$

So, by the division theorem, $\overline{5.1} = \frac{510}{99}$.

* * *

B. Find decimal names for the rational numbers listed.

1. 2/3   2. 1/9   3. 3/11   4. 14/11   5. 2/7

C. Show that these repeating decimals stand for rational numbers.

1. $0.\overline{3}$ 2. $0.\overline{8}$ 3. $0.\overline{10}$ 4. $3.\overline{10}$

5. $0.4\overline{9}$ 6. $0.\overline{49}$ 7. $3.2\overline{81}$ 8. $27.389\overline{2}$

D. Is each sum of a positive integer and a positive integer also a positive integer? The answer is 'yes' [although, as yet, we have no way of proving this]. For this reason we say that the set of positive integers is closed under addition.

1. Do you think the set of positive integers is closed under multiplication? Under subtraction? Under division?

2. Use the facts that the set of positive integers is closed under addition and multiplication and that the set of integers is closed under oppositing in proving that the set of integers is closed under addition, subtraction, and multiplication.

3. Under what operations is the set of rational numbers closed?

IRRATIONAL NUMBERS

You have seen how to show that any number which is named by a repeating decimal is a rational number. It is also the case that each rational number can be named by a repeating decimal. [Some rational numbers have two such repeating decimal names. For example, $1/2 = 0.4\overline{9}$ and $1/2 = 0.5\overline{0}$. Also, $74 = 73.\overline{9}$ and $74 = 74.\overline{0}$.]

So, any real number whose decimal name is not a repeating decimal is an irrational number. For example,

$$1.0100100010000 1\ldots \text{ is an irrational number.}$$

More interesting examples of irrational numbers are

$$\sqrt{2}, \quad \sqrt{3}, \quad \pi, \quad \sqrt{131}, \quad 5\sqrt{5}, \quad \text{and} \quad 1/\sqrt{3}.$$

Suppose you know that $\sqrt{7}$ is irrational. Does it follow that $\sqrt{7}/3$ is irrational? Yes, because if $\sqrt{7}/3$ were rational then $(\sqrt{7}/3) \cdot 3$, that is, $\sqrt{7}$, would be a product of rational numbers. And, you know that each product of rationals is rational.

(1) Given that $\sqrt{5}$ is irrational, show that $3\sqrt{5}$ is irrational.

(2) Is each product of a rational number by an irrational number an irrational number?

(3) Given that π is irrational, show that $\pi + 5$ is irrational.

(4) Is each sum of an irrational number and a rational number irrational?

(5) Given that $\sqrt{11}$ is irrational, show that $1/\sqrt{11}$ is irrational.

(6) Is the reciprocal of each irrational number irrational?

(7) Is each sum [product] of two irrational numbers irrational?

(8) Show that $\sqrt{2} + \sqrt{3}$ is irrational. [Hint: Square the number.]

(9) Is it the case that, for each $x \geq 0$, if $\sqrt{x}$ is an integer then so is x?

(10) Is it the case that, for each $x \geq 0$, if x is not an integer then $\sqrt{x}$ is not an integer?

(11) Is it the case that, for each $x \geq 0$, if x is an integer then so is $\sqrt{x}$?

*

☆ The answer to the last question is 'no' as is shown by the fact that 8 is a counter-example. 8 is an integer but, since $2^2 < 8 < 3^2$ and there is no integer between 2 and 3, $\sqrt{8}$ is not an integer. As a matter of fact, $\sqrt{8}$ is not even a rational number. Let's prove this.

Suppose $\sqrt{8}$ were rational. Then there would be many positive integers which could be divided into integers to give $\sqrt{8}$. Among these positive integers there would have to be a smallest one.

Let q be the smallest positive integer such that, for some integer p, $\sqrt{8} = p/q$. Since $2 < \sqrt{8} < 3$,

$$2 < \frac{p}{q} < 3,$$

$$2q < p < 3q,$$

and

$$0 < p - 2q < q.$$

Now, p - 2q is a positive integer <u>smaller</u> than q. So, since q is the smallest positive integer such that $\sqrt{8} \cdot q$ is an integer,

(*) $\qquad \sqrt{8}(p - 2q)$ is not an integer.

But, $\sqrt{8} = p/q$. So,

$$\begin{aligned} &\sqrt{8}(p - 2q) \\ &= \frac{p}{q}(p - 2q) \\ &= \frac{p^2}{q} - 2p \\ &= \frac{p^2}{q^2}q - 2p \\ &= 8q - 2p. \end{aligned}$$

Since q and p are integers, 8q - 2p is an integer. That is,

(**) $\qquad \sqrt{8}(p - 2q)$ is an integer.

(**) contradicts (*). Since (*) and (**) follow from the supposition that $\sqrt{8}$ is rational, this contradiction shows that $\sqrt{8}$ is <u>not</u> rational.

*

☆(12) Prove that $\sqrt{31}$ is irrational.

☆(13) What happens when you try to use the method illustrated above to show that $\sqrt{16}$ is irrational?

☆(14) Prove that, for each positive integer n, if $\sqrt{n}$ is not an integer then $\sqrt{n}$ is irrational.

☆(15) Prove that, for each positive integer n, $\sqrt{n - 1} + \sqrt{n + 1}$ is irrational.

FACTORS OF NUMBERS

We mentioned that we want a definition which will allow us to say, sometimes, that 3 is <u>not</u> a factor of 11, and, sometimes, that 3 <u>is</u> a factor of 11. We accomplish this by saying that

> 3 is not a factor of 11 <u>with respect to the set of integers</u> because 11 is not the product of 3 by any <u>integer</u>,

and by saying that

> 3 is a factor of 11 <u>with respect to the set of rationals</u> because 11 is the product of 3 by some <u>rational number</u> [11/3].

Is −5 a factor of 15? This question is ambiguous. −5 is a factor of 15 with respect to the set of integers [Explain], but −5 is not a factor of 15 with respect to the set of positive integers [Explain]. Is −5 a factor of 15 with respect to the set of rationals? With respect to the set of irrationals? With respect to the set of reals?

In general,

for each set S of numbers,

x is a factor of y with respect to S
if and only if
x and y are in S, and there is a z in S
such that y = xz.

EXERCISES

<u>A</u>. Complete each of these sentences to true ones in at least one way.

<u>Sample</u>. 4 is a factor of 7 with respect to the set of _______ because 4, 7, and ____ belong to this set and 7 = 4 · ____.

<u>Solution</u>. One completion:

4 is a factor of 7 with respect to the set of <u>*rationals*</u> because 4, 7, and <u>7/4</u> belong to this set and 7 = 4 · <u>7/4</u>.

Another completion:

4 is a factor of 7 with respect to the set of <u>*reals*</u> because 4, 7, and <u>7/4</u> belong to this set and 7 = 4 · <u>7/4</u>.

(continued on next page)

1. 3 is a factor of 39 with respect to the set of __________
because 3, 39, and ____ belong to this set and $39 = 3 \cdot$ ____.

2. 5 is a factor of -20 with respect to the set of __________
because 5, -20, and ____ belong to this set and $-20 = 5 \cdot$ ____.

3. -10 is a factor of -20 with respect to the set of __________
because -10, -20, and ____ belong to this set and $-20 = -10 \cdot$ _____.

4. -6 is a factor of -8 with respect to the set of __________
because -6, -8, and ____ belong to this set and $-8 = -6 \cdot$ _____.

5. $\frac{1}{2}$ is a factor of 9 with respect to the set of __________
because $\frac{1}{2}$, 9, and ____ belong to this set and $9 = \frac{1}{2} \cdot$ _____.

6. $\frac{3}{5}$ is a factor of $\frac{2}{17}$ with respect to the set of __________
because $\frac{3}{5}$, $\frac{2}{17}$, and _____ belong to this set and $\frac{2}{17} = \frac{3}{5} \cdot$ _____.

7. 8 is a factor of 64 with respect to the set of __________
because 8, 64, and ____ belong to this set and $64 = 8 \cdot$ _____.

8. $\sqrt{10}$ is a factor of 10 with respect to the set of __________
because $\sqrt{10}$, 10, and _____ belong to this set and $10 = \sqrt{10} \cdot$ ____.

9. 9 is a factor of 0 with respect to the set of __________
because 9, 0, and ____ belong to this set and $0 = 9 \cdot$ ____.

10. 0 is a factor of ____ with respect to the set of __________
because 0, ____, and ____ belong to this set and ____ $= 0 \cdot$ ____.

11. 783 is a factor of $783 \cdot 429$ with respect to the set of __________
because 783, $783 \cdot 429$, and ________ belong to this set and
$783 \cdot 429 = 783 \cdot$ _______.

12. 17 is a factor of $17 \cdot 89 + 17 \cdot 11$ with respect to the set of
__________ because 17, $17 \cdot 89 + 17 \cdot 11$, and _______ belong
to this set and $17 \cdot 89 + 17 \cdot 11 = 17 \cdot$ ______.

13. 11 is a factor of 33 with respect to the set of integers. So, 11 is a factor of 33 • ____ with respect to the set of integers.

14. 13 is a factor of 26 and of 39 with respect to the set of integers. So, $\_\_\_\_^2$ is a factor of 26 • 39.

[More exercises are in Part I, Supplementary Exercises.]

B. 1. Exercise 12 of Part A suggests an interesting generalization. You noticed there that 17 is a factor of both 17 • 89 and 17 • 11 with respect to the set of integers [and perhaps other sets]. From this you probably concluded that 17 is a factor of the sum of 17 • 89 and 17 • 11 with respect to the set of integers. In fact, you probably suspect that the following generalization is a theorem:

$\forall_a \forall_b \forall_c$, with respect to the set of integers,

if a is a factor of b and of c then a is a factor of b + c.

(a) Prove it. [Hint: If a is a factor of b and of c with respect to the set of integers, then there are integers m and n such that b = am and c = an. ....]

(b) In the theorem above, replace 'integers' by 'positive integers'. Is the resulting generalization a theorem?

2. Exercise 13 of Part A suggests a generalization about a factor of the product of two integers. State and prove it.

3. Complete this to a theorem equivalent to the one just proved in Exercise 2:

$\forall_a \forall_b \forall_c$, with respect to the set of integers,

if a is a factor of b and b is a factor of c

then ________________________________.

4. State the theorem suggested by Exercise 14 of Part A.

5. Is it the case that, with respect to the set of integers, if a first number is a factor of a second, then the square of the first is a factor of the square of the second?

EVEN AND ODD NUMBERS

An <u>even number</u> is one which has 2 as a factor with respect to the set of integers.

Is 0 an even number? Is $^{-}4$? Is $\frac{2}{3}$? Is 3? Is 0.02?

Is the set of even numbers closed under oppositing?

Is the set of even numbers closed under addition? Prove that it is. How about subtraction? Multiplication? Division? Squaring? Square rooting?

Is it the case that each product of an even number by an integer is even? Prove that it is.

*

An <u>odd number</u> is an integer which is not even.

Is 7 an odd number? [From your earlier experience, you know that the answer is 'yes'. But, let's see if this fits the definition we just gave.] According to the definition, to show that the integer 7 is an odd number, we must show that 7 is not even. Let's do so.

If 7 were even then 7/2 would be an integer, and a positive one at that. We described the set of positive integers [page 4-43] as the set of real numbers consisting of 1 together with the real numbers obtainable by successive additions of 1. The first three positive integers are 1, 2, and 3. Since each of these is less than 7/2, 7/2 is not one of them. The next positive integer is 4 and the remaining positive integers are greater than 4. So, since $7/2 < 4$, 7/2 is not one of these. So, 7/2 is not a positive integer, and, therefore, 7 is not even.

This is a lot of work to show that a given integer is odd. There should be a better way. What we need is a theorem.

Given an integer n. Either n/2 is an integer or n/2 is not an integer. n/2 is an integer if and only if n is an even number. So, n is an odd number if and only if n/2 is not an integer. Now, if n/2 is not an integer, there is a pair of consecutive integers which bracket n/2. That is, there is an integer k such that

$$(1) \qquad k < \frac{n}{2} < k + 1.$$

Conversely, if there is such an integer k, n/2 is not an integer.

So, (1) is equivalent to:

$$0 < \frac{n}{2} - k < 1,$$

and so to:

$$(2) \qquad 0 < n - 2k < 2.$$

But, since n and 2k are integers, n - 2k is an integer. And, since 1 is the only integer between 0 and 2, sentence (2) is the case if and only if n - 2k = 1, that is, if and only if

$$(3) \qquad n = 2k + 1.$$

So, n is an odd number if and only if there is an integer k such that n = 2k + 1. For example, 7 is odd because $7 = 2 \cdot 3 + 1$. Also, $2 \cdot 9835416 + 1$ is an odd number.

We have proved the following theorem:

$\forall_n$ [n is an odd number
if and only if
there is an integer k
such that n = 2k + 1].

There is a similar theorem for even numbers which follows easily from the definition of 'even number'.

$\forall_n$ [n is an even number
if and only if
there is an integer k
such that n = 2k].

These two theorems are the justification for the quick way of telling when an integer is an odd number--just find out if the integer is 1 more than an even number.

*

There is another way of telling quickly whether an integer is even or odd. Take a look at the standard decimal numeral for the integer. If the last digit is a '0', a '2', a '4', a '6', or an '8', the integer is even. Otherwise, it is odd. Can you explain why this test works?

*

Pick a pair of integers. If neither is even then their sum is even. Why? This question and others like it will be very easy to answer after you read the next page.

The two theorems on page 4-53 about even and odd numbers make it easy to prove other theorems about evenness and oddness.

Example. Prove that each sum of an odd number and an even is odd.

First, let's state the theorem in a form which will help us write the proof:

$$\forall_m \forall_n \text{ if } m \text{ is odd and } n \text{ is even then } m + n \text{ is odd.}$$

We want to start with the premiss:

m is odd and n is even

and derive from it the conclusion:

m + n is odd.

The two theorems we have proved give us "standard forms" for odd and even numbers. For example, to show that m + n is odd, it is sufficient to show that m + n is 1 more than the product of 2 by an integer.

Proof.

Suppose that m is odd and n is even. Then there are integers x and y such that

$$m = 2x + 1 \text{ and } n = 2y.$$

So,

$$\begin{aligned} m + n &= 2x + 1 + 2y \\ &= 2(x + y) + 1. \end{aligned}$$

Since x and y are integers, and since the set of integers is closed under addition, it follows that there is an integer k such that

$$m + n = 2k + 1.$$

So, m + n is odd.

Therefore, if m is odd and n is even then m + n is odd.

Is it the case that if the sum of two integers is odd then one of them is even and the other odd?

EXERCISES

A. Prove these theorems.

1. Each sum of an odd number and an odd number is even.

2. The set of even numbers is closed under multiplication.

3. Each product of an odd number by an odd number is odd.

4. Each product of two consecutive integers is even.

   [Hint: Suppose m and m + 1 are consecutive integers. Either m is even or m is odd. Suppose m is even. Then . . . . Suppose m is odd. Then . . . .]

5. Each sum of two consecutive integers is odd.

6. For each integer p, if p^2 is odd then p is odd.

   [Hint: Suppose that p is not odd. From this, what can you say about p? And from that, what can you say about p^2?]

7. For each integer p, if p^2 is even then p is even.

☆8. Use the result of Exercise 7 to prove that $\sqrt{2}$ is irrational.

   [Hint: If $\sqrt{2}$ is rational then there are integers x and y such that $\sqrt{2} = x/y$, and such that x and y are not both even [Why?]. So, $x^2 = 2y^2$ . . . . Continue the argument, showing first that x must be even, and then that it follows that y^2 must be even. Complete the proof.]

EXPLORATION EXERCISES

1. What are all the factors of 12 with respect to the set of positive integers?

2. What are all the factors of 12 with respect to the set of integers?

3. What are all the factors of −12 with respect to the set of integers?

4. What are all the factors of −12 with respect to the set of negative integers?

5. What numbers are not factors of 12 with respect to the set of rationals? The set of reals?

6. What are the factors of 1 with respect to the positive integers?

(continued on next page)

7. Show that each positive integer other than 1 has at least two factors with respect to the set of positive integers.

8. Give five numbers which have exactly three factors with respect to the set of positive integers.

9. Give five numbers which have exactly two factors with respect to the set of positive integers.

10. Give five positive integers other than 1, which have no factors with respect to the set of positive-integers-greater-than-1.

PRIME NUMBERS

A positive integer which has exactly two factors with respect to the set of positive integers is a <u>prime number</u>. So, for example, 5 is a prime number since the only positive integers which are factors of 5 with respect to the set of positive integers are 1 and 5. 7 is another prime number. Is 1 a prime number? One factor of 1 with respect to the set of positive integers is 1. Does 1 have <u>another</u> factor with respect to the set of positive integers?

EXERCISES

<u>A</u>. Which of these numbers are prime numbers?

2, 6, 11, 1, 3, 17, 15, 84972, 61

<u>B</u>. 1. Show that a positive integer other than 1 is a prime number if, with respect to the set of positive integers, it has no factors other than itself and 1.

2. Show that a positive integer other than 1 is a prime number if it has no factor with respect to the set of positive-integers-greater-than-1.

C. 1. On a picture of "Quadrant I of the number plane lattice", draw a picture of $\{(x, y): x$ is a factor of y with respect to the set of positive integers$\}$. [Choose your scale so that your picture includes the graph of (30, 30).]

2. On each horizontal line indicate, by writing a numeral to the left of the vertical axis, the number of factors of the corresponding number.

3. How many numbers have just one factor with respect to the set of positive integers?

4. Which numbers have just two factors with respect to the set of positive integers?

5. Describe the numbers which have just three factors with respect to the set of positive integers.

☆6. Repeat Exercise 5 for four factors, instead of three.

D. A positive integer which is neither a prime number nor 1 is a composite number. Each composite number has at least three factors with respect to the set of positive integers [Explain]. Give five examples of composite numbers.

*

Let's agree that from now on when we talk about factors of numbers we mean factors with respect to the set of positive integers (unless we say otherwise).

*

1. All of the factors of the composite number 12 are 1, 2, 3, 4, 6, and 12. List all numbers which are factors of the composite number 24.

2. Find a composite number which has just four factors exactly two of which are prime factors. [A prime factor is a factor which is a prime number. For example, 2 is a prime factor of 16.]

3. Find a composite number which has exactly three factors only one of which is a prime factor.

(continued on next page)

4. Find a composite number which has exactly four factors only one of which is a prime factor.

5. Find a composite number which has exactly five factors only one of which is a prime factor.

6. Find a composite number which has exactly seven factors only one of which is a prime factor.

7. Find a composite number which has 2 and 3 as its only prime factors. Find four more such composite numbers.

8. The factors of 72 are 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, and 72. Of these, the only prime factors are 2 and 3. We can use these prime factors to factor '72'.

$$72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$$

We shall call the expression '$2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$' <u>the prime factorization of the number 72</u>. So, the prime factorization of a positive integer is a particular kind of numeral for the integer. Here are other examples of prime factorizations.

$18 = 2 \cdot 3 \cdot 3$	$40 = 2 \cdot 2 \cdot 2 \cdot 5$	$35 = 5 \cdot 7$
$130 = 2 \cdot 5 \cdot 13$	$1400 = 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 7$	$29 = 29$

It can be proved that each positive integer other than 1 has just one prime factorization.

Give the prime factorization for each of the listed numbers. [Can you find a systematic way of solving exercises like these?]

(a) 98	(b) 108	(c) 847
(d) 144	(e) 900	(f) 901

9. Use the prime factorizations you found in Exercise 8 in making a list of all the factors of each of the given numbers. [Can you find a systematic way of solving exercises like these?]

(a) 98	(b) 108	(c) 847
(d) 144	(e) 900	(f) 901

4.05 Exponents.--In Unit 3 we abbreviated expressions such as 'xx' to 'x^2', and '$3 \cdot 3$' to '3^2'. The raised numeral is called an exponent symbol [or: an exponent]. We can use numerals for other positive integers as exponent symbols in order to simplify more complicated expressions. For example, we can abbreviate

'$5 \cdot 5 \cdot 5$' to '5^3', 'zzzz' to 'z^4',

'$\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3}$' to '$\left(\frac{2}{3}\right)^4$' '(xx)(xx)' to '$(x^2)^2$',

'$2 \cdot 2 \cdot 2\,(3 \cdot 3)(4 \cdot 4 \cdot 4 \cdot 4)$' to '$2^3 \cdot 3^2 \cdot 4^4$',

'$(2x + y)(2x + y)(2x + y)$' to '$(2x + y)^3$',

'aaaaaaaaaaaa' to 'a^{12}',

and '(xxy)(xxy)y' to '$(x^2y)^2y$'.

EXERCISES

A. Use exponents to abbreviate each expression.

1. $6 \cdot 6 \cdot 6(aaaa)$ 2. xxxx(yyyy) 3. $36 \cdot 36 + xxx$

4. yyyy - aaaa 5. (xy)(xy)(xy)(xy) 6. (nnn)(nnn) - (rrs)(rrs)

7. (abb)(abb)a 8. $(a + bb)(a + bb)$ 9. $\dfrac{2 \cdot 2 \cdot 2(xx) - yyyyy}{3 \cdot 3(xxx) + 6 \cdot 6 \cdot 6 \cdot 6y}$

[More exercises are in Part J, Supplementary Exercises.]

B. Use exponent notation to abbreviate prime factorizations for the listed numbers.

Sample. 5760

Solution. $5760 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$

$= 2^7 \cdot 3^2 \cdot 5.$

1. 2700 2. $4^3 \cdot 9^2$ 3. $10^3 \cdot 15^2$

4. 2548 5. $1287 \cdot 10^2$ 6. 1000000

* * *

Using the commutative and associative principles for multiplication, you can see that

$(5 \cdot 4)(5 \cdot 4) = 5^2 \cdot 4^2$, $\quad 8^3 \cdot 8^2 = 8^5$, $\quad (x^3 y^2)^2 = x^6 y^4$,

and that '$z^3 \cdot z^2 \cdot z$' and 'z^6' are equivalent.

Also, $\frac{4^7}{4^3} = 4^4$, and '$\frac{x^3}{x^5}$' and '$\frac{1}{x^2}$' are equivalent, $[x \neq 0]$.

* * *

<u>C</u>. Examine the following expressions, and sort them into sets of equivalent expressions such that all the expressions (and only those) equivalent to a given expression are in the same set with it. [The domains of the pronumerals are such that no denominator can be converted into a name for 0.]

$y \cdot y^5$	x^3y^4	$\left(\frac{8}{2}\right)^2$	the square of 6	$y(y^2)^2$	$2 \cdot 2^2$	
$2 \cdot 3^2$	$x^2 \cdot x$	$y^2y^2y^2$	$xy^2x^2y^2$	xx^2x	$4 + 2$	y^5
$y^5 \cdot y$	$2^5 \div 2$	x^3	y^6	$x^2 \cdot x^2$	$x^3 \cdot x$	4^2	xy^2
$2 \cdot 2 \cdot 3^2$	$y^2 \cdot y^3 \cdot y$	$(y^2)^2$	$\frac{8^2}{2^2}$	$y^2 \cdot y^2 \cdot y$	$(xy)^2$	4
$(x^2y)^2y$	$(x^2)^2y^3$	$6 \cdot 6$	$2^2 \cdot 3^2$	the square of y^2	6^2	
$2(2)^2$	$\frac{4^2}{2}$	the square of $(2 \cdot 3)$	$\frac{x^4y^4}{x^2y^2}$	$\frac{2^8}{2^4}$	$x^2y^2x^2y$	
$(xy)^3y$	x^4y^3	$xy^2 \cdot xy^2 \cdot x$	$(xy)^3x$	y^4y^2	$2 \cdot 2 \cdot 2 \cdot 2$	
$\frac{x^2y^3}{xy}$	$x^4y^3 \cdot y^3$	$3 \cdot 2^2 \cdot 3$	$9 \cdot 4$	xyy	$(y^2)^3$	
the square of the square of x	$\frac{x^4y^4}{x^3y^2}$	$x(x^2)(y^4)$	$(x^2)^2$			
$2 \cdot 2 \cdot 2$	xxx	36	$2^2 \cdot 2^2$	$\frac{8^2}{2^3}$	$\frac{x^6y^6}{x^4y^4}$	$x(xy^2)^2$
8	16	$\frac{12^2}{2^2}$	$y(xy)$	$2(2 \cdot 3^2)$	$xx \cdot yy$	$(xy^2)x$

D. Simplify. [Look for short cuts.]

Sample 1. $(4a^6)6a^4)$

Solution. $$\begin{aligned}(4a^6)(6a^4) &= (4 \cdot 6)(a^6 \cdot a^4)\\ &= 24[(aaaaaa)(aaaa)]\\ &= 24(aaaaaaaaaa)\\ &= 24a^{10}.\end{aligned}$$

Sample 2. $\dfrac{2x^{12}y^3}{5x^{15}y}$

Solution. $$\begin{aligned}\frac{2x^{12}y^3}{5x^{15}y} &= \frac{2(xxxxxxxxxxxx)(yyy)}{5(xxxxxxxxxxxxxxx)y}\\ &= \frac{2(yy)(xxxxxxxxxxxxy)}{5(xxx)(xxxxxxxxxxxxy)}\\ &= \frac{2(yy)}{5(xxx)}\\ &= \frac{2y^2}{5y^3}, \quad [x \neq 0,\ y \neq 0].\end{aligned}$$

1. $a^5 \cdot a^{10}$

2. $3^4 \cdot 3^5$

3. $2^3 \cdot 2^2 \cdot 2 \cdot 2^5$

4. $x^2 \cdot x^3 \cdot x \cdot x^4$

5. $(2n^2)(3n^3)$

6. $(-7r^3)(2r^4)$

7. $(3a^2b)(-3ab^2)$

8. $(5cd^2)(-4c^2d^3)$

9. $(.4nr^2)(.5rs)$

10. $(.3c^2d)(.6d^2e)$

11. $\dfrac{x^5}{x^2}$

12. $\dfrac{4^4}{4^2}$

13. $\dfrac{-7^5}{7^2}$

14. $\dfrac{r^8}{-r^3}$

15. $\dfrac{3x^2y}{6xy^3}$

16. $\dfrac{6hm^2}{2hm}$

17. $\dfrac{21a^2b^3c}{-3ab^4c^2}$

18. $\dfrac{-32cd^2c^4}{4c^2d^2e}$

19. $\dfrac{-7x^2y^5z}{-7x^2y^5z}$

20. $\dfrac{-12a^3b^3c}{24a^3bc}$

21. $(c^6)(c^3) + c^2$

22. $(x^4)(x^5) - x^8$

(continued on next page)

23. $(-2x^2)(-4x^3y)$ 24. $(-3y^3)(-5yz)$ 25. $(a^2b)(2ab^2)$

26. $(cd^2)(3c^2d)$ 27. $\frac{14^6}{14^2}$ 28. $\frac{32^5}{32^3}$

29. $\frac{-15n^3s^2t}{3nst^4}$ 30. $\frac{20h^3j^4k^5}{-4h^4j^2k^4}$

31. $\frac{5x^2y^3z^2}{4ab^3c^2} \times \frac{24a^2bc^5}{25xyz^5}$ 32. $\frac{-2mp^2q}{3a^2b^3c^5} \times \frac{-9a^3b^2c^5}{4m^2pq^3}$

33. $\frac{3(a+b)^5(x+y)^7}{7(r+s)\ (u+v)^5} \times \frac{28(r+s)^2(u+v)^7}{9(x+y)^5(a+b)^4}$

<u>Sample 3.</u> $(x^2)^3$

<u>Solution.</u> $(x^2)^3 = x^2 \cdot x^2 \cdot x^2$

$= (xx)(xx)(xx)$

$= x^6$.

<u>Sample 4.</u> $(2x)^5$

<u>Solution.</u> $(2x)^5 = (2x)(2x)(2x)(2x)(2x)$

$= (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2)(xxxxx)$

$= 2^5x^5$ [or: $32x^5$].

<u>Sample 5.</u> $\left(-\frac{x}{2}\right)^3(2x)^4$

<u>Solution.</u> $\left(-\frac{x}{2}\right)^3(2x)^4 = \left[\left(-\frac{1}{2}\right)^3 x^3\right][2^4x^4]$

$= -\frac{x^3}{2^3} \cdot 2^4x^4$

$= -2x^7$.

34. $(c^4)^5$ 35. $(x^4)^5$ 36. $(-2x^2)^3$

37. $(-3y^3)^4$ 38. $(-2x^3)^2$ 39. $(-3y^4)^3$

40. $(9x^3y^4)^2$

41. $(7a^4b^2)^3$

42. $(a^2b)(2ab^2)^2$

43. $(cd^2)(3c^2d)^2$

44. $\left(-\frac{1}{2}a^5\right)^3$

45. $\left(-\frac{1}{3}b^4\right)^2$

46. $\left(\frac{y}{3}\right)^3 y^2$

47. $\left(-\frac{a}{4}\right)^2\left(-\frac{a}{2}\right)^3$

48. $\left(\frac{1}{5}x\right)^2(-2x^3)$

49. $\left(-\frac{1}{4}b\right)^3(-4b)^3$

50. $(3x^2y^3z^4)^3$

51. $(2a^4b^2c^2)^3$

52. $(-2r^4s^5t^6)^5$

53. $(-3r^5s^6t^7)^4$

54. $(x^2)^3(x^3)^2$

55. $(a^4)^5(a^5)^4$

56. $(2n^2s^3)^2(3n^3s^2)^2$

57. $(-3c^3d^2)^2(2c^2d^3)^2$

[More exercises are in Parts K and M, Supplementary Exercises.]

E. Each exercise contains two pronumeral expressions. For each pair of expressions, tell whether the expressions are equivalent or nonequivalent. If they are nonequivalent, give a counter-example.

1. $(x^2)^3$, x^6

2. $2a^2$, $(2a)^2$

3. $(xy)^2$, xy^2

4. $(x^4)^5$, $(x^5)^4$

5. $(x + 2)^2$, $x^2 + 4$

6. $(a^3b^5)^2$, a^5b^7

7. $(2z)^2$, $4z^2$

8. $y^3 \cdot y^2$, y^5

9. $\frac{a^2x^4}{(ax)^3}$, $\frac{x}{a}$, $[ax \neq 0]$

10. $\frac{a^2 + b^2}{ab}$, $a + b$, $[ab \neq 0]$

(continued on next page)

11. $\frac{a^2 + b^2}{a + b}$, $a + b$, $[a \neq -b]$

12. $\frac{a^2 - b^2}{a + b}$, $a - b$, $[a \neq -b]$

13. $\sqrt{x^4}$, x^2

14. $\sqrt{y^6}$, y^3

15. $\sqrt{(-3ab^2)(-12ab^2)}$, $6ab$

16. $x^9 \div x^{13}$, -4, $[x \neq 0]$

[More exercises are in Part L, Supplementary Exercises.]

F. The number 8 is called the third power of 2 because $8 = 2^3$. Complete each of the following sentences.

1. 16 is the ______ power of 2 because $16 = 2^{—}$.
2. ______ is the third power of 7 because ______ $= 7^3$.
3. 729 is the sixth power of ______ because $729 =$ ______^{6}.
4. The tenth power of 2 is ______ because ______$^{—}$ = ______.

G. A number which is either a prime number or a power of a prime number is called a prime power. Here is a list of some prime powers:

$$3^4, \quad 49, \quad 29^6, \quad 125, \quad 1024, \quad 4^3, \quad 53^2, \quad 11^{273}, \quad 41.$$

You have seen that you can name a positive integer by its prime factorization. You can also name a positive integer by its prime power factorization. For example:

$$72 = 2^3 \cdot 3^2, \qquad 98 = 2 \cdot 7^2, \qquad 624 = 2^4 \cdot 3 \cdot 13$$

$$900 = 2^2 \cdot 3^2 \cdot 5^2, \qquad 847 = 7 \cdot 11^2, \qquad 108 = 2^2 \cdot 3^3.$$

1. Give the prime power factorization of each number.

   (a) 36 (b) 135 (c) 78

   (d) 216 (e) 981 (f) 6240

2. For each number listed in Exercise 1, list all of its factors, other than 1, by writing the prime power factorization for each factor. [Example: The factors, other than 1, of $2^2 \cdot 3^3$ are 3, 3^2, 3^3, 2, $2 \cdot 3$, $2 \cdot 3^2$, $2 \cdot 3^3$, 2^2, $2^2 \cdot 3$, $2^2 \cdot 3^2$, and $2^2 \cdot 3^3$.]

SCIENTIFIC NOTATION

Scientists and engineers often deal with very large numbers. For example, the distance between the Earth and the Sun is about 93,000,000 miles, the speed of light is about 671,000,000 miles per hour, and the mass of the Earth is about 6,595,000,000,000,000,000,000 tons. The three numerals which you have just seen are awkward to write and difficult to read. For these reasons, scientists use a more compact notation. You will learn about this scientific notation in the exercises which follow.

EXERCISES

A. Write an equivalent numeral without using exponents, parentheses, or multiplication signs.

1. 2×10^3 2. 0.00263×10^3 3. 6.21×10^6
4. 53×10^5 5. 1.14×10^4 6. 4.3×10^{21}
7. 6.038×10^5 8. 1.3189×10^3 9. 2.01×10^{30}

* * *

A number is named in scientific notation by expressing it as:

(a number between 1 and 10) × (a power of 10).

For example, the distance in miles between the Earth and the Sun is given in scientific notation by:

$$9.3 \times 10^7.$$

Most of the numbers listed in Part A are listed in scientific notation. Which of them are not?

* * *

B. Express in scientific notation.

Sample 1. 276.9

Solution. $276.9 = 2.769 \times 10^2$

1. the speed of light in miles per hour

(continued on next page)

2. the mass, in tons, of the Earth

3. 728

4. 1657

5. 4, 378, 000

6. 2, 736, 000, 000

7. 5764.6

8. 375.4

Sample 2. 67.5×10^3

Solution. $67.5 \times 10^3 = (6.75 \times 10) \times 10^3$

$= 6.75 \times 10^4$.

9. 423.4×10^4

10. 2376×10^2

11. 78.32×10^2

12. $(58 \times 10^2) \times (91 \times 10^3)$

Sample 3. 0.632×10^4

Solution. $0.632 \times 10^4 = \frac{6.32}{10} \times 10^4$

$= 6.32 \times \frac{10^4}{10}$

$= 6.32 \times 10^3$.

13. 0.751×10^4

14. 0.546×10^5

15. 0.085×10^7

16. 0.00742×10^6

17. 0.546×10^3

18. 0.0546×10^3

19. 0.00546×10^3

20. 0.000546×10^3

* * *

When you worked the last two or three exercises in Part B, you may have found it difficult to express each number as the product of

a number between 1 and 10 and a power of 10. Let us reconsider Exercises 17-20 of Part B.

$$0.546 \times 10^3 = \frac{5.46}{10} \times 10^3 = 5.46 \times \frac{10^3}{10} = 5.46 \times 10^2$$

$$0.0546 \times 10^3 = \frac{5.46}{10^2} \times 10^3 = 5.46 \times \frac{10^3}{10^2} = 5.46 \times 10$$

$$0.00546 \times 10^3 = \frac{5.46}{10^3} \times 10^3 = 5.46 \times \frac{10^3}{10^3} = 5.46 \times 1$$

$$0.000546 \times 10^3 = \frac{5.46}{10^4} \times 10^3 = 5.46 \times \frac{10^3}{10^4} = 5.46 \times \frac{1}{10}$$

These results suggest that we write '10^1' for '10', '10^0' for '1', and '10^{-1}' for '$\frac{1}{10}$'. In general,

for each real number a, $a^1 = a$, and $a^0 = 1$, and

for each real number $a \neq 0$, for each positive integer n,

$$a^{-n} = \frac{1}{a^n}.$$

In particular,

$$0.0546 \times 10^3 = 5.46 \times 10^1,$$

$$0.00546 \times 10^3 = 5.46 \times 10^0, \text{ and}$$

$$0.000546 \times 10^3 = 5.46 \times 10^{-1}.$$

So, among the powers of 10 we include 10 [which is 10^1], 1[which is 10^0], $\frac{1}{10}$[which is 10^{-1}], $\frac{1}{100}$[which is 10^{-2}], etc. [Read the new numerals as 'the first power of 10', 'the zeroth power of 10', 'the negative first power of 10', etc.]

* * *

C. For each number listed below, write its name in scientific notation.

1. 27.3
2. 2.73
3. 0.273
4. 0.0273
5. 0.00273
6. 0.000273
7. 0.754
8. 0.0000000162

D. Simplify.

Sample 1. $10^3 \times 10^7 \times 10^{-6}$

Solution.
$$\begin{aligned} & 10^3 \times 10^7 \times 10^{-6} \\ &= 10^{10} \times 10^{-6} \\ &= 10^{10} \times \frac{1}{10^6} \\ &= \frac{10^{10}}{10^6} \\ &= 10^4. \end{aligned}$$

Sample 2. $\dfrac{10^5 \times 10^{-2}}{10^{-8}}$

Solution.
$$\begin{aligned} & \frac{10^5 \times 10^{-2}}{10^{-8}} \\ &= 10^5 \times \frac{1}{10^2} \times \frac{1}{\frac{1}{10^8}} \\ &= 10^5 \times \frac{1}{10^2} \times 10^8 \\ &= \frac{10^5 \times 10^8}{10^2} \\ &= 10^{11}. \end{aligned}$$

1. $10^8 \times 10^3 \times 10^{-4}$

2. $10^{-3} \times 10^{-1} \times 10^5 \times 10^{-6}$

3. $10^0 \times 10^{-2} \times 10^3$

4. $10^7 \times 10^{-2} \times 10^{-3}$

5. $10^6 \div 10^{-2}$

6. $10^{-7} \div 10^3$

7. $\dfrac{10^4 \times 10^{-3}}{10^2}$

8. $\dfrac{10^5 \times 10^{-9}}{10^3}$

9. $\dfrac{10^{-7} \times 10^{-3}}{10^{-7} \times 10^6}$

10. $\dfrac{10^4 \times 10 \times 10^{-3}}{10^{-5} \times 10^2 \times 10^{-1}}$

E. Simplify, and use scientific notation for the results.

Sample 1. $9800000000 \times 0.00025$

Solution. $9800000000 \times 0.00025$

$$(9.8 \times 10^9) \times (2.5 \times 10^{-4})$$

$$= (9.8 \times 2.5) \times (10^9 \times 10^{-4})$$

$$= 24.5 \times 10^5$$

$$= 2.45 \times 10^6.$$

Sample 2. $\dfrac{(65 \times 10^5) \times (9 \times 10^{-2})}{(15 \times 10^4) \times (26 \times 10^{-4})}$

Solution. $\dfrac{(65 \times 10^5) \times (9 \times 10^{-2})}{(15 \times 10^4) \times (26 \times 10^{-4})}$

$$\frac{(65 \times 9) \times (10^5 \times 10^{-2})}{(15 \times 26) \times (10^4 \times 10^{-4})}$$

$$= \frac{\overset{\overset{1}{\cancel{13}}}{\cancel{65}} \times \overset{3}{\cancel{9}}}{\underset{\underset{1}{\cancel{3}}}{\cancel{15}} \times \underset{2}{\cancel{26}}} \times \frac{10^3}{10^0}$$

$$= 1.5 \times 10^3.$$

1. $7200 \times 1500 \times 0.002$

2. $(68 \times 10^4) \times 0.0003$

3. $(4 \times 10^5) \times (7 \times 10^{-3})$

4. $(6 \times 10^{-7}) \times (8 \times 10^{-5})$

5. $\dfrac{(9 \times 10^{-2}) \times (24 \times 10^3)}{(8 \times 10^{-5}) \times (27 \times 10^4)}$

6. $\dfrac{(81 \times 10^{-5}) \times (64 \times 10^{-2})}{(54 \times 10^{-3}) \times (30 \times 10^5)}$

7. $(7 \times 10^{-3})^2$

8. $(9 \times 10^4)^3$

9. $(0.0008)^2$

F. Solve these problems.

1. Is $10^{15} + 10^{2}$ closer to 10^{15} or closer to 10^{17} ?

2. Is 0.000578 closer to 10^{-4} than to 10^{-3}? Than to 10^{-5}?

3. A movie is shown at the rate of 24 frames per second. The number of individual frames needed for a film which lasts 90 minutes is closest to which of these numbers?

   (a) 10^{3} (b) 10^{4} (c) 10^{5} (d) 10^{6}

4. A watch ticks 5 times each second. The number of times it ticks in one year is closest to which of these numbers?

   (a) 10^{6} (b) 10^{7} (c) 10^{8} (d) 10^{9}

5. The human heart beats day and night 60 times per minute, on the average. The number of times it beats during a 70-year lifespan is closest to which of these numbers?

   (a) 10^{8} (b) 10^{10} (c) 10^{12} (d) 10^{14}

6. Light travels 3×10^{8} meters each second. How many centimeters does it travel in 1 hour? [1 centimeter = 10^{-2} meters.]

7. A .30-caliber bullet takes about 10^{-1} seconds to travel 100 meters, and a fly can beat its wings once in about 10^{-3} seconds. How many times can a fly beat its wings during the time it takes a .30-caliber bullet to travel 100 meters?

8. The length of a diameter of a red blood corpuscle is about 10^{-5} meters and the average distance between the Earth and the Moon is about 10^{8} meters. About how many red blood corpuscles would it take to make a chain of them which stretched from the Earth to the Moon?

EXPLORATION EXERCISES

A. In each exercise you are given two or more numbers. For each number in the exercise, find the set of its factors. Then, find the intersection of these sets.

Sample. 36, 45, 90

Solution. $36 = 3^2 \cdot 2^2$, $45 = 3^2 \cdot 5$, $90 = 3^2 \cdot 2 \cdot 5$.

The factor-set of 36 is $\{1, 3, 9, 2, 6, 18, 4, 12, 36\}$, the factor-set of 45 is $\{1, 5, 3, 15, 9, 45\}$, and the factor-set of 90 is $\{1, 5, 3, 15, 9, 45, 2, 10, 6, 30, 18, 90\}$. The intersection of these factor-sets is $\{1, 3, 9\}$.

1. 12, 18 2. 50, 75, 90 3. 8, 16, 40 4. 3, 5, 7

B. For each exercise, find the intersection of the factor-sets of the numbers listed. [That is, find the set of common factors of the numbers listed in the exercise.]

1. 18, 30
2. 12, 30, 42
3. 16, 36, 72
4. 9, 90, 900
5. 20, 30, 40
6. 48, 14, 28
7. 64, 108, 300
8. 12, 33, 66, 132

* * *

7 is a factor of 21 and 21 is a multiple of 7. 36 is a multiple of 4 and a multiple of 3. A first number is a multiple of a second number if and only if the second number is a factor of the first. So, the notion of multiple, like the notion of factor, depends upon the set of numbers under discussion. For example, 7 is a multiple of 3 with respect to the set of rationals but not with respect to the set of positive integers.

Some multiples of 5 with respect to the set of positive integers are

5, 10, 15, 20, 25, and 30.

What are four more such multiples of 5? Three multiples of 7 with respect to the set of integers are

$^-14$, 0, and 700.

Of course, each real number is a multiple of each nonzero real number with respect to the set of real numbers. How many multiples of 0 are there with respect to the set of real numbers? How many multiples of 7 are there with respect to the set of reals? How many multiples of 2 are there with respect to the set of positive integers?

As we agreed earlier for the word 'factor', unless otherwise specified, when we speak of multiples of numbers, we shall mean multiples with respect to the set of positive integers.

* * *

C. 1. What are the multiples of 3 which are less than 20?

2. What are the multiples of 7 between 50 and 100?

3. How many multiples of 4 are there between 201 and 299?

4. How many multiples of 13 are there between 7241 and 8573?

5. What numbers less than or equal to 15 are common multiples of 2 and 3, that is, numbers which are multiples of both 2 and 3?

6. What numbers less than or equal to 4×25 are common multiples of 4 and 25?

7. What numbers less than or equal to 10×21 are common multiples of 10 and 21?

8. What numbers less than or equal to 6×15 are common multiples of 6 and 15?

9. Each of the following exercises lists a pair of numbers. Draw a loop around [or copy the letter of] each exercise for which the numbers have a common multiple which is smaller than their product.

(a) 3, 5 (b) 8, 12 (c) 4, 8 (d) 10, 12

(e) 10, 24 (f) 6, 9 (g) 9, 20 (h) 20, 30

10. For the pairs given in Exercise 9, tell which have a common <u>factor</u> greater than 1.

11. Do the numbers 856254 and 739876 have a common multiple less than their product?

12. Draw a loop around [or copy the letter of] each exercise whose numbers have a common multiple smaller than their product.

(a) $3 \cdot 5,\ 3 \cdot 7$ (b) $2 \cdot 5,\ 3 \cdot 7$

(c) $2^3 \cdot 5,\ 2^2 \cdot 3$ (d) $3^2 \cdot 5,\ 3^2 \cdot 5 \cdot 7$

(e) $2^3 \cdot 11,\ 3^2 \cdot 13$ (f) $2^2 \cdot 3 \cdot 5^2,\ 3 \cdot 7 \cdot 13^2$

4.06 <u>Factoring</u>. --Consider the factor-sets of 18 and 30, that is, the sets

$$\{1, 3, 9, 2, 6, 18\} \quad \text{and} \quad \{1, 5, 3, 15, 2, 10, 6, 30\}.$$

Do you see that the common factors of 18 and 30 are 1, 3, 2, and 6? Notice that one of these common factors is a multiple of each of them [Which one?]. Use the results of Parts A and B of the Exploration Exercises to check the generalization that among the common factors of two or more positive integers there is one which is a multiple of each common factor. This common factor is called <u>the highest common factor</u> [HCF, for short] <u>of the given positive integers</u>. [Sometimes the highest common factor is called <u>the greatest common divisor</u> [GCD, for short] because it is the largest positive integer by which each of the integers can be divided "exactly".]

EXERCISES

<u>A</u>. For each exercise, find the HCF of the numbers listed.

1. 20, 30
2. 25, 55
3. 7, 11
4. 100, 120
5. 36, 54, 99
6. 42, 70, 140

(continued on next page)

<u>Sample</u>. 900, 6240

<u>Solution</u>. $900 = 2^2 \cdot 3^2 \cdot 5^2$,

$6240 = 2^5 \cdot 3 \cdot 5 \cdot 13$

The HCF of 900 and 6240 is $2^2 \cdot 3 \cdot 5$.

7. 36, 135 8. 108, 144 9. 900, 1080

10. 36, 42, 105 11. 84, 45, 105 12. 35, 36

<u>B</u>. 1. Use the fact that the HCF of 4147 and 10672 is 29 to reduce to lowest terms the fraction: 4147/10672.

2. Use the fact that the GCD of 630 and 924 is 42 to reduce to lowest terms the fraction: 630/924.

3. Use the fact that the HCF of 2184 and 203375 is 1 to reduce to lowest terms the fraction: 2184/203375.

4. Is the HCF of 120 and 180 the same as the HCF of 120 and 120 + 180? 180 and 120 + 180? 120, 180, and 120 + 180?

☆5. (a) The last result of Exercise 4 suggests the theorem:

$\forall_a \forall_b$ the HCF of a and b = the HCF of a, b, and a + b.

Prove this theorem.

(b) Prove:

$\forall_a \forall_b$ the HCF of a and b = the HCF of a, b, and 85a + 370b.

(c) Generalize the results of (a) and (b).

* * *

You have seen that two or more positive integers have a <u>common factor</u> which is a <u>multiple</u> of each of their common factors.

It is also true that two or more positive integers have a <u>common multiple</u> which is a <u>factor</u> of each of their common multiples.

For example, some of the common multiples of 6 and 8 are 24, 48, 72, 96, and 120. One of these common multiples--24--is a factor of each of them. And, this number is called <u>the lowest common multiple of 6 and 8</u>. [Sometimes it is called <u>the least common multiple</u> because it is the smallest positive integer which is a multiple of both 6 and 8.]

* * *

C. For each exercise, find the LCM of the numbers listed.

Sample. 45, 50, 54

Solution. $45 = 3^2 \cdot 5$

$50 = 2 \cdot 5^2$

$54 = 2 \cdot 3^3$

The LCM of 45, 50, and 54 is $2 \cdot 3^3 \cdot 5^2$.

1. 12, 18
2. 4, 6, 15
3. 2, 3, 5
4. 2, 3, 6
5. 10, 14, 35
6. 18, 45, 100
7. 26, 65
8. 6, 8, 20
9. 4, 16, 128
10. $3, 3^2, 3^5$
11. $6^2, 12^3, 1^2$
12. $1^2, 2^2, 3^2, 4^2$

D. 1. Complete this table.

	Product	HCF	LCM	HCF × LCM
(a) 15, 20				
(b) 30, 50				
(c) 12, 18				
(d) 15, 28				
(e) 10, 14				
(f) $2^4 \cdot 3^2$, $2^2 \cdot 3^3$				
(g) 3, 12, 21				

2. State a generalization suggested by the results of Exercise 1.

☆3. Find three numbers whose product is the product of their HCF by their LCM.

FACTORING PRONUMERAL EXPRESSIONS

You have seen that in speaking of factors of a number it is necessary to specify the set of numbers to which the factors are to belong. For example, the number 34 has 1, 2, 17, and 34 as factors with respect to the positive integers. But, with respect to the set of all integers,

it has additional factors--$^-1$, $^-2$, $^-17$, and $^-34$. Also, each nonzero rational number is a factor of 34 with respect to the set of rationals.

Suppose you are asked to factor an expression such as '34'. This means that you are to find an expression "in the form of a product" which is equivalent to '34'. Clearly, there are many such expressions, and each is called a factorization of '34'. For example, one factorization of '34' is its prime factorization, '$2 \cdot 17$'. If we decide that the factors occurring in the factorization are to be names for positive integers then '$2 \cdot 17$' is a "complete" factorization of '34'. [You can't factor any more except by introducing '1's.]

If we decide that the factors occurring in the factorization are to be names for rational numbers then '$\frac{3}{5} \cdot \frac{85}{8} \cdot \frac{48}{9}$' is a factorization of '34'. It is clear that, under these conditions, there is no such thing as a complete factorization of '34'. [There are no numbers which are "prime" with respect to the set of rationals.] When we allow numerals for irrational numbers to occur as factors, we get other factorizations of '34', such as '$\sqrt{2} \cdot \sqrt{34} \cdot \sqrt{17}$'.

If you are asked to factor an expression which is a name for a positive integer, you should assume that what is wanted is the prime factorization [or the prime power factorization] of the positive integer.

Similarly, in order to speak precisely about factors and factorizations of pronumeral expressions, we would have to specify the set of expressions which were to be eligible as factors. [And, it turns out, we would also have to specify the domains of the pronumerals.] Let's take an example. Suppose we decide to use as factors expressions whose only numerals are those for integers. Under this requirement, it is not possible to factor '$2x + 1$' [if we want the domain of 'x' to contain all real numbers]. But, if we allow numerals for rational numbers to occur in factors, one factorization of '$2x + 1$' is '$2(x + \frac{1}{2})$', and another is '$\frac{3}{5}\left(\frac{10}{3}x + \frac{5}{3}\right)$'. And, if we allow restrictions to be put on the domain of 'x', there are additional factorizations; for example, '$x(2 + \frac{1}{x})$'.

As a second example, consider factorizations of '$x^4 - 9$'. This expression is equivalent to '$(x^2 - 3)(x^2 + 3)$'. If we disregard expressions which contain numerals for nonintegral numbers, and require

that the domain of 'x' be the set of real numbers, then this is a "complete" factorization. But, if we allow numerals for arbitrary real numbers to occur in the factors, we can "factor further" to obtain:

$$(x - \sqrt{3})(x + \sqrt{3})(x^2 + 3),$$

or even such a silly thing as:

$$\frac{6}{7}\left(\frac{x}{3} - \frac{\sqrt{3}}{3}\right)\left(\frac{x}{2} + \frac{\sqrt{3}}{2}\right)(7x^2 + 21).$$

And, if we are willing to restrict the values of 'x' to the set of non-negative real numbers, we can factor '$x^4 - 9$' and obtain:

$$(\sqrt{x} - \sqrt{\sqrt{3}})(\sqrt{x} + \sqrt{\sqrt{3}})(x + \sqrt{3})(x^2 + 3).$$

It is not easy [nor would it be profitable here] to describe the possible sets of pronumeral expressions with respect to which one may factor a given expression. We shall content ourselves with two general remarks which you can use as guides in deciding when an expression has been "completely" factored.

First, the factors of an expression should be simpler than the expression itself [or, at any rate, no more complicated than the given expression]. Here, as usual, the meaning of 'simpler than' is vague, and varies from one context to another. But, just as you learned to speak more or less grammatically long before you were aware of grammar, or had any inkling as to its rules, so you must now develop a feeling for the meaning of 'simpler than' in the absence of any rules. One of the reasons for working numerous factoring, expanding, and simplifying exercises is to develop such a feeling. The best we can do is to give you a few examples. Suppose, for instance, that you are merely asked to factor '$4x^2 - 1$'. There are two more or less reasonable choices for an answer, '$(2x - 1)(2x + 1)$' and '$4(x - \frac{1}{2})(x + \frac{1}{2})$'. Now, in the absence of any further instructions, one would probably choose the first of these, perhaps on the ground that '$4x^2 - 1$' contains no fractions, while '$4(x - \frac{1}{2})(x + \frac{1}{2})$' does. On the other hand, if one were factoring '$4x^2 - 1$' for the purpose of reducing the fraction '$\dfrac{x - \frac{1}{2}}{4x^2 - 1}$', it would be more appropriate to use the factorization '$4(x - \frac{1}{2})(x + \frac{1}{2})$' or the factorization '$2(x - \frac{1}{2})(2x + 1)$'.

Again, if you were asked to factor '$\frac{a^2}{4} - \frac{b^2}{9}$', it would be hard to decide between '$\left(\frac{a}{2} - \frac{b}{3}\right)\left(\frac{a}{2} + \frac{b}{3}\right)$' or '$\frac{1}{36}(3a - 2b)(3a + 2b)$'. But, if you had an ulterior motive for factoring, then one of them [and it could be either] might be much simpler to use than the other.

The second remark has to do with the fact that, in factoring pronumeral expressions, it is generally not important to factor numerals completely. Thus, if asked to factor '$36x^2 + 72$', it is enough to write '$36(x^2 + 2)$'. There is no point, in this context, in writing '$2^2 \cdot 3^2(x^2 + 2)$'.

EXERCISES

A. Factor.

Sample 1. $12xy + 15xz$

Solution. $3x(4y + 5z)$

Sample 2. $18x^2y - 24x^3y^2$

Solution. $6x^2y(3 - 4xy)$

1.	$4y + 8x$	2.	$6p - 3q$	3.	$yc - yd$
4.	$12R - 12r$	5.	$7y - 21$	6.	$8x - 8y$
7.	$7x + 7$	8.	$4 - 24c$	9.	$x^2 - 3x$
10.	$4x^2 + 5x$	11.	$9y^2 - 18y$	12.	$38x + x^2$
13.	$cy - 5cb$	14.	$10y + 15y^3$	15.	$2y - 4y^3$
16.	$25x^2y - 35y^2x$	17.	$12z^2 + 12$	18.	$rt - 5rs$
19.	$\pi r^2 + 2\pi rh$	20.	$yx^2 + 9y$	21.	$7m - 21m^3$
22.	$11y^4 + 11y^2$	23.	$3p^2 - 39p^3$	24.	$bx + bx^3$
25.	$2ab^2 + 4a^2b$	26.	$10yx - 30x^2y^2$	27.	$42m^3n^2 - 14m^2n$
28.	$15r^2s^3 - 24s^4r$	29.	$72m^2n^2 - 54m^2n$	30.	$200tv^2 - 160t^2v$

[More exercises are in Part N, Supplementary Exercises.]

Sample 3. $x^2 - y^2$

Solution. $(x - y)(x + y)$

Sample 4. $16a^4 - b^4$

Solution. $(4a^2)^2 - (b^2)^2$

$= (4a^2 - b^2)(4a^2 + b^2)$

$= (2a - b)(2a + b)(4a^2 + b^2).$

31. $t^2 - 4$	32. $m^2 - 36$	33. $100 - k^2$
34. $16 - s^4$	35. $s^4 - 49$	36. $c^2 - 0.16$
37. $4m^2 - 121$	38. $p^4 - 81$	39. $m^2 - (1/64)$
40. $t^2 - (1/100)$	41. $49m^2 - (1/16)$	42. $(9/25) - 81d^2$
43. $x^4 - y^2$	44. $4m^4 - 9n^2$	45. $36p^4 - (1/49)$

[More exercises are in Part N, Supplementary Exercises.]

Sample 5. $a^2 + 8a + 12$

Solution. $(a + 6)(a + 2)$

Sample 6. $c^4 + c^3 - 6c^2$

Solution. $c^2(c^2 + c - 6)$

$= c^2(c + 3)(c - 2).$

46. $y^2 + 9y + 8$	47. $m^2 - 6m + 8$
48. $3a^2 - 15a + 18$	49. $7x^2 - 21x - 70$
50. $72 - 32x - 2x^2$	51. $42m + m^2 - m^3$
52. $6x^3 - 32x^2 + 10x$	53. $70y^4 - 130y^3 - 20y^2$
54. $5a^2 + 15a + 10$	55. $3c^3 + 15c + 18c^2$
56. $80x + 10x^2 + 70$	57. $y^6 + 10y^5 + 9y^4$
58. $m^2 + 7mt + 10m$	59. $5x^2 - 5$

(continued on next page)

60. $8x^2 + x^4 + 6x^3$

61. $x^4 - 5x^2 + 4$

62. $y^4 - 8y^2 - 9$

63. $6x^3y^2 + 5x^2y^2 - 4xy^2$

64. $u^6 - 9y^6$

65. $(x + y)^2 - (x - y)^2$

66. $(a^2 + b^2)^2 - 4a^2b^2$

67. $4y^4 - 5y^2 - 9$

68. $a^6 - 2a^3b^3 + b^6$

69. $4c^4 - 7c^3 - 2c^2$

[More exercises are in Part N, Supplementary Exercises.]

B. Simplify.

Sample. $\sqrt{4x^6}$

Solution.
$$\begin{aligned}\sqrt{4x^6} &= \sqrt{2^2 \cdot (x^3)^2}\\ &= \sqrt{(2x^3)^2}\\ &= |2x^3|.\end{aligned}$$

1. $\sqrt{x^2y^6}$

2. $\sqrt{y^4 - 4y^2 + 4}$

3. $\sqrt{a^6 + 2a^4 + a^2}$

4. $\sqrt{9 - 6u^8 + u^{16}}$

5. $\sqrt{25x^{12}y^8}$

6. $\sqrt{(x^4 + y^4)^2 - (x^4 - y^4)^2}$

7. $\sqrt{(a + b)^2(x + 2y)^4}$

8. $\sqrt{(x^2 + y^2)^2(x^2 - y^2)^4}$

9. $\sqrt{\dfrac{x^6y^4}{z^2}}$

10. $\sqrt{\dfrac{a^4 - 2a^2 + 1}{a^4 + 2a^2 + 1}}$

* * *

Given two or more pronumeral expressions, a common factor which is a multiple of each common factor of the given expressions is called a highest common factor of the expressions. For example, both 'xy' and 'yx' are highest common factors of '2xy' and '3xy'.

* * *

C. Find an HCF of the given pronumeral expressions.

Sample 1. x^5y^4, x^3y^2, x^2y^6

Solution. An HCF of 'x^5y^4', 'x^3y^2', and 'x^2y^6' is 'x^2y^2'. [Another is 'y^2x^2'.]

<u>Sample 2</u>. $8a^3b^2c^4$, $12ab^3c^2$

<u>Solution</u>. An HCF is '$4ab^2c^2$'.

1. $10xy^2z$, $6y^3z$, $2yz^2$

2. $15r^3s^2$, $30rs^3$, 10^2s

3. $6a^2bc^3$, $5a^3b^2c$

4. x^5y^{12}, $3x^4y^9$, $9x^2y^{23}$

<u>Sample 3</u>. $6a + 18b$, $12a + 30b$

<u>Solution</u>. For each a and b,

$$6a + 18b = 6(a + 3b)$$

and $$12a + 30b = 6(2a + 5b).$$

So, an HCF is '6'.

<u>Sample 4</u>. $x^2 - y^2$, $x^2 + 4xy + 3y^2$

<u>Solution</u>. For each x and y,

$$x^2 - y^2 = (x - y)(x + y)$$

and $$x^2 + 4xy + 3y^2 = (x + 3y)(x + y).$$

So, an HCF is '$x + y$'.

5. $6x^2y + 15xy^2$, $21x^2y - 6xy^2$

6. $12m^3p^5 + 20m^2p^3r^2$, $8m^5p^4 + 24m^2p^5r$

7. $(x - 2y)^2(x + y)(x - y)^3$, $(x - 2y)(x + y)^2(x - y)$

8. $a^2 - 5a + 6$, $a^2 - 4a + 4$

9. $x^3 + 3x^2 + 2x$, $x^3 - x$

10. $12x^2 - 4x - 21$, $2x^2 + 7x - 15$

11. $x^3y - 5x^2y^2 - 14xy^3$, $x^2 - 3xy + 2y^2$

12. $n^2 + nr - 6r^2$, $n^2 - 7nr + 10r^2$

13. $5n^3 - 45n$, $10n^3 - 50n^2 + 60n$

14. $c^3d + 3c^2d^2 + 2cd^3$, $c^3d + c^2d^2 - 2cd^3$

15. $5s^2 - 125$, $5s^3 - 50s^2 + 125s$

16. $(a + b)(2a^2 - ab - b^2)$, $(2a^2 + 3ab + b^2)$

D. The idea of the HCF is useful in factoring pronumeral expressions.

Sample. Factor:

$$15x^2y^3z^2 + 20x^3y^2z^5 + 35x^2y^3z.$$

Solution. Since an HCF of the three addends is '$5x^2y^2z$', it follows that a factorization of the given expression is:

$$5x^2y^2z(3yz + 4xz^4 + 7y).$$

Factor these expressions using the idea of the HCF.

1. $7x^2y + 21xy^2$
2. $3a^2bc^4 - 5ab$
3. $6x^2y^2z^2 + 3xyz$
4. $9a^5b^3c^2 - 2ab^3c$
5. $13abc + 26a^2b^3c^4 - 65a^2b^2c$
6. $12x^2y^2 + 4y^2z^2 + 2z^2x^2$
7. $10(x + y)^2(x - y)^3 + 5(x + y)^3(x - y)^3$
8. $3(2a - b)(a + 3b) + 7(4a^2 - 4ab + b^2)(a^2 + 6ab + 9b^2)$

[More exercises are in Part O, Supplementary Exercises.]

* * *

Given two or more pronumeral expressions, a common multiple which is a factor of each common multiple of the given expressions is called a lowest common multiple of the expressions. For example, both '$6xy$' and '$6yx$' are lowest common multiples of '$3xy$' and '$2x$'.

* * *

E. Find an LCM of the given pronumeral expressions.

Sample 1. a, $2b$, $7c$

Solution. An LCM is '$14abc$'.

Sample 2. $5x^3y^5$, $20x^2y^8$

Solution. An LCM of '$5x^3y^5$' and '$20x^2y^8$' is '$20x^3y^8$'. [Another is '$20y^8x^3$'.]

1. x, $3y$, $7z$

2. $4x$, $5xy$, $5yz$

3. y, y^2, y^5

4. $2x$, $3x^2$, $4x^4$

5. $2a^2b$, $10b^2a$

6. 7, $14xy$, $21y$

7. $2a^3b^2c$, $4abc^2$, $6a^2b^3c^4$

8. x^2y, $4xyz$, $10y^2z^2$

9. $9u^3v^2$, $2uv^4$, $6u^2v$

10. $3a^2x^2$, $2abxy$, $4b^2y^2$

<u>Sample 3</u>. 7, $7x + 4$

<u>Solution</u>. An LCM of '7' and '$7x + 4$' is '$7(7x + 4)$'.
[Another is '$49x + 28$'.]

<u>Sample 4</u>. $x^2 - y^2$, $x^2 - 2xy + y^2$

<u>Solution</u>. For each x and y,

$$x^2 - y^2 = (x - y)(x + y)$$

and $$x^2 - 2xy + y^2 = (x - y)^2.$$

So, an LCM of '$x^2 - y^2$' and '$x^2 - 2xy + y^2$' is

'$(x - y)^2(x + y)$'.

11. 3, $a + b$

12. 4, $2(x - y)$

13. $6x - 6$, 3

14. $a + 5b - c$, 15

15. $a + b$, $a - b$

16. $3(x + y)$, $(x + y)^2$

17. $5(x - y)$, $10(x - y)$

18. $x - 2y$, $x - 2y$

19. $2x - y$, $x - 2y$

20. $a - b$, $a^2 - b^2$

21. $x + 1$, $2x - 1$

22. $a^4 - b^4$, $a^2 - b^2$, $a + b$

23. $x^2 - 2x - 3$, $x^2 - 1$, $x + 2$

24. $3s^2 - 7s - 10$, $30s^2 - 91s - 30$

25. $(x - y)(x + 2y)^2(x + y)$, $(2x + y)(x - y)^2$, $(x + y)^2$

[More exercises are in <u>Part P</u>, <u>Supplementary Exercises</u>.]

F. Simplify.

Sample 1. $\frac{5}{72} + \frac{17}{48}$

Solution. Since $72 = 3 \cdot 24$ and $48 = 2 \cdot 24$, the LCM of 72 and 48 is $6 \cdot 24$, or 144. $144 = 72 \cdot 2$, and $144 = 48 \cdot 3$.

$$\begin{aligned}\frac{5}{72} + \frac{17}{48} &= \frac{5}{72} \cdot \frac{2}{2} + \frac{17}{48} \cdot \frac{3}{3} \\ &= \frac{10}{144} + \frac{51}{144} \\ &= \frac{61}{144}.\end{aligned}$$

1. $\frac{3}{8} + \frac{7}{12}$
2. $\frac{7}{15} - \frac{3}{10}$
3. $\frac{5}{24} + \frac{7}{12} - \frac{17}{16}$
4. $\frac{7}{36} + \frac{5}{24}$
5. $\frac{5}{14} - \frac{23}{49}$
6. $\frac{19}{21} - \frac{6}{35} + \frac{4}{15}$

Sample 2. $\frac{3}{5x^3y^5} + \frac{2}{7x^2y^8}$

Solution. An LCM of '$5x^3y^5$' and '$7x^2y^8$' is '$35x^3y^8$'.

For each x, for each y,

$35x^3y^8 = 5x^3y^5 \cdot 7y^3$ and $35x^3y^8 = 7x^2y^8 \cdot 5x$.

$$\begin{aligned}\frac{3}{5x^3y^5} + \frac{2}{7x^2y^8} &= \frac{3}{5x^3y^5} \cdot \frac{7y^3}{7y^3} + \frac{2}{7x^2y^8} \cdot \frac{5x}{5x} \\ &= \frac{21y^3}{35x^3y^8} + \frac{10x}{35x^3y^8} \\ &= \frac{21y^3 + 10x}{35x^3y^8}.\end{aligned}$$

[Note: We assume in this sample and in the exercises which follow that values of the pronumerals which would convert a denominator into a name for 0 have been excluded from the domains of the pronumerals.]

7. $\frac{2x}{3y} + \frac{5}{2x^2y^2}$

8. $\frac{5}{x^2} - \frac{6}{x} + \frac{3}{x^3}$

9. $\frac{2x^2 + 4x - 3}{4x^2} - \frac{x + 2}{2x}$

10. $\frac{2x + y}{4xy^2} - \frac{x + 2y}{6x^2y}$

Sample 3. $\frac{x - y}{x^2 + 2xy + y^2} - \frac{x}{x^2 - y^2}$

Solution. $\frac{x - y}{x^2 + 2xy + y^2} - \frac{x}{x^2 - y^2}$

$$= \frac{x - y}{(x + y)^2} - \frac{x}{(x - y)(x + y)}$$

$$= \frac{x - y}{(x + y)^2} \cdot \frac{x - y}{x - y} - \frac{x}{(x - y)(x + y)} \cdot \frac{x + y}{x + y}$$

$$= \frac{(x - y)^2}{(x + y)^2(x - y)} - \frac{x(x + y)}{(x + y)^2(x - y)}$$

$$= \frac{(x^2 - 2xy + y^2) - (x^2 + xy)}{(x + y)^2(x - y)}$$

$$= \frac{-3xy + y^2}{(x + y)^2(x - y)} \quad \left[\text{or: } \frac{y(y - 3x)}{(x + y)^2(x - y)}\right].$$

11. $\frac{x}{4x + 12} - \frac{2x}{x + 3}$

12. $\frac{4y + 3}{y^2 - 6y + 5} + \frac{2}{y - 1}$

13. $\frac{x + 2}{x + 1} + \frac{x - 3}{x - 1} - \frac{2x}{x^2 - 1}$

14. $\frac{x + 1}{x^2 + x - 2} - \frac{x - 1}{x^2 + 3x + 2}$

15. $\frac{y}{x^2 + xy} - \frac{x}{xy + y^2}$

16. $\frac{u + 5}{9u^2 - 4} + \frac{2u - 1}{3u^2 + 4u - 4}$

17. $\frac{6a^3}{a^4 - 16} + \frac{2a}{a^2 + 5a + 6}$

18. $\frac{y}{3y - 9} + \frac{2y}{2y + 6} - \frac{3y^2}{y^2 - 9}$

(continued on next page)

<u>Sample 4</u>. $\frac{5}{x - y} - \frac{3y - 2x}{y - x}$

<u>Solution</u>. $\frac{5}{x - y} - \frac{3y - 2x}{y - x}$

$$\left.\begin{aligned} &= \frac{5}{x - y} - \frac{3y - 2x}{-(x - y)} \\ &= \frac{5}{x - y} - \frac{-(3y - 2x)}{x - y} \end{aligned}\right\} \underline{\text{[Explain]}}$$

$$= \frac{5 - -(3y - 2x)}{x - y}$$

$$= \frac{5 + 3y - 2x}{x - y}.$$

19. $\frac{7a}{a - b} - \frac{10 + b}{b - a}$

20. $\frac{2c - 3}{d - 2c} + \frac{-5 + c}{2c - d}$

21. $\frac{3s + 1}{s^2 - 9} - \frac{-2s - 3}{3 - s}$

22. $\frac{-3a + b}{3a - b - 2c} + \frac{c - 2b}{b + 2c - 3a}$

23. $\frac{2r - s}{r^2 - s^2} + \frac{4}{4s - 4r}$

24. $\frac{c - d}{d - c} - \frac{2a - 3b}{3b - 2a}$

25. $\frac{-5c + d}{2c - d} + \frac{2d - 3c}{d - 2c}$

26. $\frac{3n}{n + 2} - \frac{n}{10 + 5n}$

27. $\frac{2d - 5}{1 - 2d + d^2} + \frac{3}{d - 1}$

28. $\frac{n}{n^2 - nk} - \frac{k}{nk - k^2}$

29. $\frac{h + 1}{h + 2} - \frac{h + 2}{2 - h} - \frac{2h}{h^2 - 4}$

30. $\frac{-4t^2}{16 - t^2} + \frac{t}{4t - 16} - \frac{2t}{2t + 8}$

31. $-\frac{5n - 4t}{4t - 5n} + \frac{r - s}{s - r}$

32. $\frac{2r + 1}{4r^2 - 4r + 1} - \frac{1 - 2r}{1 - 4r^2}$

33. $\frac{3a + 2c}{bc^2 - ba^2} - \frac{5}{ba - bc}$

34. $\frac{147a}{a - b} + \frac{b + 146a}{b - a}$

☆35. $\frac{2 - 3n}{9 - 8n - n^2} - \frac{5n + 6}{n^2 + 29n - 30}$

[More exercises are in <u>Part Q</u>, <u>Supplementary Exercises</u>.]

MISCELLANEOUS EXERCISES

A. 1. (a) In Unit 3 you learned that the graph of '$x = 5$' [on a picture of the number line] consits of a single point. What is the graph of '$x = 5$' on a number plane picture?

(b) What is the graph of '$y > -1$' on a number line picture? What is the graph of '$y > 1$' on a picture of the (x, y)-plane?

2. Write two equations [in 'x' and 'y'] whose graphs on a number plane picture cross at the graph of $(-2, 5)$.

3. Write three equations [in 'x' and 'y'] whose graphs on a picture of the number plane have the graph of the origin as their common point.

4. Write two equations whose loci in (x, y) intersect in the empty set.

5. Write two equations whose loci in the number plane lattice intersect in the empty set, but whose loci in the number plane intersect in a set consisting of a single point.

6. Write two different equations which have the same loci in the number plane.

7. Use brace-notation [$\{(x, y): \ldots\}$] to name a set whose elements [members] are all the points of the number plane. Of the number plane lattice.

8. Write an equations whose solution set in (x, y) is the empty set.

9. In each of the following sentences, replace the 't' by a numeral such that the locus of the resulting sentence will include the ordered pair listed.

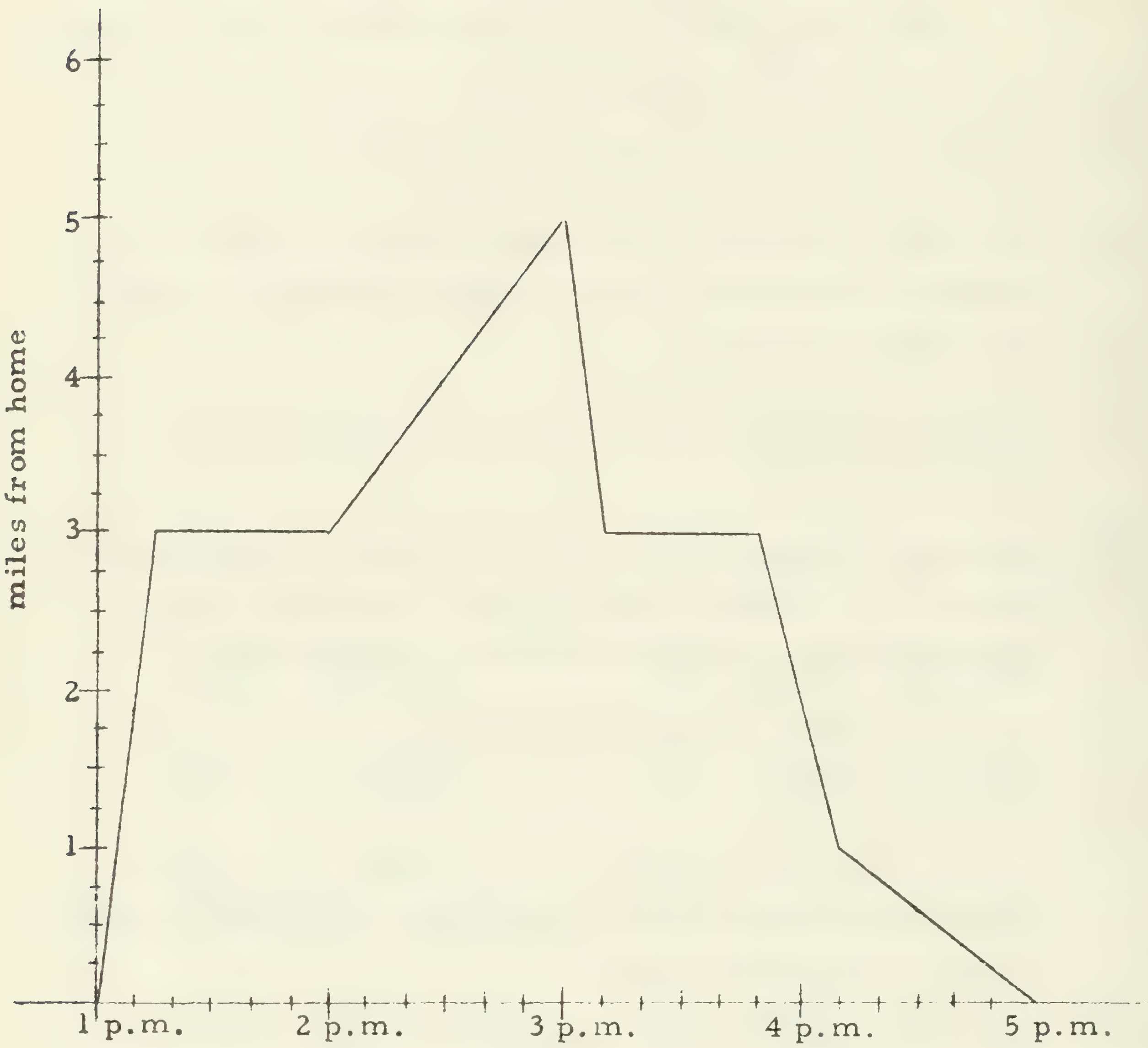
(a) $x + ty = 16$; $(6, 2)$ (b) $3y - 7x = t$; $(-3, -4)$

(c) $3tx - 5y = -18$; $(0, -7)$ (d) $2x^2 + 11ty = 13t$; $(3, -1)$

(e) $5tx - 6y + 7 = 0$; $(0, 0)$

(continued on next page)

10. Here is a graph which is an approximate record of a bicycle trip Ed took one day last summer. He started from home at 1:00 p.m. and kept track of his distance from home at various times. [For example, at 3:10 p.m. he was 3 miles from home.]



(a) How far from home was he at 2:30 p.m.?

(b) At what time was he 2 miles from home?

(c) By what time had he traveled a total of 8 miles?

(d) What was his average speed during the first 15 minutes?

(e) Did he stop during the trip?

(f) What was his average speed during the period from 3:00 p.m. to 3:10 p.m.?

(g) What was his average speed during the period from 3:00 p.m. to 3:50 p.m.?

(h) During which period was he traveling faster--from 3:50 p.m. to 4:10 p.m., or from 4:10 p.m. to 5:00 p.m.?

(i) How many miles did he travel altogether?

(j) What was his average speed for the entire trip?

(k) If, from 4:10 p.m. until arrival at home, Ed had traveled at the same average speed as that maintained from 3:50 p.m. to 4:10 p.m., at what time would he have arrived at home?

(l) Notice the "corners" at various points of the graph. If the chart were drawn accurately, is it likely that there would be corners like these? Explain.

11. Suppose you are playing a number plane game in which the "moving" rule is:

$$(x, y) \rightarrow (x, y + 3).$$

Then, if you start at a point in $\{(x, y): x = 7\}$, say, $(7, 2)$, the first move takes you to $(7, 5)$, the second move takes you to $(7, 8)$, and, in fact, no matter how many moves you make, the point you reach is in $\{(x, y): x = 7\}$.

In each of the following exercises, you are given a moving rule and several sets. Tell for which sets it is the case that, if you start from a point in the set, you can't get out of the set no matter how many moves you make.

(a) $(x, y) \rightarrow (x - 3, y)$

(1) $\{(x, y): x + 3 = y\}$ (2) $\{(x, y): y = 2\}$

(3) $\{(x, y): x + 2y = 10 + x\}$ (4) $\{(x, y): x = 3\}$

(b) $(x, y) \rightarrow (y, x)$

(1) $\{(x, y): x + y = 9\}$ (2) $\{(x, y): x - y = 9\}$

(3) $\{(x, y): x + 2xy + y = 4\}$ (4) $\{(x, y): x^2 + 3y^2 = 5\}$

(continued on next page)

(c) $(x, y) \rightarrow (x, -y)$

(1) $\{(x, y):\ x + 7 = |y|\}$ (2) $\{(x, y):\ |x| - y = 4\}$

(3) $\{(x, y):\ x^2 + y^4 = 17\}$ (4) $\{(x, y):\ x^4 + y^3 = 17\}$

(d) $(x, y) \rightarrow (x + 2,\ y + 3)$

(1) $\{(x, y):\ y = \frac{3}{2}x + 5\}$ (2) $\{(x, y):\ y = \frac{3}{2}x + 4\}$

(3) $\{(x, y):\ 3x - 2y + 1 = 0\}$ (4) $\{(x, y):\ x = 2\}$

12. Suppose U, V, W, and Z are four points in the number plane such that

$$U = (85,\ 16), \qquad V = (97, -101),$$
$$W = (97,\ 16), \qquad Z = (85, -101).$$

(a) What is the midpoint of $\overline{UW}$? That is, give the ordered pair which is the midpoint of $\overline{UW}$.

(b) What is the midpoint of $\overline{UZ}$?

(c) Suppose A is the midpoint of $\overline{UW}$, B is the midpoint of $\overline{UZ}$, C is the midpoint of $\overline{ZV}$, and D is the midpoint of $\overline{VW}$. What points if any are in

(1) $\overleftrightarrow{AC} \cap \overleftrightarrow{BD}$? * (2) $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$?

(3) $\overleftrightarrow{UV} \cap \overleftrightarrow{WZ}$? (4) $\overleftrightarrow{VB} \cap \overleftrightarrow{ZD}$?

☆(d) What points, if any, are in

(1) $\overleftrightarrow{UV} \cap \{(x, y):\ y = 0\}$? (2) $\overleftrightarrow{ZW} \cap \{(x, y):\ x = 0\}$?

---

*In case you haven't guessed it, $\overleftrightarrow{AC}$ means the straight line through the points A and C.

B. 1. Suppose you have a die and also a "coin" which has a numeral '0' on one side and a numeral '1' on the other. By throwing the die you can get a number from $\{1, 2, 3, 4, 5, 6\}$, and by flipping the coin you can get a number from $\{0, 1\}$. The result of a single throw of both the die and the coin can be interpreted as an ordered pair in the cartesian product

$$\{1, 2, 3, 4, 5, 6\} \times \{0, 1\}.$$

Give the probability of getting, in one throw,

(a) $(2, 0)$. (b) $(2, 1)$.

(c) an element [member] of $\{(5, 0), (6, 1)\}$.

(d) an element of $\{(1, 1), (3, 0), (2, 1)\}$.

(e) $(1, 2)$.

(f) an ordered pair whose first component is 4.

(g) an ordered pair whose second component is 0.

2. Consider the equation:

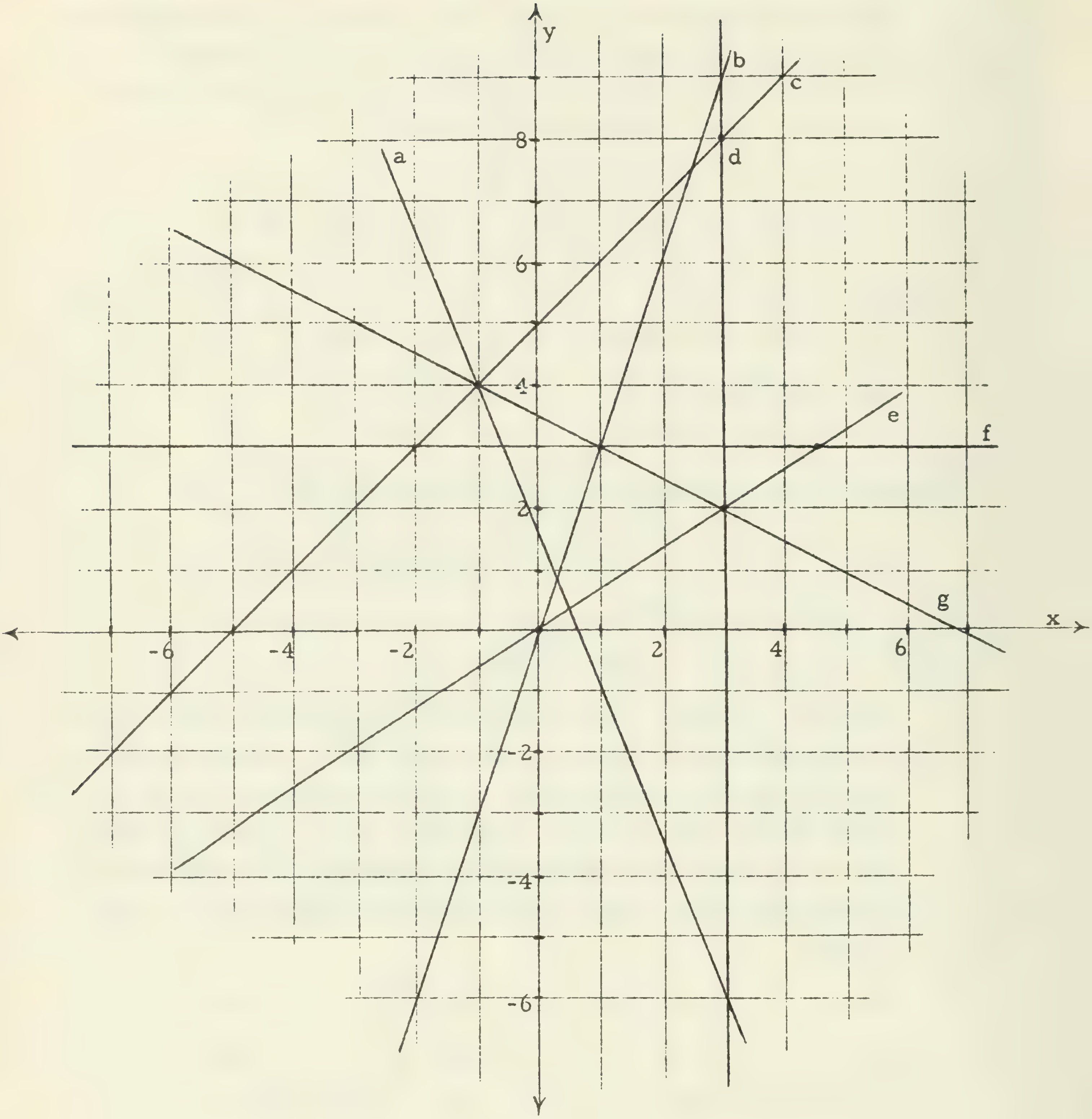
$$(*) \quad Rx = G.$$

Suppose you throw a pair of dice [one red and one green] to get an ordered pair of numbers, agreeing that the red die gives the first component of the ordered pair. If you replace the 'R' in (*) by a name for the first component of the ordered pair, and the 'G' by a name for the second component, you get an equation whose root can be found. What is the probability that the root will be

(a) 1 (b) 2 (c) 3 (d) $\frac{1}{3}$

(e) 6 (f) 7 (g) $\frac{1}{2}$ (h) 0

(i) an even number (j) an odd number


y
x
a
b
c
d
e
f
g
8
6
4
2
-2
-4
-6
-6
-4
-2
2
4
6

C. 1. Seven sets (straight lines) are pictured on the opposite page. List the points which belong to the sets listed below.

(1) $e \cap b$ [Ans: (0, 0)] (2) $d \cap c$

(3) $g \cap (b \cap f)$ (4) $c \cap a$

(5) $f \cap e$ (6) $e \cap f$

(7) $g \cap a$ (8) $c \cap g$

(9) $g \cap (a \cup b)$ (10) $(b \cap g) \cup (a \cap g)$

2. In the blank at the left of each of the following equations, write the letter which is the name of its locus.

______(1) $y = 5 + x$ ______(2) $3x - y = 0$

______(3) $5x = 3 - 2y$ ______(4) $x = 3$

______(5) $y = -\frac{1}{2}x + 3\frac{1}{2}$ ______(6) $4y - 6 - 10x = 0$

3. Use the picture to tell which ordered pairs satisfy both equations.

(1) $\left.\begin{array}{l} y = 5 + x \\ x = 3 \end{array}\right\}$ (2) $\left.\begin{array}{r} 3x - y = 0 \\ x = 3 \end{array}\right\}$ (3) $\left.\begin{array}{r} 5x = 3 - 2y \\ y = -\frac{1}{2}x + 3\frac{1}{2} \end{array}\right\}$

(4) $\left.\begin{array}{l} y = 3x \\ y = -\frac{1}{2}x + 3\frac{1}{2} \end{array}\right\}$ (5) $\left.\begin{array}{r} y - x = 5 \\ y + \frac{1}{2}x = 3\frac{1}{2} \end{array}\right\}$

D. 1. Suppose A, B, and C are sets of numbers such that

$A = \{1, 3, 5, 7, 9, 11\}$,

$B = \{-3, -2, -1, 0, 1, 2, 3\}$, and

$C = \{3, 4, 5, 6, 7\}$.

Tell the elements of the sets listed below.

(1) $A \cap B$ (2) $B \cap C$ (3) $C \cup A$

(4) $C \cap B$ (5) $A \cup (B \cap C)$ (6) $A \cap (B \cup C)$

(7) $A \cup \emptyset$ (8) $B \cap \emptyset$ (9) $(A \cup B) \cup C$

(continued on next page)

2. Given these sets of ordered pairs

   A: $\{(1, -9), (2, -13), (0, -5), (-1, -1), (-2, 3)\}$,

   B: $\{(0, 0), (-1, 4), (-2, 8), (3, -12)\}$,

   C: $\{(-1, -1), (0, 0), (2, 2), (4, 4), (6, 6)\}$,

   D: $\{(0, 5), (1, 6), (3, 8), (-2, 3), (-1, 4), (-5, 0)\}$,

   determine the ordered pairs in the sets listed below.

(1) $A \cap B$	(2) $A \cap D$	(3) $B \cap C$
(4) $B \cap D$	(5) $A \cup (B \cap C)$	(6) $(A \cup B) \cap C$
(7) $(C \cap D) \cup B$	(8) $A \cap (B \cup C)$	(9) $B \cup (A \cap D)$

<u>E</u>. Make labeled drawings of these loci.

1. the locus in (m, n) of '$2m - n = 8$'
2. the locus in (a, b) of '$3a + 2b = 10$'
3. the locus in (c, d) of '$c \geq 2d$'
4. the locus in (p, q) of '$2q - 10 = 3p$'
5. the locus in (r, s) of '$8 + |r| = 2s$'.

*

[Recall the convention explained on page 4-31 regarding sentences which contain the pronumerals 'x' or 'y'; then graph the sentences which follow.]

6. $-2y = x + 5$	7. $y = -x + 3$
8. $x \geq y + 2$	9. $y \leq -3x - 1$
10. $2x + 3y - 2 = 0$	11. $y + 5 > -2x$
12. $\lvert x\rvert + \lvert y\rvert \geq 6$	13. $\lvert x\rvert + \lvert y\rvert \leq 6$ or $\lvert x\rvert - \lvert y\rvert \geq 6$

*

On a picture of the number plane, graph the sets listed below.

14. $\{(g, h): g = -4h + 5\}$

15. $\{(m, k): m = -4k + 13\}$

16. $\{(b, a): a = \frac{1}{2}b - 3\}$

17. $\{(r, s): 6 = -2s + r\}$

18. $\{(x, y): x \geq 3 \text{ and } y \leq -2\}$

19. $\{(y, x): y + 3x + 2 = 0\}$

20. $\{(y, x): y + 3x + 2 = 0 \text{ and } y = x + 2\}$

21. $\{(x, y): x = y + 2 \text{ or } x + 3y + 2 = 0\}$

22. $\{(p, q): p \geq q + 1 \text{ and } q < p - 1\}$

F. Fill in the blanks with the simplest expressions to make true sentences.

1. For each a, the sum of $3a + 2$ and $-17a - 15$ is ________.

2. For each c, for each x, the sum of cx and $-15cx$ is ________.

3. For each m, the sum of $32 + m$ and $18 - 72m$ is ________.

4. For each r, the sum of $\sqrt{16} + r\sqrt{25}$ and $-5r + 343$ is ________.

5. For each a, for each b, the sum of $3a - ab + b$ and $2ab - b + 17a$ is ________.

6. For each h, for each d, for each x, the sum of $hx + 2d - x + dx$ and $(2 - h)x + 7d - dx$ is ________.

7. For each a, for each b, the sum of $3a - 2b + 7$, $-15b + 2 + a$, and $a - 100b$ is ________.

8. For each a, for each b, if I subtract $a^2 + 2ab + b^2$ from $-a^2 - 7ab - 2b^2$, I get ________.

9. For each m, for each n, the sum of $2m^2 + mn - n^2 + m - n + 10$ and $-m^2 - 10mn + 10n^2 - m + 3n + 20$ is ________.

10. For each t, the sum of $3t^2 + 2t - 10$, $-4t^2 - t - 15$, and $15t^2 + 32t - 5$ is ________.

11. For each x, for each y, for each p, the sum of $3x^2 + 2y^2 - p^2$ and $4y^2 + 2p^2 - 15x$ is ________.

(continued on next page)

12. For each x, for each y, the sum of $43x^2 + 2xy - y^2 + 1$ and $x - xy + y + 2$ is ___________.

13. For each a, for each c, the difference of $a^2 + ac - 2c^2$ from the sum of $3a^2 - 5ac + 2c^2$ and $a^2 + 7ac - 3c^2$ is ___________.

14. For each a, for each b, the difference of $14a + 15b$ from $10a + 6b$ is ________.

15. For each x, for each m, subtracting $x^2 + mx$ from $x^2 - mx + 2$ gives ___________.

16. For each p, if I subtract $-17p$ from -1, I get ___________.

17. For each r, the difference of $3r^2 - 10r + 15$ from $17r^2 + 17r - 5$ is ___________.

18. For each h, for each j, for each k, the difference of $h - j + k$ from the sum of $2h - j$ and $j + k$ is ___________.

19. For each x, for each y, the difference of $x^2 + xy - y^2 - 2x - y + 15$ from $2x^2 - 30xy + y^2 + x - 10$ is ___________.

20. For each r, the product of $2r + 1$ by $-2r^3$ is ___________.

21. For each a, for each b, the product of $a^2 + 2ab + b^2$ by $-3ab$ is ___________.

22. For each g, for each k, the product of $14gk$ by $-14g + k + gk - 53$ is ___________.

23. For each p, for each y, for each t, the product of $-py + yt + t - p$ by pyt is ___________.

24. For each b, the product of $7b - 1$ by $7b + 4$ is ___________.

25. For each c, the product of $-c + 2c - 3$ by $-4c$ is __________.

26. For each $a \neq 0$, for each $b \neq 0$, the quotient of ab by ab is ________.

27. For each $a \neq 0$, for each $c \neq 0$, the quotient of $3a^2c$ by ac^2 is ___________.

28. For each $r \neq 0$, for each $s \neq 0$, for each $t \neq 0$, the quotient of $32\ rst^2$ by $2rst$ is __________.

29. For each $k \neq 0$, for each $m \neq 0$, for each c, the quotient of $2km^2 \cdot 3cm$ by km is __________.

30. For each $n \neq 0$, for each $m \neq 0$, the quotient of $15n^2 \cdot m \cdot m$ by $-2m^2n^2$ is ___________.

31. For each a, the product of $a + 2$ by $a - 1$ is __________.

32. For each a, for each b, the product of $3a - 2b$ by $15a + b$ is __________.

33. For each x, for each y, the product of $x - y$ by one half of itself is __________.

34. For each m, the product of $2m - 15$ by three times itself is __________.

35. For each x, for each t, the product of $2x + 4t$ by $x + 2t$ is __________.

36. For each S, for each V, the product of $3S + VS$ by $S - VS$ is __________.

37. For each a, for each b, for each c, the product of $3ab + c$ by $ab - 3c$ is __________.

38. For each r, for each w, for each t, for each a, the product of $3rt - 2aw$ by $4rt + 15aw$ is __________.

39. For each a, for each b, for each c, the product of $a + b + c$ by itself is ___________.

40. For each p, the sum of $-\frac{1}{16}$ and the square of $4p - \frac{1}{4}$ is _______.

41. For each a, for each b, the sum of $a^2 - ab + b^2$ and the product of $-a + b$ by $a - b$ is __________.

42. For each $a \neq 0$, for each b, the quotient of $4a^2 + 8ab + 16ab^2$ by $8a$ is ___________.

43. For each j, for each k, the product of $2j - \frac{1}{3}k$ by itself is _______.

44. For each x_0, for each x_1, the product of $x_0 - 3x_1$ by $x_0 + 5x_1$ is ____________.

(continued on next page)

45. For each y, for each z, the difference of $3y^2 + yz - 25z^2$ from the product of $3y + z$ by $y - 25z$ is ___________.

46. For each a, for each $z \neq a$, the quotient of $a - z$ by $z - a$ is ___________.

47. For each m, for each n, the product of $m + n + 1$ by $-m + 2n$ is ___________.

48. For each $b \neq 0$, for each $c \neq 0$, the product of $\frac{3}{b} - \frac{4}{c}$ by $\frac{2}{b} + \frac{1}{c}$ is ___________.

49. For each $d \neq 0$, for each m, the product of $\frac{1}{d} + m$ by $4m - \frac{3}{d}$ is ___________.

50. For each r, for each t, the product of $3r + t + 1$ by $-r - t - 1$ is ___________.

51. For each x, ___________ exceeds x by 1.43.

52. For each y, for each x, y increased by the difference of y from x is ____________.

G. Evaluate each of the following pronumeral expressions using the given values of the pronumerals.

1. $\sqrt{2as}$; '14' for 'a', '7' for 's'.

2. $\sqrt{9a^2 - 6ab + b^2}$; '−1' for 'a', '10' for 'b'.

3. $\sqrt{\dfrac{3 + a}{a - 1}}$; '2' for 'a'.

4. $\sqrt{\dfrac{a - 2\sqrt{b} + 1}{a + b}}$; '4' for 'b', '12' for 'a'.

5. $\frac{h}{2}(a^{-3} + A^{-3})$; '4' for 'h', '2' for 'a', '10' for 'A'.

6. $\pi r^2 + ar$; '$\frac{1}{\pi}$' for 'r', 'π' for 'a'.

7. $\dfrac{\frac{1}{2}ax + \frac{1}{3}bx}{1 + 2}$; '2.2' for 'a', '7.9' for 'b', '32' for 'x'.

8. $s_0 + v_0 t + \frac{1}{2}at$; '4' for 's_0', '-2' for 'v_0', '32' for 'a', '9' for 't'.

9. $-\frac{1}{p} + \frac{1}{2} \cdot \frac{pq^2}{p + q^2}$; '$-\frac{1}{2}$' for 'p', '$\frac{1}{2}$' for 'q'.

10. $\pi\ell(r_1 + r_2)$; '1' for 'ℓ', '2' for 'r_1', '.01' for 'r_2'.

H. Solve these equations for the pronumeral indicated.

1. $s = \frac{2t - v}{w}$; v

2. $s = \frac{2t - v}{w}$; w

3. $7t - \frac{1}{3}r = r - 5t$; t

4. $\frac{2}{m} + \frac{1}{n} = \frac{3}{p}$; p

5. $A = \frac{1}{2}h(a + B)$; B

6. $s = s_0 + vt + \frac{1}{2}at^2$; a

7. $\frac{9}{y} = \frac{9 + a}{2y + 1}$; y

8. $\frac{9 - a}{y} = \frac{9 + a}{2y + 1}$; y

9. $ax^2 + bx + c = 0$; b

10. $a(b + c) + d(e + f) - 15 = 0$; f

I. 1. What are all the factors of 28 with respect to the set of positive integers?

2. What are all the factors of 28 with respect to the set of integers?

3. What are all the factors of -28 with respect to the set of integers?

4. What are all the factors of -28 with respect to the set of negative integers?

5. What numbers are not factors of 28 with respect to the set of rationals? The set of reals?

6. What are the factors of -1 with respect to the set of integers?

7. Give 4 numbers which have exactly 3 factors with respect to the set of positive integers.

8. Give 1 number which has exactly 5 factors with respect to the set of integers.

9. List all the prime numbers which are greater than 100 but less than 110.

10. Give the prime power factorization of 190080.

J. Factor.

1. $4x - 4y$	2. $a^2x - bx^2$
3. $x^2 - 3x - 18$	4. $d^2 + 7d + 10$
5. $x^2 - 10x + 16$	6. $t^2 - 49$
7. $x^2 - 16x + 64$	8. $5m^2 + 3m - 2$
9. $6x^2 + 7x - 3$	10. $8x^2 + 26x - 45$
11. $4r^2 - 20r + 25$	12. $16 - 24p + 9p^2$
13. $16 + 29x - 6x^2$	14. $5z^2 - 45$
15. $8y^2 + 12y - 36$	16. $24x^2 + 24x + 6$
17. $x^4 + 5x^2 + 4$	18. $k^4 - 1$
19. $x^4 - 7x^2 - 18$	20. $2x^3 - 11x^2 + 12x$
21. $(a - 3)^2 - (a + 4)^2$	22. $4x^2 - 9y^2$
23. $x^2 - 7xy + 12y^2$	24. $6x^2 - 11xy - 10y^2$
25. $x^4y - 7x^3y + 12x^2y$	26. $8x^3 - 40x^2y + 50xy^2$
27. $-50x + 28 + 12x^2$	28. $a^2c + a^3 - 12ac^2$
29. $(x^2 - 3)^2 - (x^2 - 5)^2$	30. $c^5d^5 + 36cd - 13c^3d^3$

K. For each of the following problems, make a graph which could be used to solve problems of that type, and solve the problem.

1. A taxicab company charges its customers at a rate of 35 cents for the first $\frac{2}{3}$ mile traveled, and 10 cents for each additional $\frac{1}{3}$ mile or fraction thereof. How much is the fare for a trip of 5 miles?

2. Parcels which are sent at "book rate" cost 9 cents for the first pound and 5 cents for each additional pound or fraction thereof. How much would it cost to ship a box of books which weighs 12 pounds?

3. A printing company will print pocket calendars at a rate of \$7.50 for the first 500 calendars, and 90 cents for each additional 100. How much would 2200 calendars cost at this rate?

4. A department store which accepts mail orders for gift items charges 25 cents for postage and handling on "one item" orders, and on orders for two or more items the charge is 25 cents for the first item and 5 cents for each additional item. How much would be charged for postage and handling on a gift order for 5 items?

5. A store which specializes in selling merchandise on the installment plan advertises refrigerators for "\$25 down and \$15 per month". How many months will be required to complete the payment on a refrigerator for which the "installment plan" price is \$399.95?

L. Solve.

1. $6 + z = 11$
2. $5 - x = 7$
3. $2x - 5 = 3$
4. $3 - \frac{1}{3}x = -1$
5. $5t + 6 > 2t$
6. $6(3 - x) = 12x$
7. $\frac{x}{2} - 2 = \frac{x}{6}$
8. $\frac{x}{3} - \frac{1}{2} < \frac{1}{6}$
9. $x - \frac{1}{3} = -(x - \frac{1}{3})$
10. $2(x - 3) = 2(x - 4)$
11. $|3 - 2x| = 19$
12. $3 - |2 + 7t| = -10$
13. $8m + 30m - 2m - 10 = 2m + 12 + 36m - 12$
14. $4x - (12 - 3x) = -5x + 12$
15. $2(2z - 3.1) + (z - 1) - 3.3 = 0$
16. $14(b - 3) + b - 12(5b + 1) = -72b$
17. $3(\frac{7}{9}r + 2) - 4r + 15 = 32 - 7r - 5\frac{2}{3}$

(continued on next page)

18. $100(p + 4) + 10p + p = 111p + 400$

19. $100(p + 3) + 10p + p = 111p + 400$

20. $-7(2a - 3) + 3(\frac{1}{3} - a) = 14 + 2(4 - 4379a)$

21. $13(s + 2) - 5(2s - 10) = -3(s + 4) + 10$

22. $x(x - 3) + x^2 = 2(x^2 - 6)$

23. $17(x - 6) \leq -5 - 16(x + 4)$

24. $60x^2 - 10x + 200 = 60x^2 + 6x + 203$

25. $x(x + 11) = 2(x^2 - 6)$

26. $2a^2 + 2(6a - 3) + 6(2a - 1) - a(2a + 3) = -5$

27. $14c - 1 + 4(-c - 2) + c^2 = 12c(c + 1) - 11c^2$

28. $33 - n^2 + 3n = 3 + 4(3n + 2)$

29. $9y(y + 1) - 2(y^2 - \frac{1}{2}) = y$

30. $(6x + 25)(6x - 25) = -49$

31. $(2x - 3)(2x - 3) + 2(2x - 3) + 1 = 0$

32. $\frac{1 - k}{12} + \frac{k}{5} - \frac{k - 1}{5} = \frac{1}{12}$

33. $\frac{y}{y - 1} + 2 = \frac{3}{5}$

34. $\frac{2}{3}(j + 1) - \frac{j - 8}{2} + 1 = \frac{1}{2}$

35. $\frac{700 + 300x + 10,000\% \text{ of } x}{15} = 100$

36. $1 - \frac{1}{d} = \frac{6}{d^2}$

37. $\frac{1}{2a} - \frac{a}{3} = \frac{-5}{12}$

38. $\frac{179}{2t - 13} = 1 - \frac{2t}{2t - 13}$

39. $4 - \frac{7}{z + 2} = \frac{z + 9}{z + 2} + z - 4$

40. $y(y - 6) = -5 - 16(y - \frac{29}{16})$

41. $\frac{37}{t} - 2 = \frac{37}{\frac{2}{3}t} - 1$

42. $\frac{3x + 1}{4x + 1} = \frac{6x}{8x - 1}$

43. $\frac{p + 2}{p - 3} = \frac{p + 4}{p - 1}$

44. $\frac{x - 2}{x - 3} - 1 = \frac{3}{x + 3}$

45. $\frac{3}{4}(\frac{2}{1 - h}) + \frac{1}{2} = \frac{24h - 31}{4(1 - h)}$

46. $\frac{1}{5}(r - 1) - \frac{1}{4}(r - 1) + \frac{1}{3}(r - 1) - \frac{1}{2}(r - 1) + (r - 1) = 0$

47. $43(r + 1)^2 - 52(2r + 2)^2 = 0$

48. $\frac{r + 1}{r - 2} - \frac{r + 3}{r - 4} = \frac{-6}{6r - r^2 - 8}$

M. Simplify.

1. $2x + 5x + 3 + 1$
2. $4z - 3z + 6 - 5z - 7$
3. $2a - 5 + 3a - 2 - 1$
4. $2(3a - 2) + 3(a - 2)$
5. $4(2d - 1) - (d - 2) + (d - 4)$
6. $-[(2x - 3) - (4x - 1)] - 2(3 - 2x)$
7. $6y - -3 + -2y - 4$
8. $4a - 2b + 3a - 6b - 2b$
9. $x^2 + 3x - x^4 - 4x$
10. $-2(a - 3) + 3(a + b) - 2b$
11. $x(x - 3) + x^2(3 - x) + 3(x^2 - x)$
12. $(x - 2)(x + 3) + 2(x - 4)$
13. $-2x(x^2y) - 3x^3y^2(-xy)$
14. $-3ab(-2a^3b^2 + 4ab^3)$
15. $-(2x - 3)(x + 2) - (x - 4)^2$
16. $3x^2[(x - 3)(x - 4) + (4 - x)(x - 3)]$
17. $\frac{-24a}{2}$
18. $\frac{6a^3b^2}{2a^2b}$
19. $\frac{6a^3b^2 - 4ab^2}{2ab^2}$
20. $\frac{2x^2}{3y} \cdot \frac{9y^2}{4x}$
21. $\frac{3x^2y^3}{4xy^4} \cdot \frac{24x^3y^2}{9x^4y}$
22. $\frac{20x^2y}{-6xy^3z} \div \frac{10xy^4}{3xz}$
23. $18x(\frac{2}{9x} + \frac{1}{6})$
24. $24x^2y(\frac{3y}{8x} - \frac{2x + 1}{12x^2} + \frac{1}{6x})$
25. $\frac{2}{3x} + \frac{3}{4x}$
26. $\frac{2}{x - 3} + \frac{3}{x - 4}$

(continued on next page)

27. $\dfrac{a - 1}{a + 4} - 2 + \dfrac{a}{2a + 4}$

28. $\dfrac{1 - \frac{x}{y}}{1 + \frac{x}{y}}$

29. $\dfrac{\frac{2}{3k} + \frac{1}{k}}{\frac{1}{2k} - \frac{1}{6}}$

30. $\dfrac{1 - \frac{1}{c - d} - \frac{1}{d - c}}{\frac{c + d}{c - d}}$

31. $\sqrt{(n + 2)^4(s - 1)^2}$

32. $\sqrt{4c^2 + 12cd + 9d^2}$

33. $\sqrt{2a^2}\,(\sqrt{18a^2} - \sqrt{32a^2})$

34. $\sqrt{2}\,(\sqrt{26} - \sqrt{22}) + \sqrt{3}\,(\sqrt{33} - \sqrt{39})$

N. In these exercises, write in the blanks the principles and generalizations (<u>not just their names</u>) which justify the steps in the proof.

1. For each x, for each y, $(-x)y + xy = 0$.

$$\begin{aligned} &(-x)y + xy \\ &= y(-x) + yx \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \\ &= y(-x + x) \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \\ &= y(x + -x) \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \\ &= y \cdot 0 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \\ &= 0. \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \end{aligned}$$

2. For each a, for each b, $2a + 3b - 2a = 3b$.

$$\begin{aligned} &2a + 3b - 2a \\ &= 2a + 3b + -(2a) \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \\ &= 3b + 2a + -(2a) \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \\ &= 3b + [2a + -(2a)] \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \\ &= 3b + 0 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \\ &= 3b. \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \end{aligned}$$

3. For each r, for each $s \neq r$, $\frac{1}{r - s} = -\frac{1}{s - r}$.

$$\begin{aligned} & -\frac{1}{s - r} \\ &= \frac{1}{-(s - r)} \\ &= \frac{1}{r - s}\,. \end{aligned}$$

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4. For each a, for each b, for each c, for each d,
$a - b - (c - d) = a - b - c + d$.

$$\begin{aligned} & a - b - (c - d) \\ &= a - b + -(c + -d) \\ &= a - b + (-c + - -d) \\ &= a - b + (-c + d) \\ &= a - b + -c + d \\ &= a - b - c + d\,. \end{aligned}$$

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5. For each r, for each s, for each t, $7r - 2s - 5s = 7r - 7s$.

$$\begin{aligned} & 7r - 2s - 5s \\ &= 7r + -(2s) + -(5s) \\ &= 7r + [-(2s) + -(5s)] \\ &= 7r + [(-2)s + (-5)s] \\ &= 7r + (-2 + -5)s \\ &= 7r + -7s \\ &= 7r + -(7s) \\ &= 7r - 7s\,. \end{aligned}$$

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O. Solve these problems.

1. Mr. Adams paid $127.00 for a plane ticket. This price included 10% Federal tax. How much was the tax?

2. Two numbers differ by 58.5. The smaller number is 55% of the larger. What are the two numbers?

3. The smallest of seven consecutive integers is $\frac{6}{7}$ of the largest. What are the seven integers?

4. Gundo owns goondols and Rassler owns ramlers. In all they own 48 goondols and ramlers. The number of ramlers that Rassler owns is $\frac{9}{7}$ the number of goondols that Gundo owns. How many ramlers does Rassler own?

5. In one of the redwood forests a certain tree is calculated to be twice as old as another; 150 years ago the first tree would have been 200 years older than the second tree was then. Find the present age of the two trees.

6. Alphatown and Zilchville are 76 miles apart. At 8:00 a.m. on a certain day Zeke left Zilchville on his bicycle and started toward Alphatown. He traveled at the rate of 12 miles per hour, but rested 2 hours on the way. On the same day, Alex, who lives in Alphatown, left there at 9:00 a.m. on his bicycle and traveled toward Zilchville at a rate of 10 miles per hour all the way, with no stops to rest. How far from Zilchville were the boys when they met?

7. Charlotte went to the candy store and spent 57 cents for candy bars, bubble gum, and fruit drops. The candy bars were 6 cents each; the bubble gum was priced at two pieces for 1 cent; each package of fruit drops sold for 4 cents. Charlotte counted the number of candy bars, pieces of bubble gum, and packages of fruit drops and found that she had 57 in all. How many of each kind had she purchased?

8. Mrs. Beccardi has a rectangular flower garden which is 3 feet longer than it is wide. She wants to plant a new variety of tulips; so, she decides to enlarge the garden by making it 1 foot wider and 3 feet longer. This, of course, gives more area in the garden plot. Mrs. Beccardi calculates that she will have 22 square feet more in her garden after she makes the changes. What are the dimensions of the garden before Mrs. Beccardi enlarges it?

9. Suppose the sum of the degree-measures of the angles of a triangle is 180. The measures of angle A and angle B are in the ratio 3:4. The degree-measure of angle C is 2 less than the sum of the degree-measures of angles A and B. Find the degree-measure of each angle.

10. A dealer wishes to prepare 150 gallons of an oil blend to sell at 41 cents a quart. He plans to mix oil selling for 45 cents a quart with a cheaper grade that sells for 39 cents a quart. How many <u>gallons</u> of each kind should he use?

11. Mr. Jones received a check for $23,210 for the sale of a house. The buyer included the real estate agent's commission when he wrote the check. This commission was computed at a rate of 5% on the first ten thousand dollars [of the selling price] and $2\frac{1}{2}$% on the remainder. After giving the real estate agent his commission, how much did Mr. Jones have left? That is, what was the selling price?

12. A money box contains nickels, dimes, and quarters. The number of dimes is 1 less than twice the number of quarters; the number of nickels is 9 more than twice the number of dimes. Altogether the coins are worth $4.80. How many coins of each kind are in the box?

13. David and his cousin Dick can do a certain piece of work in 3 days when they work together. David can do this job in 5 days when he works alone. He starts on it one morning when Dick has gone hunting, and works all day. The next day Dick joins him at the task, and works with him until the job is done. How many days did Dick work <u>with</u> David?

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14. A jeweler who makes rings has two alloys of gold, one being 85% pure gold and the other 55% pure gold. How many ounces of each of the alloys must be taken to make 20 ounces of a new alloy that will be 76% pure gold?

15. A group of paratroopers are making their first practice jumps. One sixth of them jumped at the first drop zone; 25 jumped at the second; twenty per cent of the remainder left the plane at the third drop zone, and three-fourths of those remaining jumped at the fourth drop zone. There were 25 fellows remaining on the plane; they decided not to jump, even if this meant they would be transferred out of the division. How many men were in the original group ?

16. A druggist has a gallon of an alcohol solution that is 85% pure alcohol, and wishes to reduce it to a solution which is 65% pure alcohol. How much distilled water must he add to the original solution to obtain the "65% pure" solution?

P. Find the approximation correct to the nearest .01.

1. $\sqrt{6} \cdot \sqrt{2}$
2. $\dfrac{\sqrt{250}}{\sqrt{2}}$
3. $\sqrt{32} - \sqrt{8}$
4. $\sqrt{27} + \sqrt{75}$
5. $\sqrt{32} \cdot \sqrt{14} \cdot \sqrt{21}$
6. $\sqrt{108} + \sqrt{48} - \sqrt{3}$

Q. Simplify. [Use scientific notation to express the result.]

1. $\sqrt{25 \cdot 10^8} + \sqrt{36 \cdot 10^6}$
2. $\dfrac{(.144)(14000)}{(.000012)(.006)}$
3. $\dfrac{14000(10^{24})}{.007(10^{-20})}$
4. $\dfrac{\sqrt{.00000009} \cdot (2000)^8}{(.0006)^4 \cdot \sqrt{40000}}$
5. $\dfrac{.236(10^{16}) + 2.36(10^{18})}{.00236(10^{20})(10^5)}$

TEST

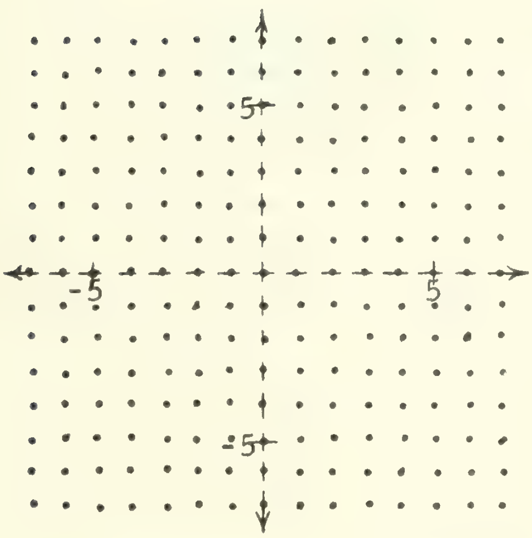
I. Suppose A = {2, 3, 5, 6, 7} and B = {2, 6, 15, 24, 35, 48, 63}.

1. How many ordered pairs of numbers are in A $\times$ B?
2. How many ordered pairs of numbers in A $\times$ B will have first component 3?
3. How many ordered pairs of numbers in A $\times$ B will have first component 6?
4. How many ordered pairs of numbers in A $\times$ B will have second component 6?
5. If you listed the ordered pairs of numbers in B $\times$ A, how many would have second component 6?

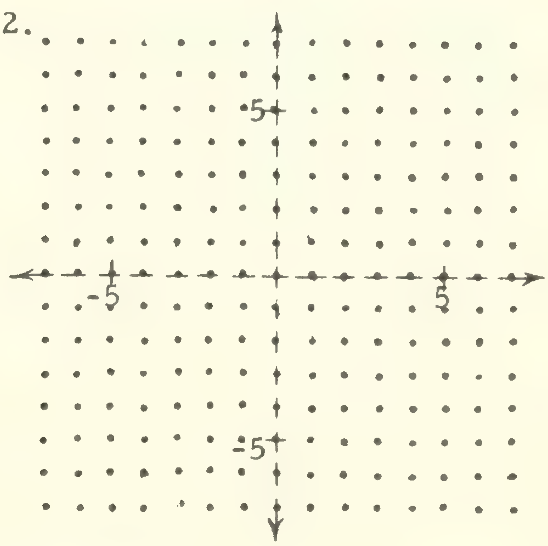
II. Use these pictures of the number plane lattice and plot the sets of points described below.

1. The set of all ordered pairs of integers which correspond with dots that have first coordinate 4.
2. The set of all ordered pairs of integers with second component -3.
3. $\{(x, y),\ x \text{ and } y \text{ integers: } x = 2 + 3y\}$
4. $\{(x, y),\ x \text{ and } y \text{ integers: } x + y \leq 5\}$
5. $\{(x, y),\ x \text{ and } y \text{ integers: } x < -4 \text{ and } y > 3\}$
6. $\{(x, y),\ x \text{ and } y \text{ integers: } x > 2 \text{ or } y < -2\}$

1.



2.



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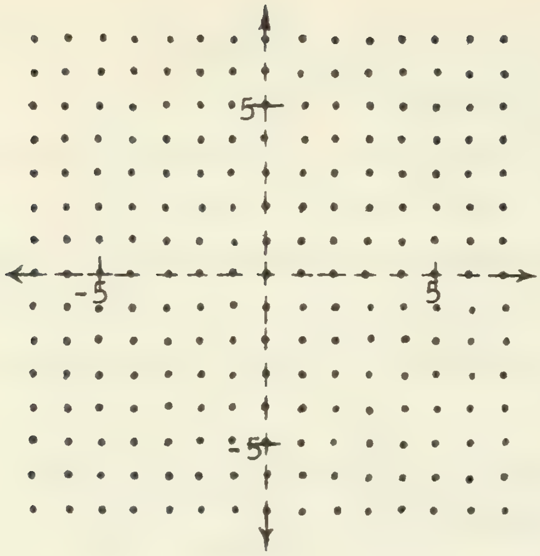

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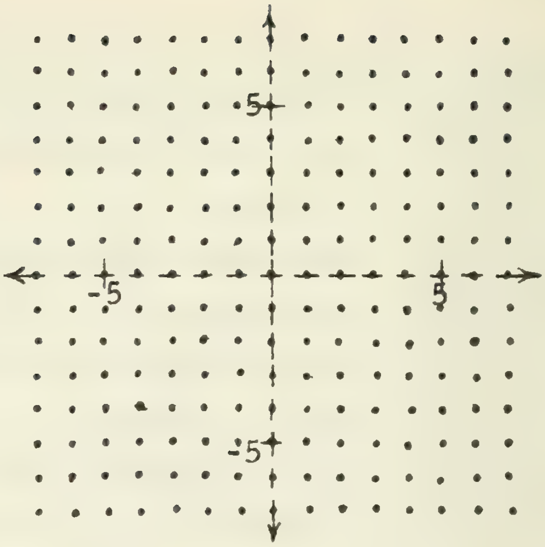

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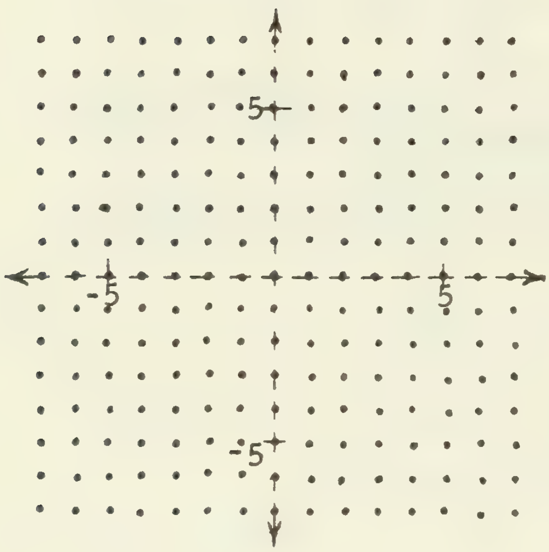

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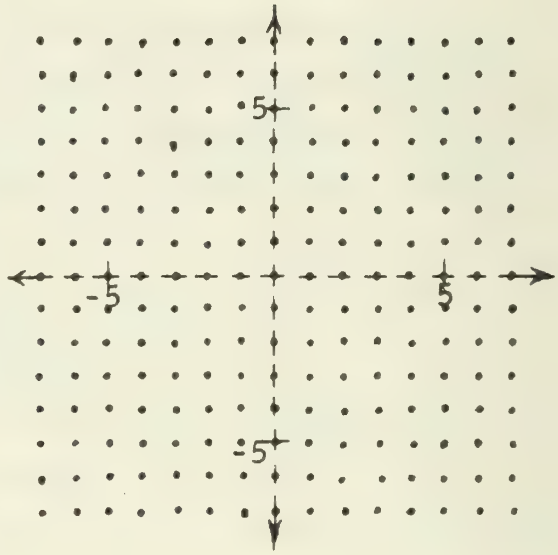
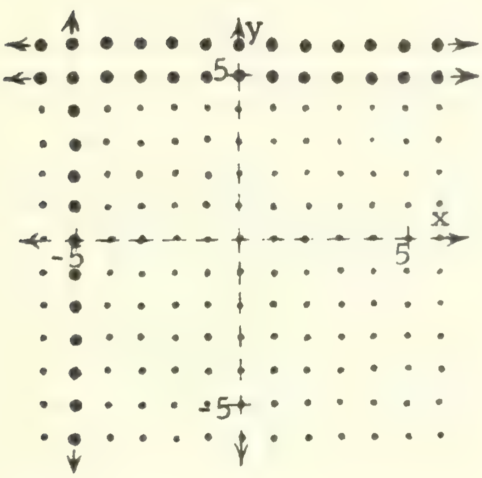
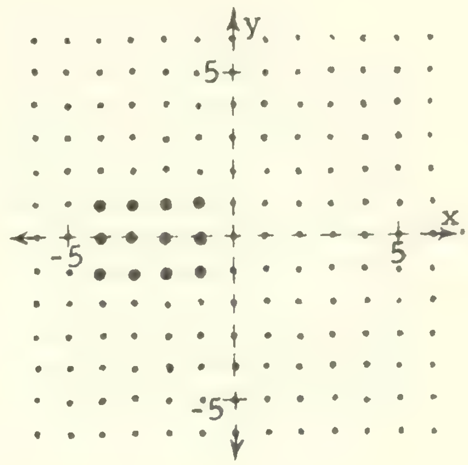


III. Here are pictures of sets of ordered pairs of integers. For the first two charts, write word-descriptions of the sets pictured. For the third and fourth charts, use brace-notation to describe the set pictured.

1.



2.



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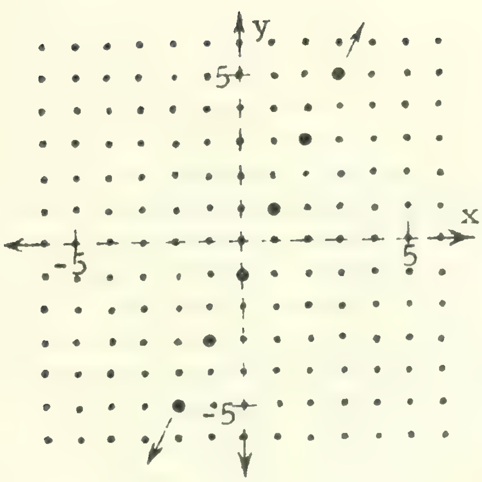

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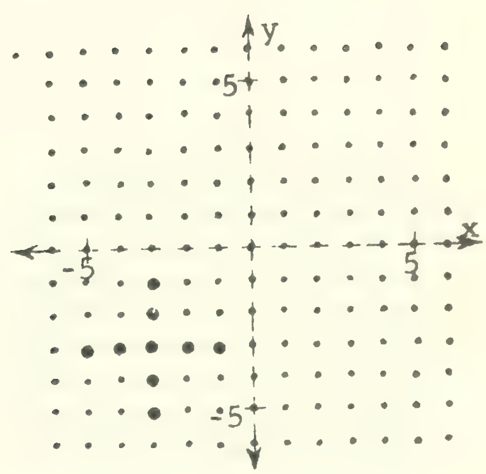


IV. In each of the following exercises, you are given descriptions of a set P and a set Q. For each exercise,

(a) plot the points in each set on the same diagram,

(b) tell the number of points in each set,

(c) tell the number of points in the intersection, and

(d) tell the number of points in the union.

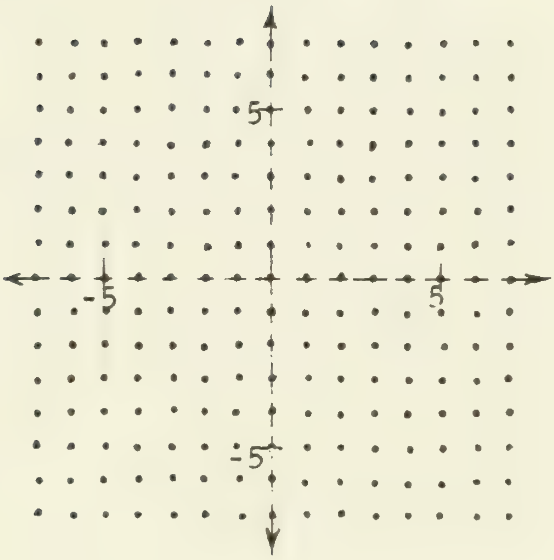
1. $P = \{(x, y),$ x and y integers: $-5 < x < 1$ and $5 > y > 1\}$

   $Q = \{(x, y),$ x and y integers: $|x| \leq 3$ and $|y| \geq 4\}$

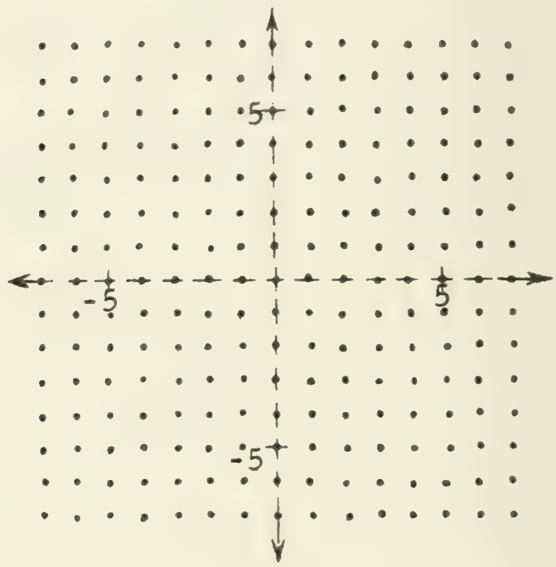
2. $P = \{(x, y),$ x and y integers: $6 > x > 3$ and $0 > y > -3\}$

   $Q = \{(x, y),$ x and y integers: $0 < x < 3$ and $3 < y < 6\}$

1.

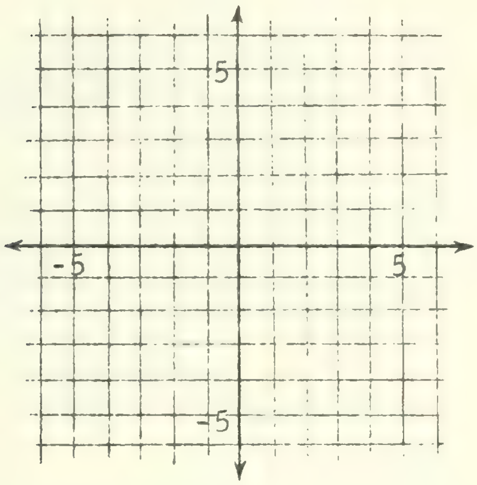


2.

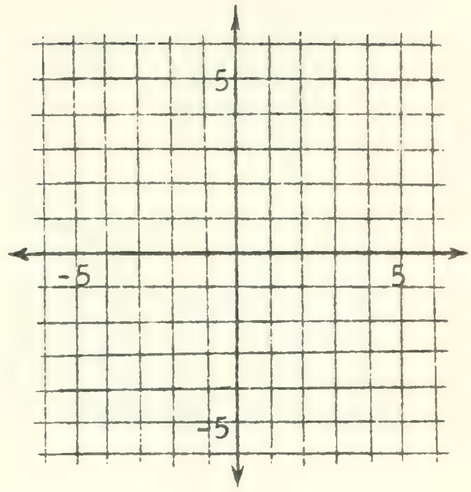


V. Graph these sets.

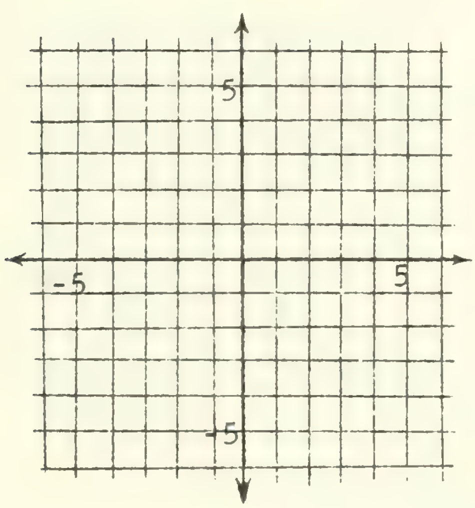
1. $\{(x, y): x \leq -3\}$



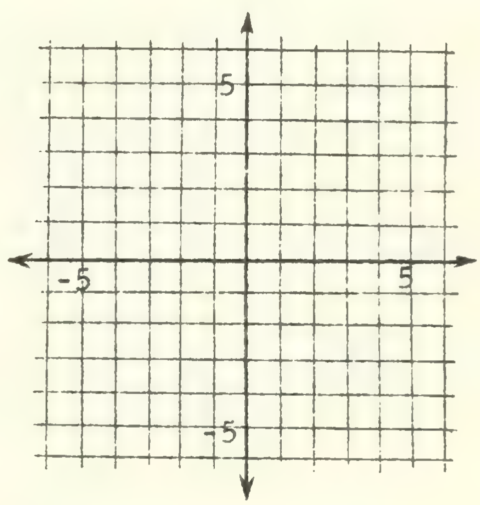
2. $\{(x, y): x \geq 5 \text{ and } y \geq 4\}$



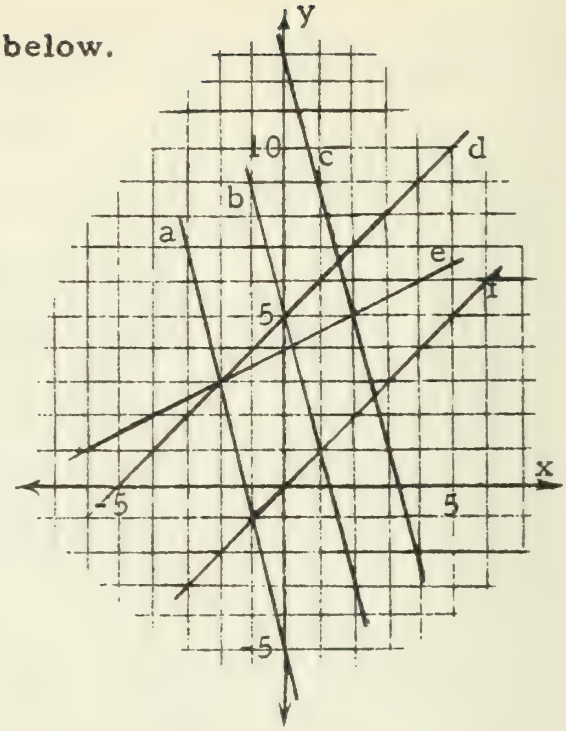
3. $\{(m, k): m > \frac{1}{2} \text{ or } k > 2\frac{1}{2}\}$



4. $\{(a, b): b = \frac{1}{2}a + 4\}$



VI. Six sets (straight lines) are pictured below.



1. List the points which belong to the sets.

(1) $a \cap e$ (2) $a \cap d$

(3) $c \cap e$ (4) $b \cap a$

(5) $d \cap (a \cup b)$ (6) $f \cap b$

(7) $f \cap$ (the x-axis)

(8) $d \cap$ (the y-axis)

(9) $(f \cap a) \cup (f \cap b)$

(10) $e \cap (a \cup c)$

2. In the blank at the left of each of the following equations, write the letter which is the name of its locus.

________(1) $y = x$ ________(2) $y = -4x - 5$

________(3) $y = -4x + 5$ ________(4) $y = \frac{1}{2}x + 4$

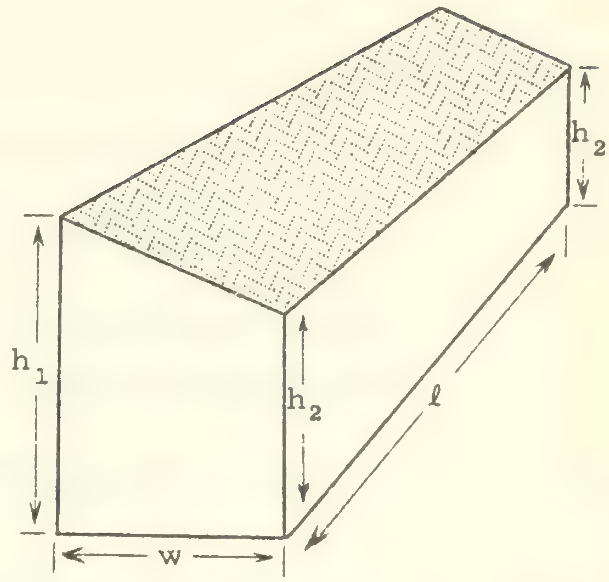
________(5) $y = x + 5$ ________(6) $y = -4x + 13$

3. Use the picture to tell which ordered pairs satisfy both equations.

(1) $\left.\begin{array}{l} y = -4x - 5 \\ y = -4x + 13 \end{array}\right\}$ (2) $\left.\begin{array}{l} y = x \\ y = -4x + 5 \end{array}\right\}$

(3) $\left.\begin{array}{l} y = -4x - 5 \\ y = \frac{1}{2}x + 4 \end{array}\right\}$ (4) $\left.\begin{array}{l} y = -4x + 5 \\ y = x + 5 \end{array}\right\}$

VII. Here is a picture of a granary seen on an Iowa farm. In order to get an estimate on the cost of painting the exterior walls (not the roof) the farmer needs to know how many square feet of wall surface is to be painted.



1. Write a formula which would enable the farmer to determine the area measure of the exterior walls.

2. Suppose that the granary is 20 feet long and 8 feet wide, and that the longer and shorter heights are 16 feet and 12 feet respectively. How many square feet of exterior wall surface would there be?

VIII. Complete each of these sentences to true ones in at least one way.

1. 4 is a factor of 56 with respect to the set of ______________ because 4, 56, and ______ belong to this set and $56 = 4 \cdot$ _____.

2. -8 is a factor of -72 with respect to the set of _____________ because -8, -72, and _______ belong to this set and $-72 = -8 \cdot$ _____.

3. -6 is a factor of 93 with respect to the set of ___________ because -6, 93, and ________ belong to this set and $93 = -6 \cdot$ _______.

4. $\frac{2}{3}$ is a factor of $\frac{3}{5}$ with respect to the set of ______________ because $\frac{2}{3}$, $\frac{3}{5}$, and ________ belong to this set and $\frac{3}{5} = \frac{2}{3} \cdot$ _______.

5. _____^{2} is a factor of $35 \cdot 50$ with respect to the set of ____________ because _____^{2}, $35 \cdot 50$, and 70 belong to this set and $35 \cdot 50 =$ ____$^2 \cdot 70$.

IX. 1. Which of these numbers are prime numbers?

5, 8, 13, 19, 21, 31, 42, 57682

2. List all numbers which are factors of the composite number 36.

3. Give the prime factorization for the number 72; for the number 150.

4. Use the prime factorization you found in Exercise 3 in making a list of <u>all</u> the factors of the number 150.

5. Give the prime <u>power</u> factorization of the number 1800.

X. 1. Use exponents to simplify each expression.

(a) $(rs)(rs)(rr)$ (b) $(dd + e)(d + ee)$ (c) $7 \cdot 7 + nnn$

(d) $\dfrac{3 \cdot 3 \cdot 3aaaa}{bbb}$ (e) $\dfrac{5 \cdot 5nn + 7 \cdot 7 \cdot 7kk}{3 \cdot 3 \cdot 3aaa - 2 \cdot 2bb}$

2. Simplify.

(a) $n^5 \cdot n^7$ (b) $4^2 \cdot 4^4$ (c) $2^3 \cdot 2^4 \cdot x^3 \cdot x^5$

(d) $(6c^3d)(-6cd^2)$ (e) $\dfrac{-14n^3p^5}{42np^7}$

(f) $(2ab^2)^3(3a^2b)^2$ (g) $(-\frac{1}{2}e^3fg^2)^3$

(h) $\dfrac{28^2xy^4}{28^3x^3y^5}$ (i) $\left(\dfrac{2a^6b^2}{3ac}\right)^3$

(j) $\left(\dfrac{4x^2}{3y^3}\right)^2 \cdot (3y)^2$

3. Factor each of these pronumeral expressions.

(a) $a^4 - 16$ (b) $36 - 12c + c^2$ (c) $5a^3b - 25ab$

(d) $10n^2 - 11n - 6$ (e) $42 + 22d - 4d^2$

4. Find an HCF of the given pronumeral expressions.

(a) $12y,\ 18y^2$ (b) $7h^3j^2k^4,\ 6hj^2k^2$

(c) $3 + n,\ 9 - n^2$ (d) $r^2 - 2rs + s^2,\ r^2 - s^2$

(e) $a^2 - ab - 2b^2,\ 3a^2 - 2ab - 15b^2$

5. Find an LCM of the given pronumeral expressions.

(a) $12u^2v,\ 2uv^2,\ 6u^3v^3$ (b) $5ab^2,\ 2ab,\ 3a^2b$

(c) $625 - n^4,\ n^2 + 25,\ n - 5$ (d) $\frac{1}{4} - n + n^2,\ .25 - n^2$

(e) $3 + c,\ 2 - 3c,\ c - 5$

XI. Simplify.

1. $\frac{3a}{5b} + \frac{6}{3a^2b^2}$ 2. $\frac{r}{r^2 + 6r + 9} - \frac{r + 2}{r + 3}$

3. $\frac{9c - 21}{3c + 12} + \frac{7 - 3c}{4 + c}$ 4. $\frac{2}{5y - 10} + \frac{5}{3y + 6} - \frac{y}{20y^2 - 80}$

XII. For each number listed below, write its name in scientific notation.

1. 1792 2. 584.3 3. 3,456,700

4. 0.369×10^3 5. 0.0468×10^4 6. 0.00579×10^5

7. 0.000679×10^2 8. 0.00345 9. $7,654,000 \times 10^{-2}$

XIII. Simplify, and use scientific notation to express the results.

1. $480,000 \times 3200 \times .0015$ 2. $(57 \times 10^3) \times 0.005$

3. $(11 \times 10^{-2})^2$ 4. $(12^2 \times 10^3)^2 \times (\frac{1}{12} \times 10^{-2})^3$

5. $\frac{(8 \times 10^{-3}) \times (45 \times 10^2)}{(9 \times 10^{-6}) \times (32 \times 10^7)}$

SUPPLEMENTARY EXERCISES

A. Consider the cartesian product

$$\{-3, -2, -1, 0, 1, 2, 3\} \times \{-5, -4, -3, -2, -1, 0, 1, 2, 3\}.$$

Make a picture of this lattice.

Tell how many dots are graphs of ordered pairs with

1. first component -2.
2. second component 3.
3. first number less than or equal to -2.
4. second component greater than or equal to 0.
5. first number 4.
6. first number less than -1 and second number greater than -1.
7. first number less than -1 or second number greater than -1.
8. first number 2 less than second number.
9. second component 3 more than first component.
10. second number equal to 2 less than 3 times first number.
11. second component 2 less than 3 times first component and with second component twice the first component.

B. Draw a picture of the number plane lattice [your diagram should contain enough dots so that you can plot the points $(-6, 6)$, $(-6, -6)$, $(6, -6)$, and $(6, 6)$.] Plot the sets of points described below. Mark the dots in some particular fashion so that you can tell the sets apart. [We abbreviate 'real integers' to 'integers'.]

The set of all ordered pairs of integers such that...

1. first component is 1 more than second component.
2. first component is 2 less than second component.
3. the sum of the components of each ordered pair is 7.
4. 6 is the sum of the first component and twice the second component.

5. second component is −5.

6. their graphs have first coordinate 5.

7. the first component is more than 3 <u>and</u> the second component is less than −4.

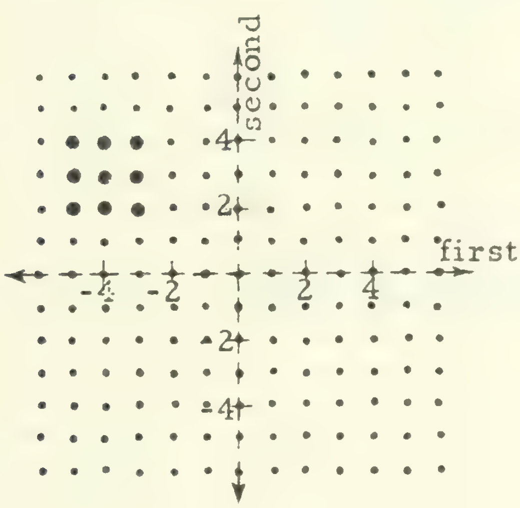
8. Repeat Exercise 7 using 'or' instead of 'and'.

9. the second component is less than 1 but greater than −2 <u>and</u> the first component is greater than −4 but less than 0. [How many points are there in this set?]

10. the second component is greater than 0 but less than 5 and the first component is −4.

<u>C</u>. Here are pictures of sets of ordered pairs of integers. Write descriptions of the sets pictured, using the type of wording of the exercises of Part B on page 4-6.

1.



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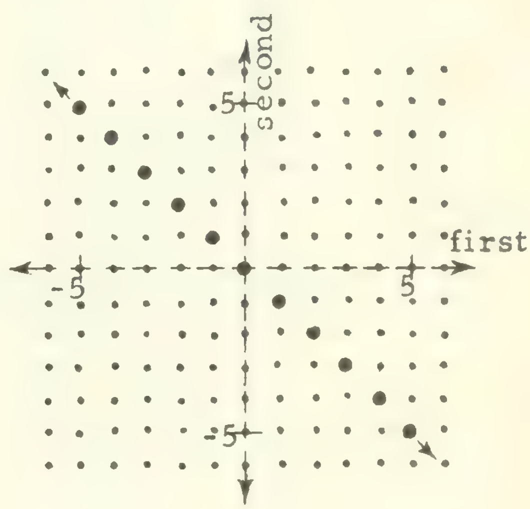

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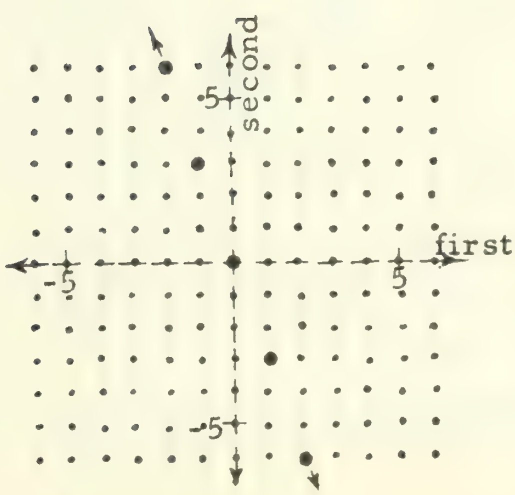

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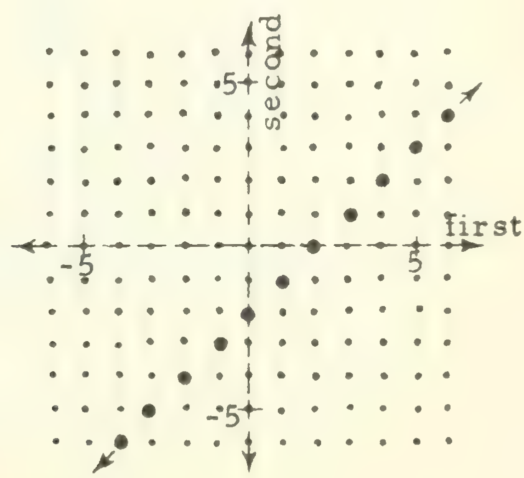

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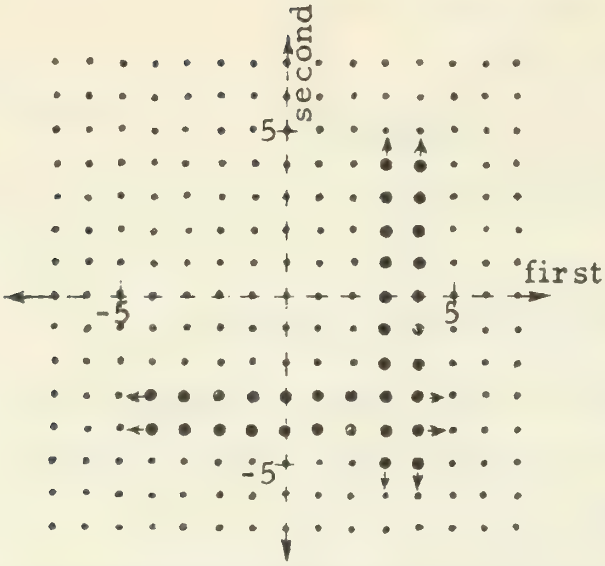

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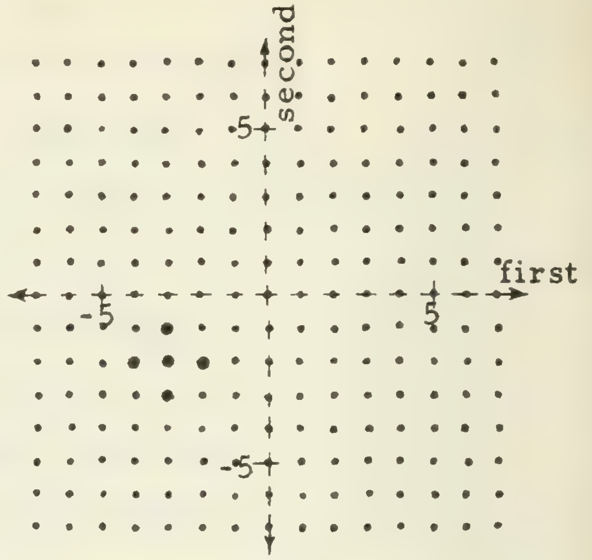
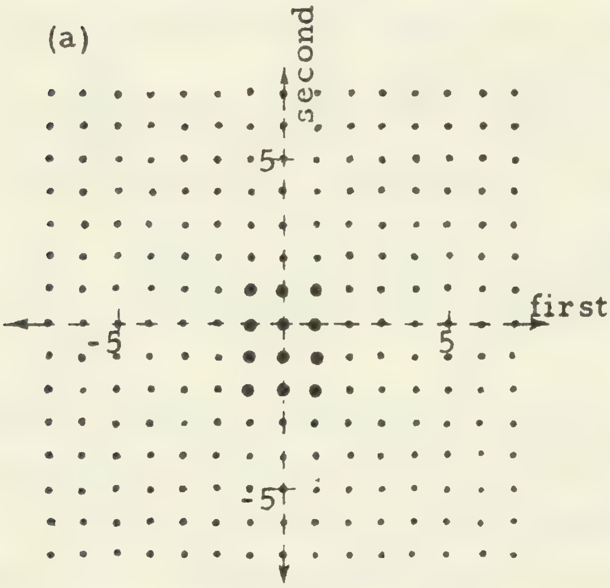


D. 1. Rewrite the descriptions of Part B of the Supplementary Exercises using the brace-notation mentioned in Part C of page 4-9.

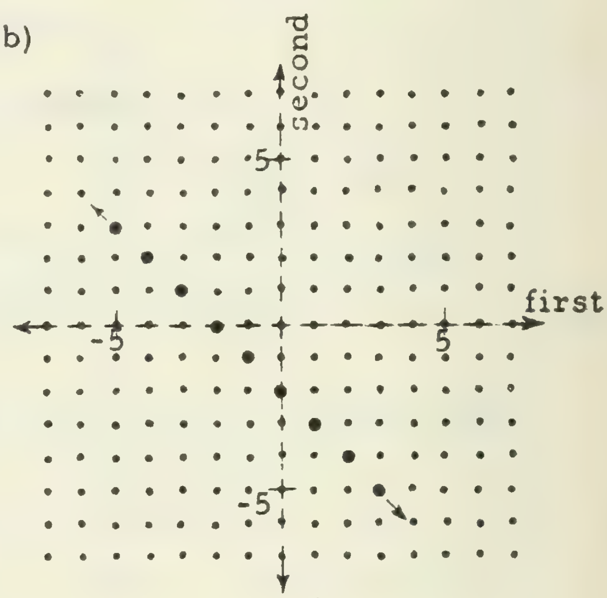
2. Rewrite the descriptions of the sets pictured in Part C of the Supplementary Exercises, using the brace-notation.

3. Describe the sets pictured below [by using the brace-notation].

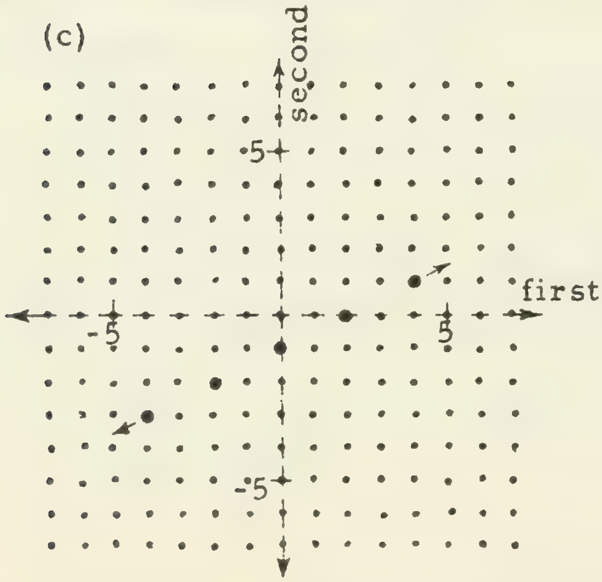
(a)



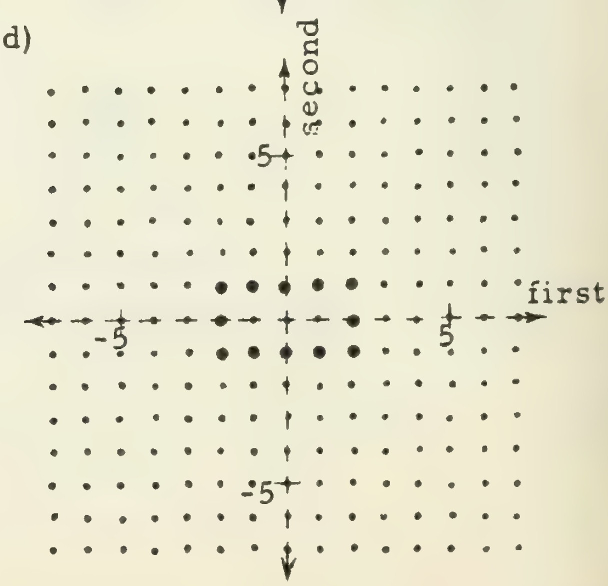
(b)



(c)



(d)



E. In each of the following exercises you are given a description of a set R and a set S. For each exercise,

(a) plot the points in each set on the same diagram,

(b) tell the number of points in each set,

(c) tell the number of points in the intersection, and

(d) tell the number of points in the union.

1. $R = \{(x, y),\ x \text{ and } y \text{ integers}:\ -3 < x < 0 \text{ and } -5 < y < -2\}$

   $S = \{(x, y),\ x \text{ and } y \text{ integers}:\ -1 < x < 3 \text{ and } 1 < y < 5\}$

2. $R = \{(x, y),\ x \text{ and } y \text{ integers}:\ 2 < x < 6 \text{ and } 3 < y < 6\}$

   $S = \{(x, y),\ x \text{ and } y \text{ integers}:\ 1 < x < 4 \text{ and } 2 < y < 5\}$

3. $R = \{(x, y),\ x \text{ and } y \text{ integers}:\ -6 < x < -1 \text{ and } 0 < y < 4\}$

   $S = \{(x, y),\ x \text{ and } y \text{ integers}:\ -3 < x < 1 \text{ and } -2 < y < 2\}$

4. $R = \{(x, y),\ x \text{ and } y \text{ integers}:\ |x| > 4 \text{ and } |y| < 3\}$

   $S = \{(x, y),\ x \text{ and } y \text{ integers}:\ 3 < x < 6 \text{ and } |y| < 2\}$

5. $R = \{(x, y),\ x \text{ and } y \text{ integers}:\ -6 < x < -3 \text{ and } -6 < y < -3\}$

   $S = \{(x, y),\ x \text{ and } y \text{ integers}:\ -3 < x < 0 \text{ and } -3 < y < 1\}$

6. $R = \{(4, -5), (4, -6), (4, -7), (5, -5), (5, -6), (5, -7)\}$

   $S = \{(x, y),\ x \text{ and } y \text{ integers}:\ |x| > 3 \text{ and } |y| > 4\}$

7. $R = \{(x, y),\ x \text{ and } y \text{ integers}:\ -6 < x < -3 \text{ and } 3 < y < 6\}$

   $S = \{(x, y),\ x \text{ and } y \text{ integers}:\ -5 < x < -2 \text{ and } 4 < y < 6\}$

8. $R = \{(x, y),\ x \text{ and } y \text{ integers}:\ x + y > 6,\ x < 6, \text{ and } y < 6\}$

   $S = \{(x, y),\ x \text{ and } y \text{ integers}:\ x + y \leq 6,\ x \geq 0, \text{ and } y \geq 0\}$

9. $R = \{(x, y),\ x \text{ and } y \text{ integers}:\ |x| \leq 7 \text{ and } y \leq 7\}$

   $S = \{(x, y),\ x \text{ and } y \text{ integers}:\ |x| + |y| \leq 6\}$

10. $R = \{(x, y),\ x \text{ and } y \text{ integers}:\ x - |y| = 5\}$

    $S = \{(x, y),\ x \text{ and } y \text{ integers}:\ 3 < x < 8 \text{ or } 1 \leq y < 3\}$

F. For each of the sets of ordered pairs described below, plot as many of the ordered pairs as you can on a picture of the number plane.

1. The set of all ordered pairs of real numbers such that the first component is equal to the sum of 2 and the product of -1 by the second component.

2. The set of all ordered pairs of real numbers such that the first component is 3 less than the second component.

3. $\{(x, y): \ 3x = 6 - y\}$

4. $\{(a, b): \ b = -a + 7\}$

5. The set of all ordered pairs of real numbers such that the second component is -2.

6. $\{(r, s): \ r > 1 \text{ and } s > -2\}$

7. $\{(s, r): \ s > 1 \text{ and } r > -2\}$

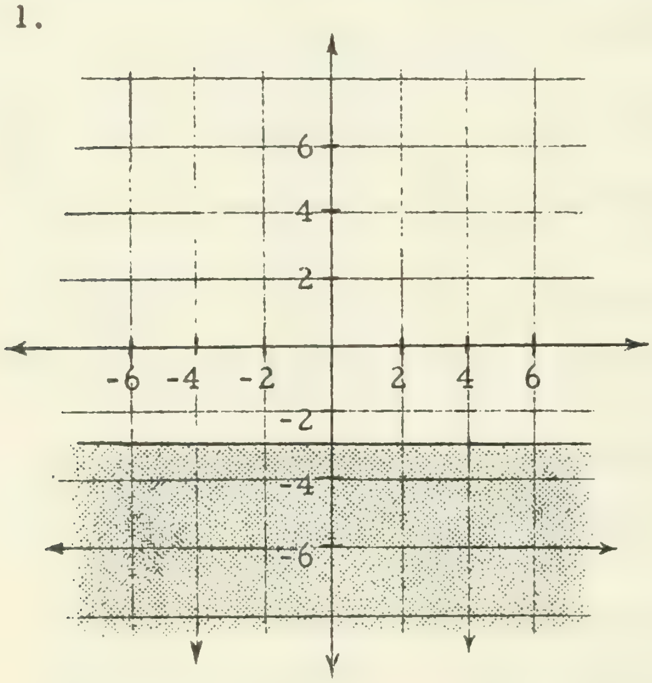
8. $\{(x, y): \ x \leq -1\}$

9. $\{(x, y): \ y \geq 4\}$

10. $\{(x, y): \ x \geq 3 \text{ or } y \leq 4\}$

G. For each set pictured below, write its description.

1.



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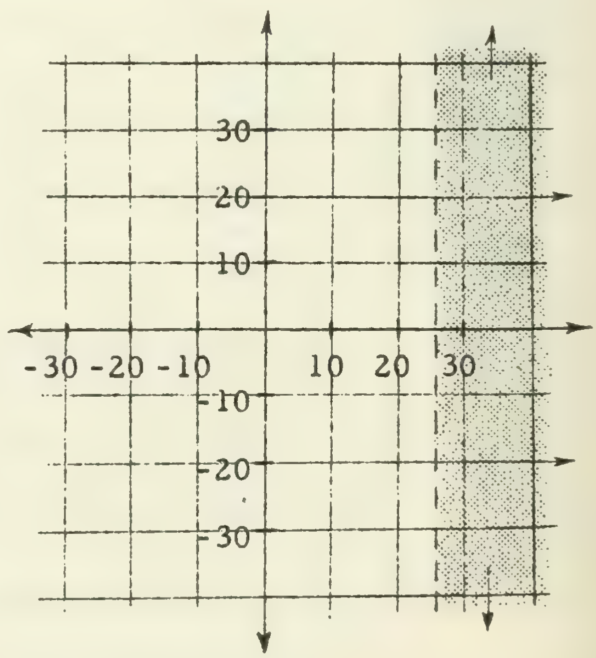

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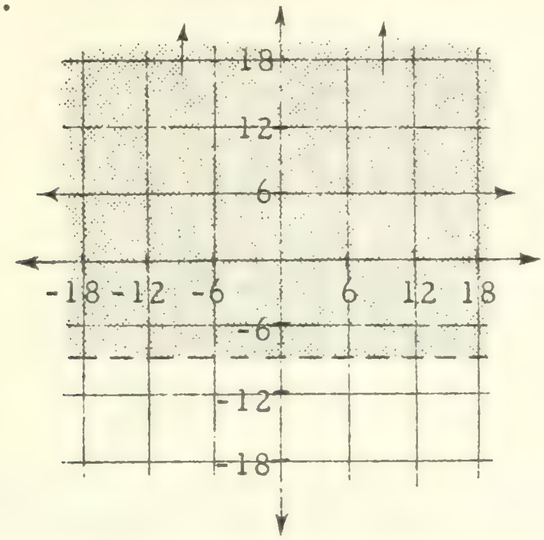

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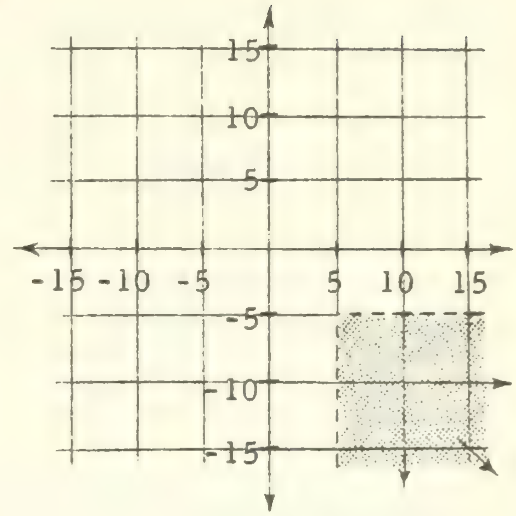

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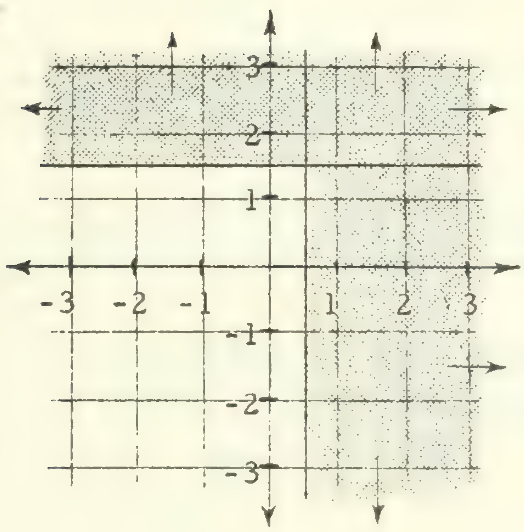
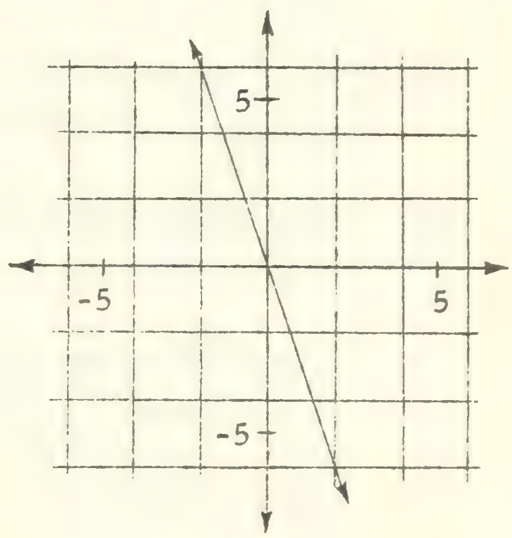


6.



H. Each of the following exercises contains a pair of equations. Graph each of the two equations and tell the components of the points in the intersection of their solution sets.

1. (a) $y - 2x = 2$

   (b) $y + x = 8$

2. (a) $3x - 2y = 0$

   (b) $x + y = 10$

3. (a) $x = 7$

   (b) $3 + y = \frac{5}{7}x$

4. (a) $y = -2x$

   (b) $x = -5y$

(continued on next page)

5. (a) $3x - y = 7$
   (b) $4x + 3y = -8$

6. (a) $y + 2 = 3x$
   (b) $y = 3x + 4$

7. (a) $xx = 49$
   (b) $yy = 16$

8. (a) $|y| = 4$
   (b) $x - 3y = 3$

9. (a) $2x = 5 - y$
   (b) $\frac{1}{2}y = x + 7$

10. (a) $|x| = 3$
    (b) $|y| = 6\frac{1}{2}$

I. Complete each of these sentences to true ones in at least one way.

1. 4 is a factor of 56 with respect to the set of ________________ because 4, 56, and ______ belong to this set and $56 = 4 \cdot$ ________.

2. -3 is a factor of 42 with respect to the set of ________________ because -3, 42, and ______ belong to this set and $42 = -3 \cdot$ ______.

3. -8 is a factor of -72 with respect to the set of __________ because -8, -72, and _______ belong to this set and $-72 = -8 \cdot$ ________.

4. -9 is a factor of -12 with respect to the set of __________ because -9, -12, and ______ belong to this set and $-12 = -9 \cdot$ _______.

5. $\frac{1}{3}$ is a factor of 15 with respect to the set of ____________ because $\frac{1}{3}$, 15, and _____ belong to this set and $15 = \frac{1}{3} \cdot$ _________.

6. $\frac{2}{3}$ is a factor of $\frac{3}{16}$ with respect to the set of _______________ because $\frac{2}{3}$, $\frac{3}{16}$, and ________ belong to this set and $\frac{3}{16} = \frac{2}{3} \cdot$ _____.

7. 9 is a factor of 81 with respect to the set of ___________ because 9, 81, and ______ belong to this set and $81 = 9 \cdot$ ___________.

8. $\sqrt{21}$ is a factor of 21 with respect to the set of _________ because $\sqrt{21}$, 21, and ______ belong to this set and $21 = \sqrt{21} \cdot$ _______.

9. 52 is a factor of 0 with respect to the set of __________ because 52, 0, and _____ belong to this set and $0 = 52 \cdot$ ______.

10. 71 is a factor of -71 with respect to the set of __________ because 71, -71, and _______ belong to this set and $-71 = 71 \cdot$ _______.

11. 937 is a factor of $937 \cdot 578$ with respect to the set of ___________ because 937, $937 \cdot 578$, and __________ belong to this set and $937 \cdot 578 = 937 \cdot$ _________.

12. 31 is a factor of $31 \cdot 78 + 31 \cdot 22$ with respect to the set of __________ because 31, $31 \cdot 78 + 31 \cdot 22$, and __________ belong to this set and $31 \cdot 78 + 31 \cdot 22 = 31 \cdot$ __________.

13. 17^2 is a factor of $(17 \cdot 4) \cdot (17 \cdot 6)$ with respect to the set of ____________ because 17^2, $(17 \cdot 4) \cdot (17 \cdot 6)$, and _____________ belong to this set and $(17 \cdot 4) \cdot (17 \cdot 6) = 17^2 \cdot$ __________.

14. $\_\_\_\_\_^2$ is a factor of $21 \cdot 35$ with respect to the set of ____________ because $\_\_\_\_^2$, $21 \cdot 35$, and 15 belong to this set and $21 \cdot 35 = \_\_\_^2 \cdot 15$.

J. Use exponents to abbreviate each expression.

1. $-3 \cdot -3 \cdot -3(bbb)$
2. $ccc(ddd)$
3. $27 \cdot 27 + rrrr$
4. $aaaaa - (-x \cdot -x \cdot -x \cdot -x)$
5. $(rs)(rs)(rs)(rs)$
6. $(ddg)(ddg) - (kkk)(kkk)$
7. $(xyy)(xyy)x$
8. $(rr + s)(rr + s)$
9. $\dfrac{-4 \cdot -4 \cdot -4(aa) - bbbb}{2 \cdot 2(aaa) + 8 \cdot 8 \cdot 8 \cdot 8b}$

K. Simplify.

1. $b^7 \cdot b^3$

2. $(-2)^4(-2)^7$

3. $4^2 \cdot 4 \cdot 4^4 \cdot 4^2$

4. $x^3 \cdot x^7 \cdot x^2 \cdot x^4$

5. $(2k^3)(5k^4)$

6. $(-2r^5)(5r^6)$

7. $(5xy^2)(-3x^2y)$

8. $(4ts^3)(-5t^3s^5)$

9. $(.3n^2r)(.7rn^3)$

10. $(\frac{1}{5}c^3d^5)(\frac{1}{7}c^5d^4)$

11. $\frac{k^7}{k^3}$

12. $\frac{5^5}{5^3}$

13. $\frac{-6^3}{6^2}$

14. $\frac{k^{10}}{-k^7}$

15. $\frac{9a^3b}{18a^5b^7}$

16. $\frac{70ts^3}{35ts^2}$

17. $\frac{45r^3t^5u^7}{-5u^6t^5r^7}$

18. $\frac{-100c^3de^5}{-5c^3d^2e^4}$

19. $\frac{18x^{70}y^{50}z^2}{-18x^{70}y^{50}z^2}$

20. $\frac{-5a^8b^8c}{10a^7b^7c^7}$

21. $(x^2)(x^5) - x^3$

22. $(x^3)(x^3) - x^3$

23. $(-5a^2)(-3a^3b^2)$

24. $(-7y^7)(-2yz^2)$

25. $(m^3n)(2mn^2)$

26. $(x_1^2x_2^3)(x_2^5x_1^3)$

27. $\frac{305^{98}}{305^{97}}$

28. $\frac{(-32)^5}{(-32)^6}$

29. $\frac{-105nrtu^5}{-1050n^2r^3t^4u^2}$

30. $\frac{-50x^2y^3z}{-25x^2y^2z^4}$

31. $\frac{24a^2bc^3}{-8x^2yz^3} \times \frac{15xy^2z}{-3ab^2c^2}$

32. $\frac{40mpr^3}{15k^2jt^3} \times \frac{-3kj^3t}{-4m^2pr^7}$

33. $\frac{2(r+t)^2(k+j)^3}{5xy} \times \frac{20x^3y^4}{4(r+t)^2}$

34. $(x^2)^5$

35. $(-y^2)^3$

36. $(-z^5)^4$

37. $(-5t^2)^4$

38. $(-y)^{171}$

39. $(-xy)^{204}$

40. $(2x^2y^3)^5$

41. $(-3a^2 b^4)^2$

42. $(3a^3b^4)(5ab^2)^2$

43. $(-xy^3)(4x^4y^2)^5$

44. $(-\frac{1}{4}a)^3$

45. $(-\frac{1}{4}a)^2$

46. $(\frac{x}{5})^3(5x)^2$

47. $(-\frac{t}{3})^2(-\frac{t}{3})^3$

48. $(\frac{1}{8}x^5)^2(-8x)^2$

49. $(-0.5a)^3(-2a)^2$

50. $(2ab^3c^4)^3$

51. $(-3a^7bc^3)^2$

52. $(-r^2t^3u)^3$

53. $(7tk^2s)^2$

54. $(-y^2)^5(y^4)^2$

55. $(-a^5)^4(-a^4)^5$

56. $(4a^2b^3)^2(2ab^5)^3$

57. $(-x^2yz)^4(-x^5y^2z^4)^7$

L. Each exercise contains two pronumeral expressions. For each pair of expressions, tell whether the expressions are equivalent or nonequivalent. If they are nonequivalent, give a counter-example.

1. $(y^4)^3$, y^7

2. $(3a)^7$, $3a^7$

3. $(ab)^3$, a^3b^3

4. x^4y^7, $(xy)^{11}$

5. $(-x)^3$, $-x^3$

6. $(-x)^4$, $-x^4$

7. $(x^2+y^3)^2$, x^4+y^6

8. $(x^2y^3)^2$, x^4y^6

9. $\frac{3+a^2}{a^2}$, 3

10. $\frac{3a^2}{a^2}$, 3

11. $\frac{27y^4 - 21y^2x^2}{15y^2}$, $\frac{9y^2 - 7x^2}{5}$

M. Simplify.

1. $4m^3 \cdot -5m^2$
2. $-3x^3 \cdot -6x^4$
3. $-5y^4 \cdot 7y^5$
4. $-z \cdot 12z^2$
5. $8mt^2 \cdot -6mt$
6. $-5z^3x \cdot 7x^2z^2$
7. $-4n^2p^3 \cdot 6n^4p^2$
8. $6t \cdot -\frac{1}{2}s$
9. $-\frac{2}{5}a \cdot -3a^2b$
10. $-12rs^2 \cdot -\frac{4}{3}s^2r$
11. $-4a^2b^3 \cdot -4a^3$
12. $-0.2x^2y \cdot -0.4xy^3$
13. $-42x^3 \div (7x^2)$
14. $-66y^5 \div (6y^2)$
15. $40a^4 \div (-2a)$
16. $-22m^2n \div (-11mn)$
17. $44a^2b^2 \div (-4b^2)$
18. $18m^3n^2 \div (6m^3n^2)$
19. $5t(4t + 7)$
20. $9k(2k - 5)$
21. $-5m(m^2 - 2m)$
22. $-7c^2(10c - 4c^2)$
23. $pq(p + q)$
24. $-xy(y - x)$
25. $3r(r^2 - 2r + 8)$
26. $8s(7 - 2s + 6s^2)$
27. $-3t(2t^2 - 7t + 9)$
28. $-12m(8 - 3m + 6m^2)$
29. $4xy(x^2 + xy + y^2)$
30. $-3ab(4a^2 - 6ab + 7b^2)$
31. $24\left(\frac{1}{3}p^2 - \frac{1}{4}p + \frac{7}{8}\right)$
32. $-56\left(\frac{1}{7}u^2 - \frac{3}{8}u + \frac{1}{14}\right)$
33. $4m^2(3m^2 - 7m + 2)$
34. $-9k^2(k^2 - 7k - 3)$
35. $-5cd(3d^2 - 4cd + 9c^2)$
36. $xyz(x^2 + y^2 + z^2)$
37. $m^2n^2p^2(3mn - 4mp + 3np)$
38. $5r^2s^2t^2(5rt + 3sr - 2st)$
39. $(30m + 40n) \div 5$
40. $(15k + 36n) \div 3$
41. $(48a - 24b) \div -4$
42. $(20m^3n^2 - 30m^2n^3) \div (-20m^2n^2)$
43. $(28k^2m - 84km^2) \div 28mk$
44. $(2x^3y^2 - xy) \div (-xy)$
45. $(10m^2 - 30m + 50) \div -10$
46. $(-6p^2 - 7a + 3) \div -1$
47. $(30p^4 - 15p^3 + 20p^2) \div (5p^2)$
48. $\dfrac{(p + 7)^2 + 8(p + 7)}{p + 7}$
49. $\dfrac{2(x + y + 1)^2 - 5(x + y + 1)}{-(x + y + 1)}$
50. $\dfrac{3(x + 1)(y + 2z) - 4(x + 1)^2(y + 2z)}{(x + 1)(y + 2z)}$

N. Factor.

1. $7a + 7b$
2. $6h - 6j$
3. $tm + tn$
4. $rx - yr$
5. $2bx + 2by$
6. $tz - t$

7. $12r - 6t$
8. $30c + 20d$
9. $48x - 72y$
10. $xy^2 - 2x$
11. $7p^4 + 7p^2$
12. $\frac{1}{2}ha + \frac{1}{2}hb$
13. $\pi r^2 + \pi R^2$
14. $3p^2 - 9$
15. $ab^2 - 2a^2b$
16. $50uv - 75u^2v^2$
17. $33m^3n^2 - 22m^2n$
18. $59 \cdot 37 + 41 \cdot 37$
19. $2x^2 + 8x + 4$
20. $3p^2 - 6p - 60$
21. $ay + yb - aby$

22. $z^2 - 81$
23. $16 - p^2$
24. $9 - q^2$
25. $144 + 4c^2$
26. $25m^2 - n^2$
27. $n^2 - 4s^2$
28. $m^4 - 9$
29. $s^2 - (t^2/16)$
30. $25 - (k^2/64)$
31. $j^2 - 144$
32. $x^4 - 0.0625$
33. $\pi R^2 - \pi r^2$
34. $25m^2 - 64n^2$
35. $49x_1^2 - 64x_2^2$
36. $100r_1^2 - 81r_2^2$

37. $p^2 + 4p + 3$
38. $u^2 + 9u + 18$
39. $m^2 + 12m + 27$
40. $3c^2 + 24c + 45$
41. $5y^2 + 60y + 175$
42. $48 + 22x + 2x^2$
43. $3y^3 + 45y^2 + 108y$
44. $2z^4 + 26z^3 + 80z$
45. $12m^2 - 27n^2$
46. $630c^2 - 70$
47. $ax^2 + 3ax + 2a$
48. $5y^2x^4 - 16y^2$
49. $(x + 3)^2 - 3(x + 3) - 4$
50. $(a + b)^2 - 6(a + b) - 7$
51. $(2x + y)^2 + 4(2x + y) - 21$
52. $(k - p)^2 + (k - p) - 6$
53. $2(r + s)^2 + 5(r + s) + 2$
54. $3(x - y - 1)^2 + 10(x - y - 1) + 3$
55. $4(x + y)^3 - (x + y)(y + z)^2$
56. $9(3k - 2)(k + 7)^2 - (3k - 2)$
57. $(x + y)^2 - (x - y)^2$
58. $(3a - 2b)^2 - (7a - 5b)^2$

O. Use the idea of the HCF in factoring these expressions.

1. $2a^3b + 3ab^2$
2. $16x^2y^3 - 12xy^4$
3. $8m^3p^2 + 2p^2m^4$
4. $6a^3b^2c^2 + 15abc^7$
5. $3r^2s^2 + 3rs$
6. $7xyz - 6xy - 7xy^2z$
7. $2xy + 8xz + 4x^2$
8. $5x^2 + 25x^3 + 30x^9$

P. Find an LCM of the given pronumeral expressions.

1. $a, 3b, 5c$
2. $10m, 30mn, 30nr$
3. k, k^4, k^8
4. $3y^3, 6y, y^7$
5. $5km^3, 15k^3m$
6. $1, 3xy, 27x^2y$
7. $4cd^2, 12c^2d$
8. $9j, 18, 27jk$
9. $3x^2y^3z, 9x^3yz^2, -18xy^2z^3$
10. $6g^2h^2, 3c^2d^2, 12cdjh$
11. $r + s, 13$
12. $5(a + b), 15$
13. $r + s - 2t, 14$
14. $12k - 24, 36$
15. $6(x + y), 18x + 18y$
16. $2(3a - b), (b - 3a)^2$
17. $2r - 3s, 2r + 3s$
18. $7 + 5k, 5k + 7$
19. $3(a - 2b), 2(a - 3b)$
20. $x + y, x^2 - y^2$
21. $x^8 - y^8, x^4 + y^4, x - y$
22. $a + 2, a - 2, a^2 - 4$
23. $x + 1, 2x - 1$
24. $7x^2 + x - 1, x^2 + x - 7$
25. $(a + b)(a + 3b)^2(a - b), (a - b)^2(3a + b), (a + b)^2$

Q. Reduce these fractions.

1. $\frac{7p}{7q}$
2. $\frac{18c}{36d}$
3. $\frac{7r}{14r}$
4. $\frac{9p^2}{11p^2}$
5. $\frac{xy}{zy}$
6. $\frac{mt^3}{nt^2}$
7. $\frac{a^2b}{a^2c}$
8. $\frac{4xm^2}{8ym^2}$
9. $(12p^3)/(4p)$
10. $(27z^4)/(9z^2)$
11. $(8r^2)/(25r^3)$
12. $(7ab^2)/(21a^2b)$
13. $\frac{39t^4s^2}{52ts^3}$
14. $\frac{12m^2n}{-8mp}$
15. $\frac{-30c^2d^2}{-70cd^2}$
16. $\frac{-48k^3r^3}{48r^3x^3}$
17. $\frac{9(y + 4)}{11(y + 4)}$
18. $\frac{35(3 - m)}{55(3 - m)}$
19. $\frac{18(4 - x)}{27(x - 4)}$
20. $\frac{p^2(a + b)}{p(a + b)^2}$
21. $\frac{8y^2(r - 2s)^2}{4y(r - 2s)^2}$
22. $\frac{11(k - 9)}{11k}$
23. $\frac{3p}{3(p + 3)}$
24. $\frac{2y(m + n)}{16y^2}$
25. $\frac{7j(j + 2)}{7j}$
26. $\frac{xy + xz}{uy + uz}$

27. $\frac{sp - sq}{rp - rq}$ 28. $\frac{5x - 5}{x^2 - 1}$ 29. $\frac{y^2 - 49}{7y + 49}$

30. $\frac{64 - r^2}{24 + 3r}$ 31. $\frac{(x + y)^2}{x^2 - y^2}$ 32. $\frac{y^2 - 4}{(y - 2)^2}$

33. $\frac{6p^2 - 6}{18p + 18}$ 34. $\frac{x^2 - 3x}{x}$ 35. $\frac{y}{y^2 + y}$

36. $\frac{k^2 - 3k}{k^2 - 4k + 3}$ 37. $\frac{6a - a^2}{6 - 7a + a^2}$

*

Simplify.

38. $\frac{1}{2} \cdot 40x$ 39. $\frac{5}{k} \cdot k^2$ 40. $pq \cdot \frac{7}{p}$

41. $10x^2 \cdot \frac{5}{2x}$ 42. $ed \cdot \frac{8}{e^2d^2}$ 43. $\frac{12m}{7n} \cdot \frac{28n^2}{48m^2}$

44. $\frac{8a^2}{5b^2} \cdot \frac{10ab}{8b^3}$ 45. $\frac{36m^3d^2}{5e^3} \cdot \frac{15e^2}{9md}$ 46. $\frac{m^5}{a^2p^3} \cdot \frac{a^4p^5}{m^2}$

47. $\frac{5}{y^2 - 4} \cdot \frac{2y + 4}{15}$ 48. $\frac{4a + 12}{16a} \cdot \frac{a^2}{a^2 - 9}$ 49. $\frac{8x - 8y}{xy^2} \cdot \frac{x^2y}{x^2 - y^2}$

50. $\frac{9m^2 - 1}{12} \cdot \frac{2}{1 + 3m}$ 51. $\frac{(a + 3)^2}{x^5} \cdot \frac{5x^6}{5a + 15}$ 52. $\frac{12k^3}{(x - y)^2} \cdot \frac{x^2 - y^2}{15k^2}$

53. $\frac{5ab^2}{8cd} \div \frac{10b^3}{4c^2d^3}$ 54. $\frac{mn^2}{m^2n} \div \frac{m}{n^3}$ 55. $\frac{3a^2b^2}{8c} \div (3ab)$

56. $\frac{16}{x^2 - 1} \div \frac{4}{x + 1}$ 57. $\frac{m^2 - 36}{21} \div \frac{m - 6}{42}$ 58. $\frac{9p - 9q}{pq^2} \div \frac{p^2 - q^2}{p^2q}$

59. $\frac{4 - 2r}{r^2 - 7r + 10} \cdot \frac{r^2 - 25}{6r + 30}$ 60. $\frac{a - b}{a + b} \cdot \frac{b^2 - a^2}{b^2 - 2ab + a^2}$

61. $\frac{5xr^2 - 15xr - 50x}{xr^2 - 25x} \div \frac{10ry + 20y}{6y^2r + 30y^2}$

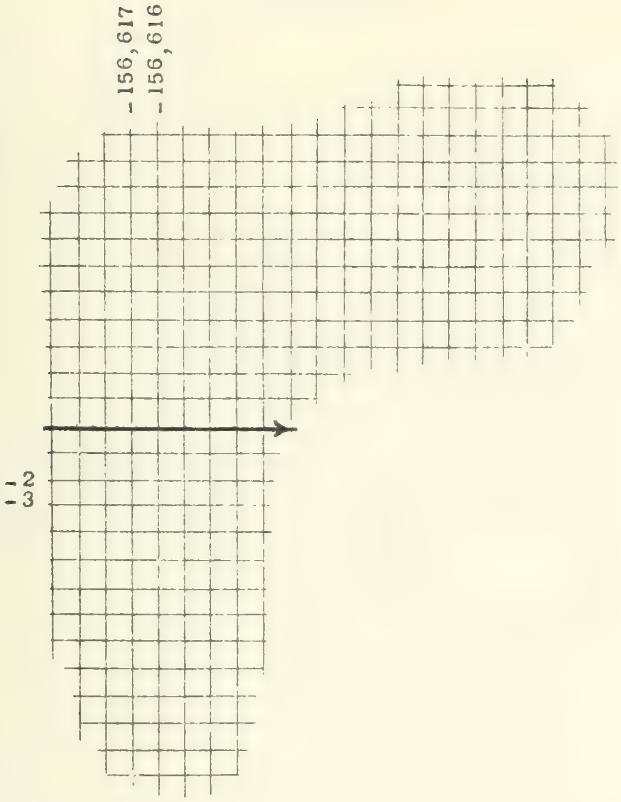
62. $\frac{5}{y^2} - \frac{2}{y}$ 63. $\frac{1}{p^2} - \frac{7}{p}$ 64. $\frac{2}{r^2} - \frac{3}{s}$

65. $\frac{2x}{c^2d} + \frac{2y}{cd^2}$ 66. $\frac{x - 2}{4} + \frac{x + 1}{8}$ 67. $\frac{2a + 1}{5a^2} - \frac{4a - 2}{4a^3}$

68. $\frac{2m}{m^2 - 9} - \frac{4}{m - 3}$ 69. $\frac{2}{x^2 - 25} - \frac{8}{5 - x}$ 70. $\frac{t + 2}{t - 2} - \frac{t - 2}{t + 2}$

NAME ______________________

Region A



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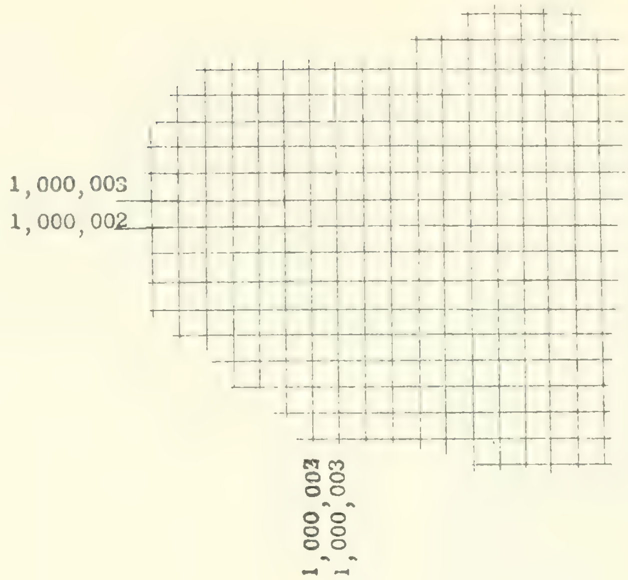

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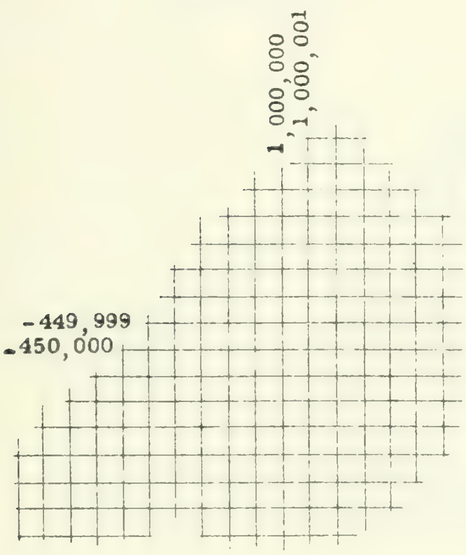

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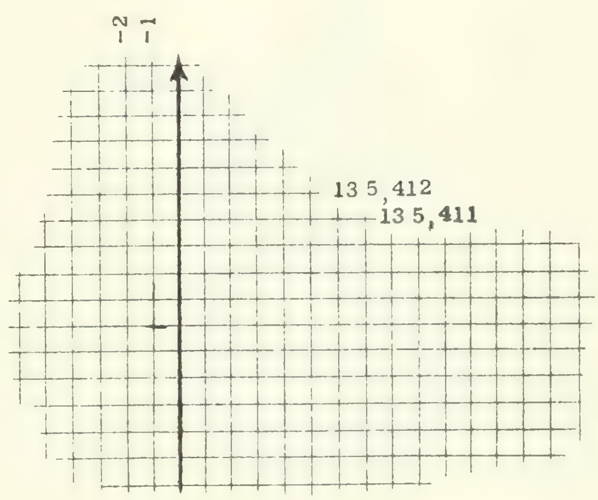

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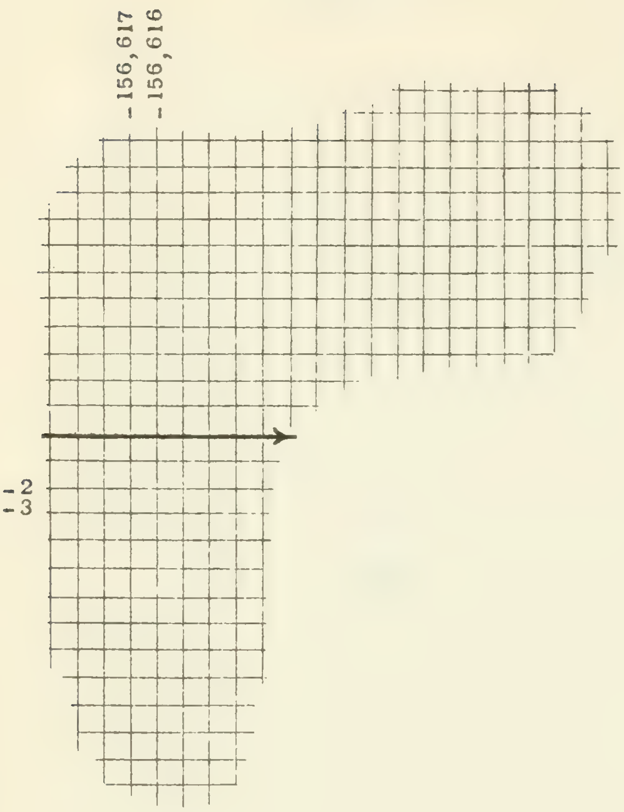

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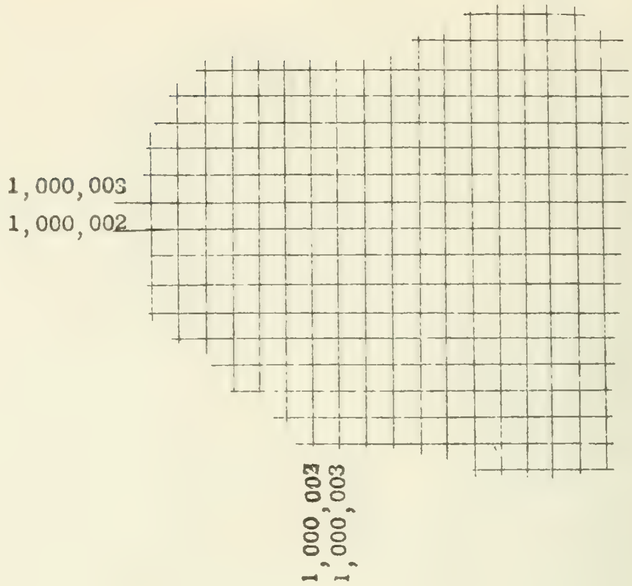

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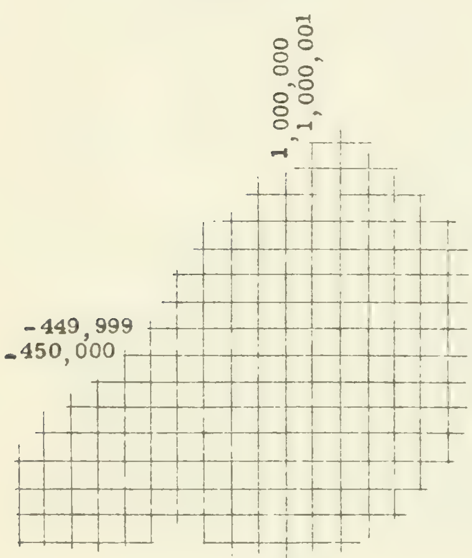

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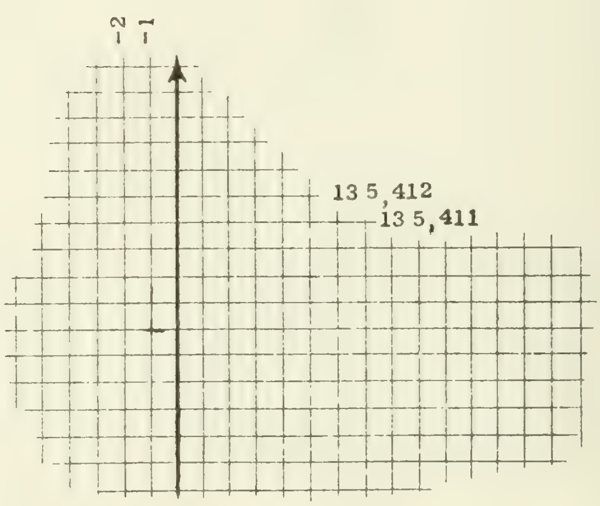

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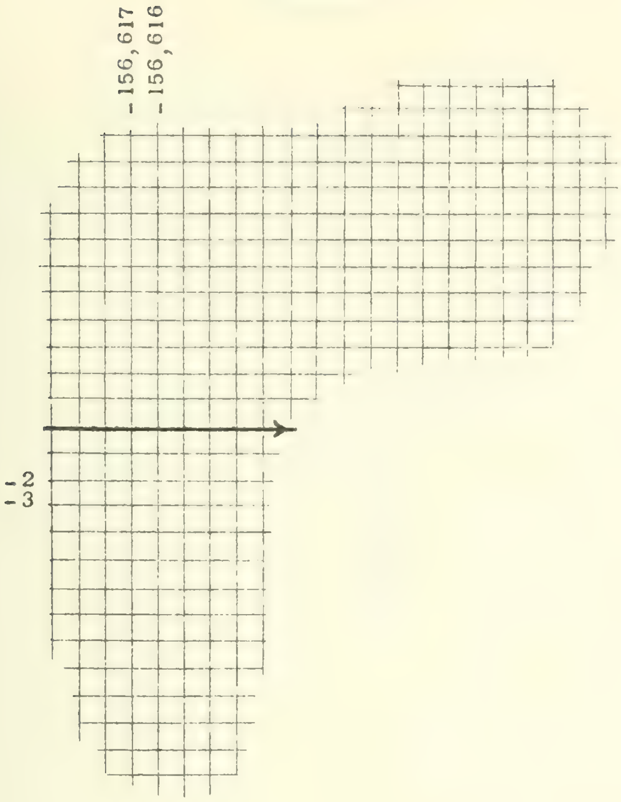

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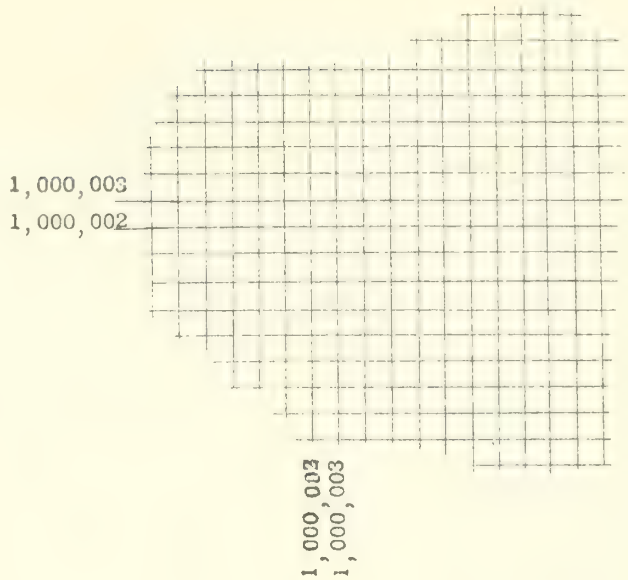

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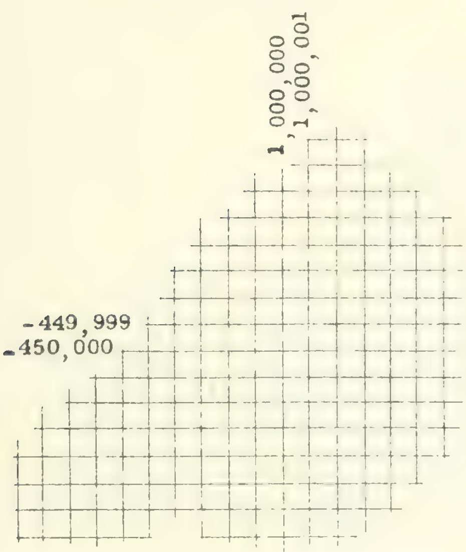

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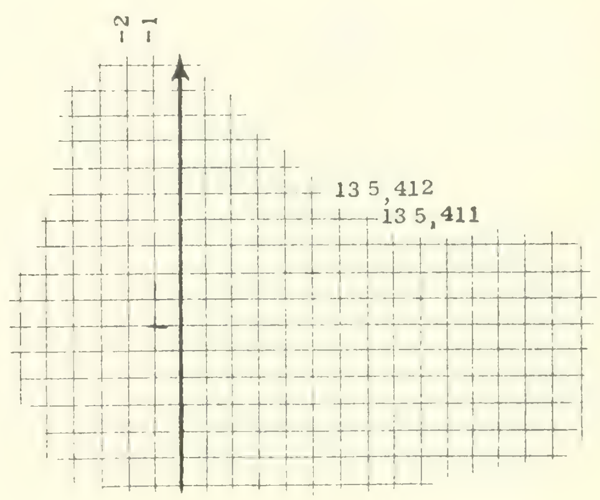

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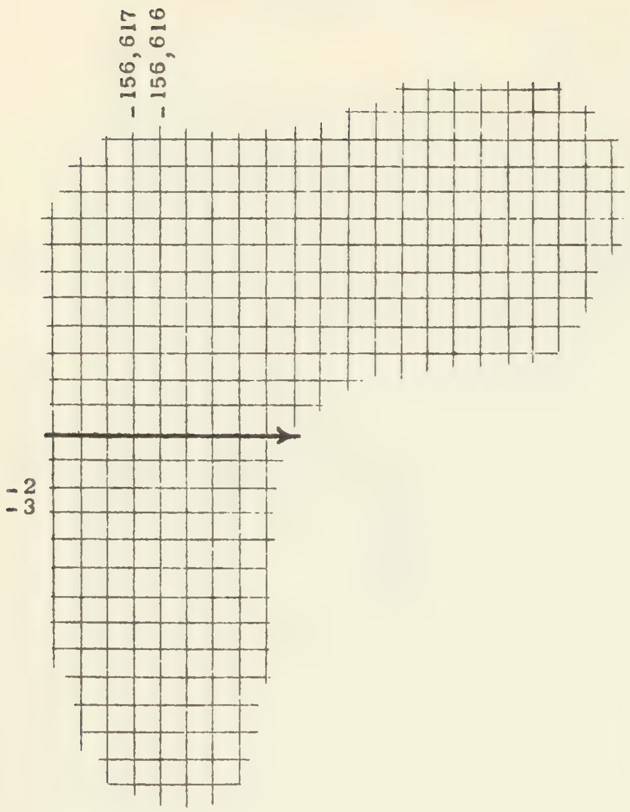

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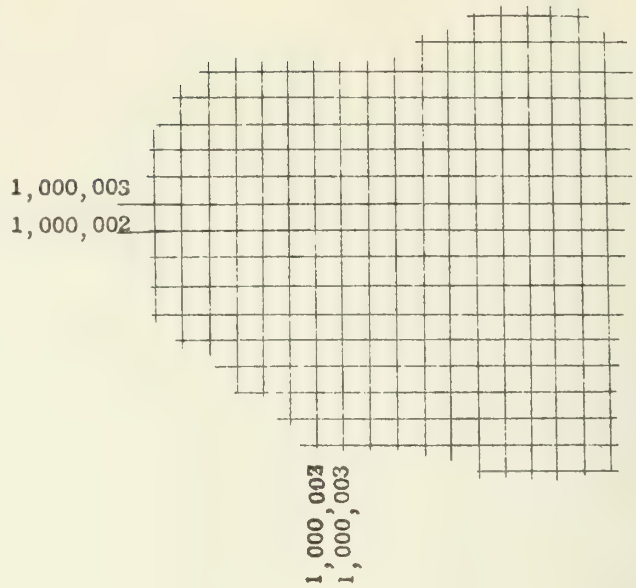

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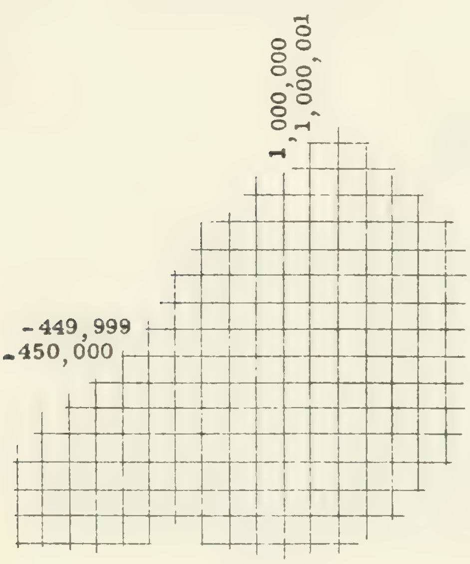

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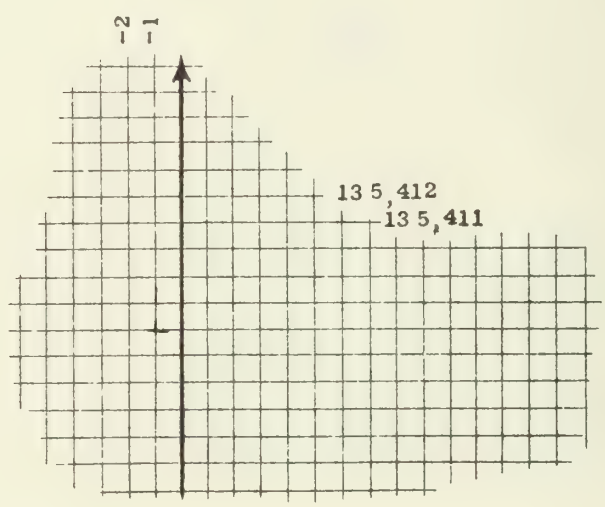

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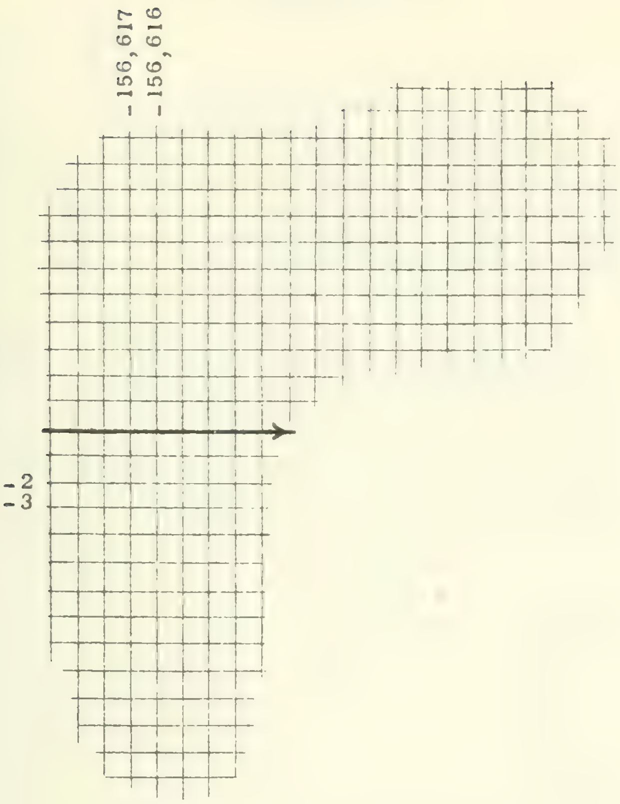

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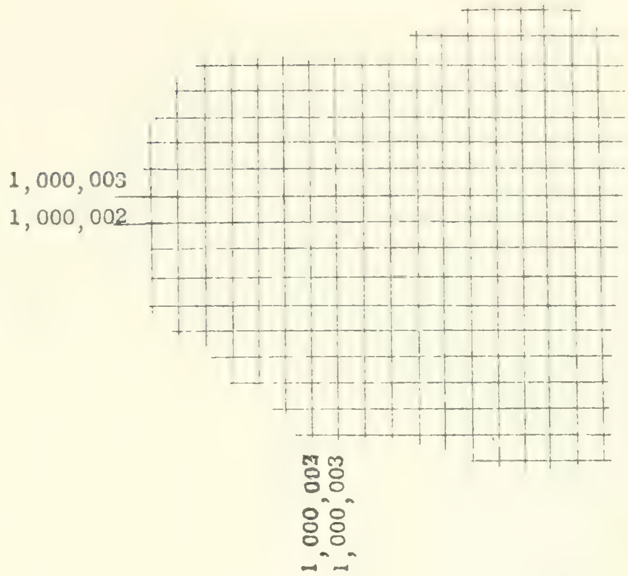

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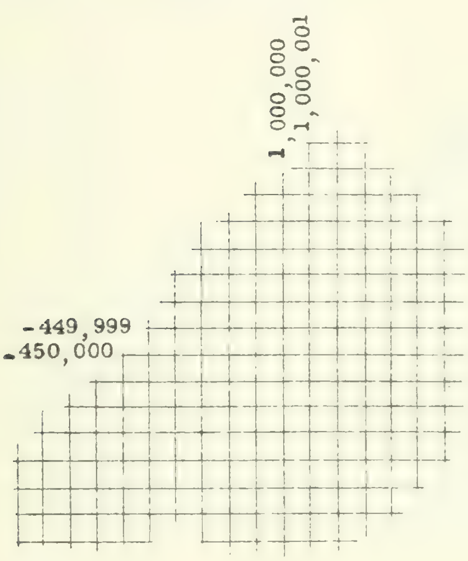

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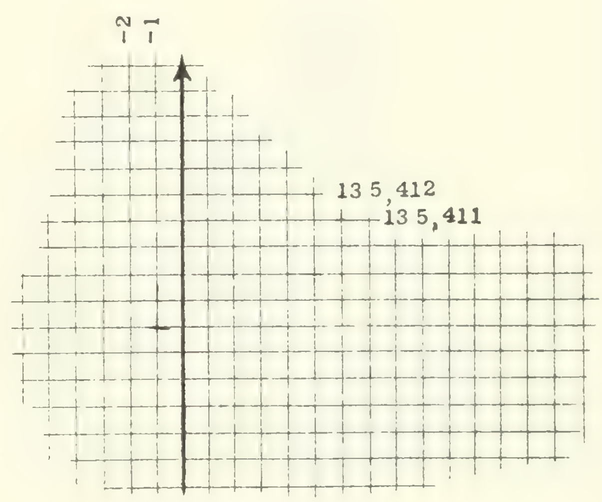

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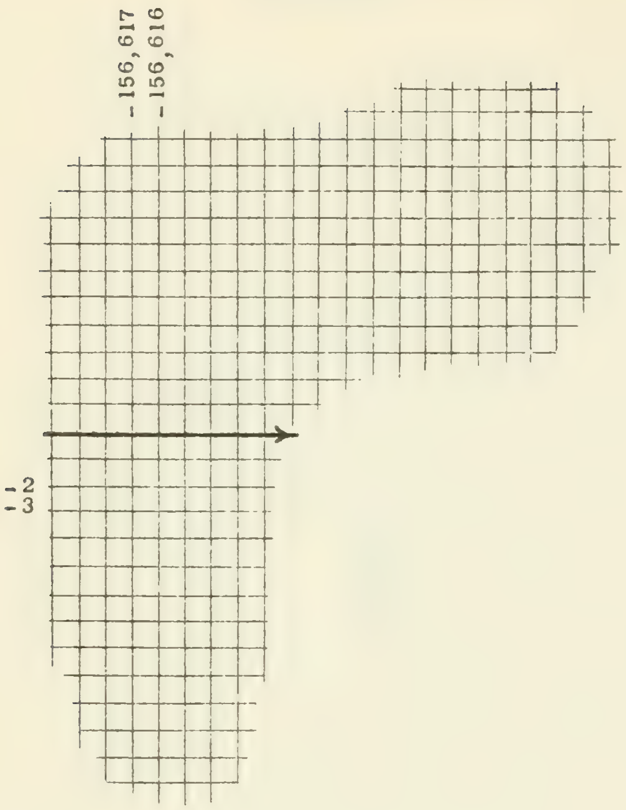
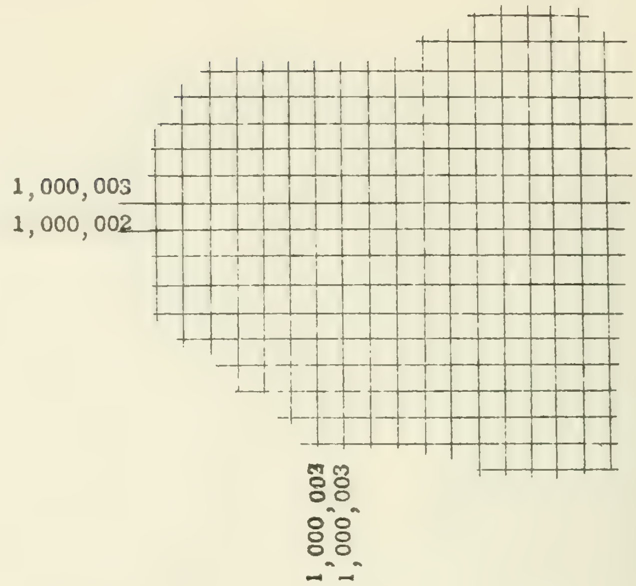


Region B



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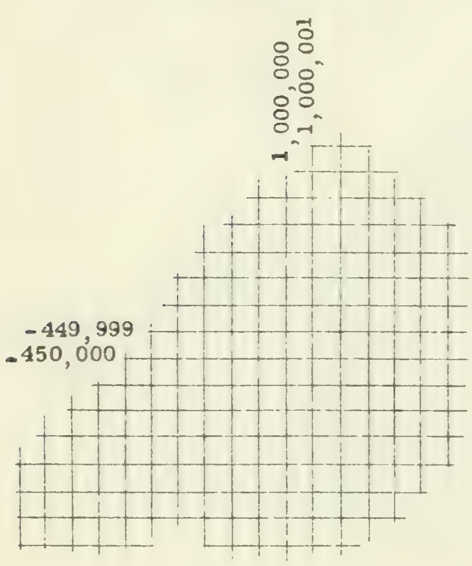

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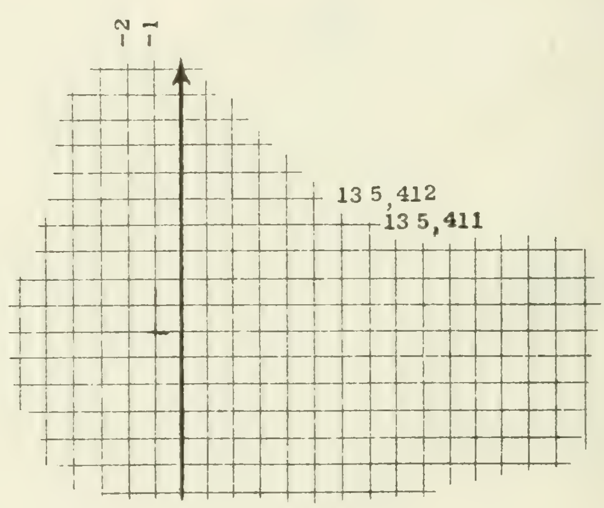

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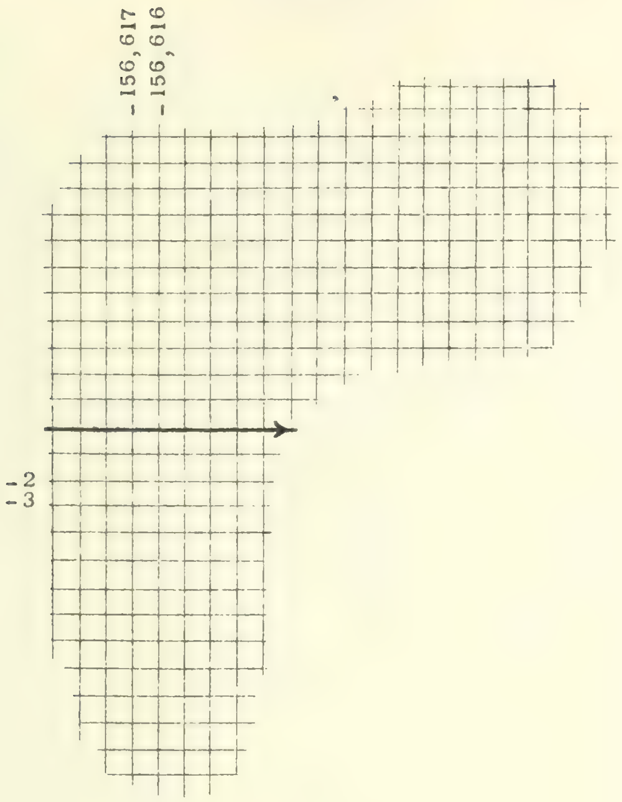

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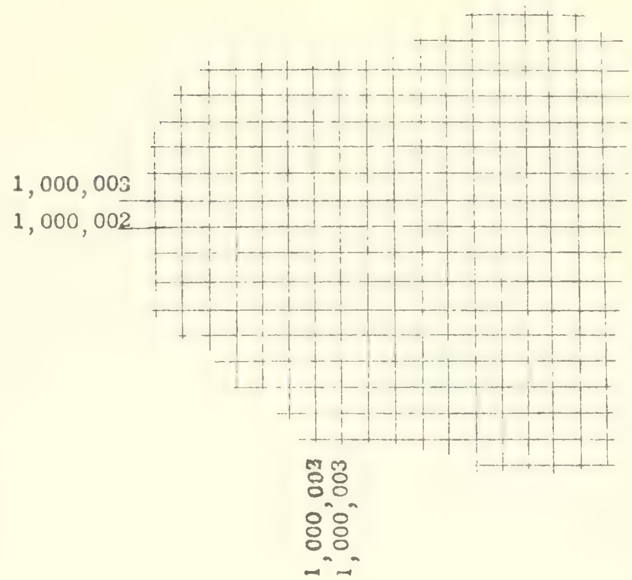

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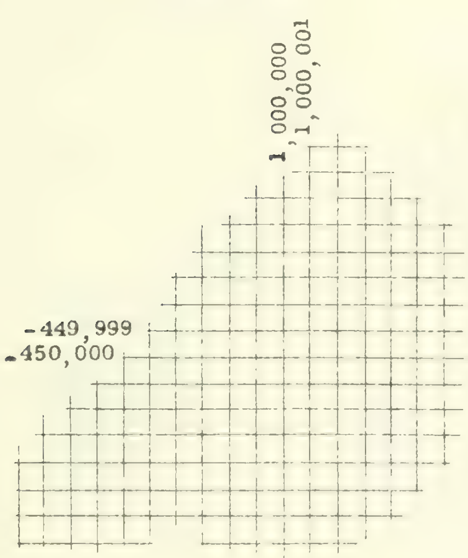

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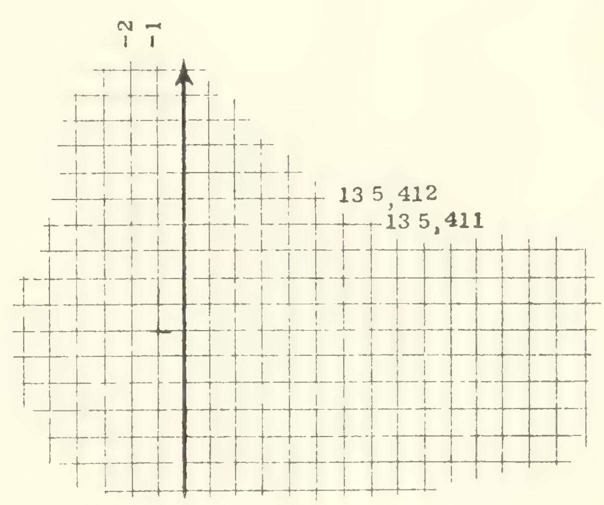

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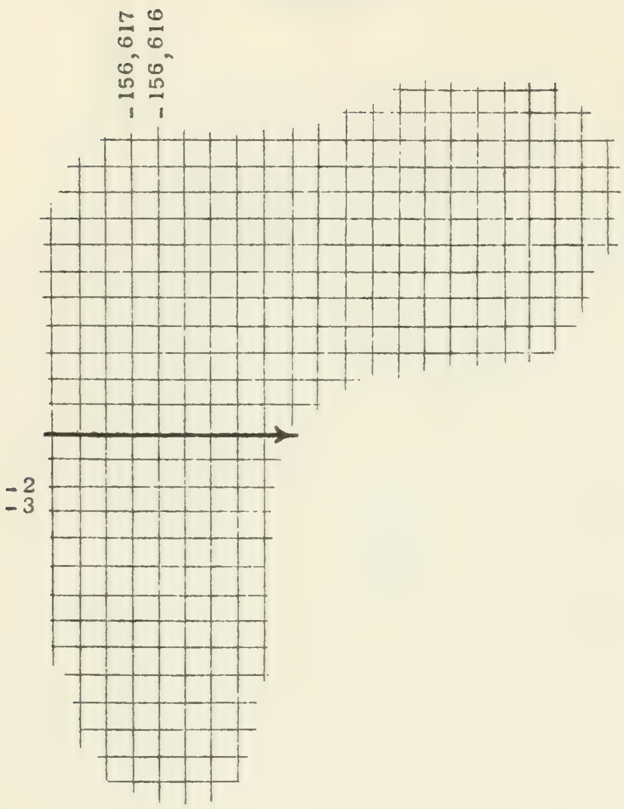

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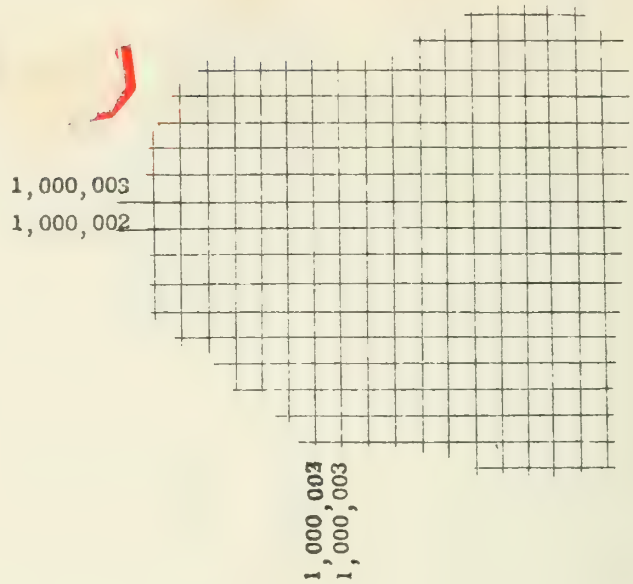
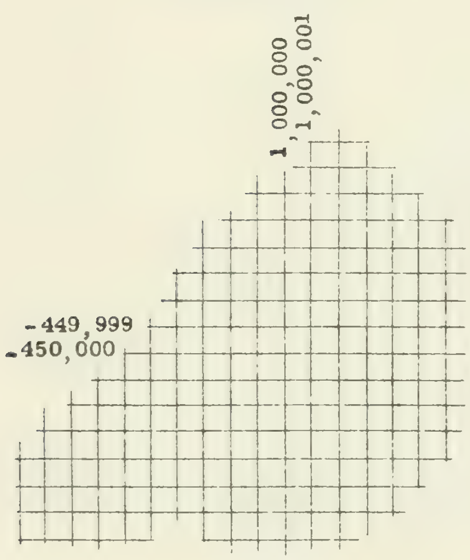


Region C



Region D